Universality in bootstrap and diffusion percolation

Marcelo C. Medeiros, C.M. Chaves*

Departamento de Física, Pontifícia Universidade Católica do Rio de Janeiro, C.P. 38071,
22452-970 Rio de Janeiro RJ, Brazil

Received 3 July 1996

Abstract

Critical concentrations and exponents of bootstrap and diffusion site-percolation models are presented for the triangular lattice. Results are based on numerical simulations and are consistent with universal exponents for random, bootstrap and diffusion percolation.

1. Introduction

Some physical systems are best described by correlated percolation models rather than by the usual standard random percolation (RP).

For example, in the “bootstrap percolation” model (BP) [1,2] sites are occupied at random with probability \( p \) and then each occupied site with less than \( \ell \) occupied neighbors is permanently changed to an empty site. This procedure continues until all occupied sites have at least \( \ell \) occupied neighbors or all sites are empty. The percolation threshold \( p_{BP}(\ell) \) is the value of \( p \) at or above which an infinite cluster of occupied sites remains after the removal procedure.

The BP models dilute magnets in which, under some circumstances, a magnetic atom displays a magnetic moment only if it has enough magnetic neighbors.

In the “diffusion percolation” model (DP) [2,3] sites are initially occupied with probability \( p \). In successive time steps empty sites with at least \( k \) occupied neighbors become occupied. In DP, the percolation threshold \( p_{DP}(k) \) is the value of \( p \) at which an infinite cluster of occupied sites appears after the site-addition procedure.

The diffusion percolation rules are appropriate to describe fluid invasion in porous media and magnetic domain growth. Clearly, \( p_{BP}(0) = p_{DP}(z+1) = p_c \) where \( z \) is the coordination number of the lattice and \( p_c \) its percolation threshold for the (site) RP problem. Also, for a given lattice, \( p_{DP}(k) \leq p_c \leq p_{BP}(\ell) \).

* Corresponding author.
Adler and Aharony [3] have shown that there is a one-to-one correspondence between BP and DP models in the same lattice if one identifies removed sites with less than $\ell$ occupied neighbors to added sites with at least $k = (z - \ell + 1)$ occupied neighbors. It followed from that the inequality

$$p_{DP}(k) \geq 1 - p_{BP}(\ell).$$  \hspace{1cm} (1)

The equality holds for self-matching lattices as the triangular lattice.

For the BP problem there exists a value $\ell_1$ such that in the region $\ell \geq \ell_1$ the lattice is unstable by the formation of polygonal voids [4, 5] and $p_{BP}(\ell \geq \ell_1) = 1$. Correspondingly, for the DP problem, there is a value $k_1 = z - \ell_1 + 1$ such that in the region $k \leq k_1$, $p_{DP}(k \leq k_1) = 0^+$. For the square lattice $\ell_1 = 3$ and $k_1 = 2$, for the honeycomb $\ell_1 = 3$ and $k_1 = 1$, and for the triangular lattice $\ell_1 = 4$ and $k_1 = 3$.

In the region $\ell < \ell_1$ for BP ($k > k_1$ for DP) the transition is second order and critical exponents are defined in analogy to the definitions for the standard RP: for instance, the correlation length, defined as the average distance between sites belonging to the same cluster, diverges as $\xi \propto |p - p_c|^{-\nu}$ while the fraction $P(p)$ of occupied sites belonging to the infinite cluster behaves as $P(p) \propto (p - p_c)^{\beta}$ where $p_c$ stands for the corresponding critical concentrations $p_{BP}$ or $p_{DP}$.

For the BP problem, the $\ell = 1$ case clearly gives the same threshold and exponents as RP because only isolated sites are removed. In the case $\ell = 2$, dangling ends of the percolation cluster are also removed, which does not change the critical concentration, although the critical exponents could be affected.

Numerical simulation and renormalizations group (RG) calculations lead to the belief that $\nu$ is universal for all models while $\beta$ may be $\ell$-depend ($k$-depend) in BP(DP) (see [4] and references therein).

Recently, however, numerical simulations in the square (SQ) and honeycomb (HC) lattices, performed in larger cells [2] found also universal $\beta$ exponents for these models.

It could be argued that the BP $\ell = 2$ problem – the only non-trivial case in the SQ and HC lattices, because $\ell_1 = 3$ for these lattices – has also the same exponent $\beta$ as RP and that the removal of the dangling ends of the percolation cluster would affect, at the most, the amplitude but not the exponent governing the behavior of its “mass”. Indeed, Adler and Stauffer [6] also found RP values for the exponent $\beta$ in the BP $\ell = 2$ problem in the simple cubic lattice (but found different value for the $\ell = 3$ case). However, as reviewed in Table II of Ref. [4], most previous calculations found values of $\beta$ greater than that of RP, even for $\ell = 2$.

In order to settle this question, we present a simulation for the triangular lattice with sizes up to $17280 \times 17280$, which allows the non trivial cases BP $\ell = 3$ and DP $k = 4$ to be performed.

In Section 2 we present the details of the adopted algorithms and the numerical results for the thresholds and critical exponents. Section 3 summarizes our main conclusions.
2. Numerical simulations

2.1. Algorithms

Simulations are performed in cells with $L^2$ sites. Periodic boundary conditions are imposed in the horizontal direction and the spanning of the cell probed in the vertical direction.

A number of simulation cells of a given size were generated for each value of $p$, typically 1000 for sizes up to $L = 700$ and 100 for larger cells.

The critical value $p_c(L)$ is obtained when 50% of the sample are connected ("percolate"). (Here we refer generally to $p_c$ to represent the threshold concentration in either random or correlated percolation models.) The finite-size scaling relation [3]

$$|p_c(L) - p_c| \sim L^{-1/v}$$

allows the determination of $p_c$. The exponent $v$ was assumed universal ($v = \frac{4}{3}$ in two dimensions).

At $p = p_c$, the number $n_s$ of clusters with size $s$ decreases with cluster size as

$$n_s(p_c) \sim s^{-\tau}$$

for large $s$. In two dimensions ($d = 2$) $\tau = \frac{187}{91} \approx 2.055$ for RP. From $\tau$, assuming the validity of the hyperscaling hypothesis, $\beta$ can be obtained [7]:

$$\beta = \left(\frac{\tau - 2}{\tau - 1}\right)vd.$$ 

Each lattice site was represented by a bit of a long integer of 4 bytes (32 bits). Thus, only $L^2/32$ of these integers were needed for a cell containing $L^2$ sites. Dynamical memory allocation allowed cells up to $L = 17280$ to be generated by our program, which was written in the language C.

2.2. Results

In Fig. 1(a) we plot $p_c(L)$ versus $L^{-1/v}$ for the case $\ell = 2$ (BP) whereas in Fig. 1(b) the case $\ell = 3$ is shown. As expected, the critical threshold is the same, within numerical error, as for (site) RP in the triangular lattice ($p_c = \frac{1}{2}$) for $\ell = 2$. For $\ell = 3$ we get $p_{BP}(\ell = 3) = 0.6291$.

For the exponent $\tau$, the plot in Fig. 2 shows the case $\ell = 3$, giving $\tau \approx 2.06$. The simulations were performed for cells size $L = 16000$.

As an illustration of the DP case we present in Fig. 3 $p_c(L)$ for $k = 4$. We get $p_{DP}(k = 4) = 0.3708$. The exponent $\tau$ is also, within numerical error, the same as for RP (Fig. 4).

---

1 We grouped several cluster sizes in bins of exponentially increasing sizes; as a result, a log-log plot of $n_s$ vs. $s$ gives $1 - \tau$, rather than $-\tau$. 
We can estimate $P(p)$ by calculating the fraction of occupied sites belonging to the spanning cluster [7]. The log plot of $P(p)$ versus $p - p_c(L)$ displayed in Fig. 5(b) gives, for $L = 3840$, $\beta = \approx 0.17$, close to $\beta = \frac{\xi}{\xi_0} \approx 0.14$ of RP in $d = 2$.

The values obtained for $p_c$ in corresponding bootstrap and diffusion models are summarized in Table 1, whereas the values obtained for $\tau$ are in Table 2.
Fig. 3. Calculated $p_c(L)$ vs. $L^{-1/\eta}$ with $\eta = \frac{4}{3}$ for DP in the triangular lattice with $k = 4$. The extrapolated value is $p_{DP}(k = 4) = 0.3708$.

Fig. 4. Number of clusters vs. cluster size for $(k = 4)$ DP in the triangular lattice. Here $(L = 17.280)$ and we obtained $\tau = 2.03$, consistent with the RP result.

Table 1
Calculated threshold concentrations for corresponding BP and DP models $(k = 7 - \ell)$ in the triangular lattice

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>Bootstrap $p_c$</th>
<th>Diffusion $p_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.4997 \pm 0.0005$</td>
<td>$0.5000 \pm 0.0005$</td>
</tr>
<tr>
<td>3</td>
<td>$0.6291 \pm 0.0005$</td>
<td>$0.3708 \pm 0.0005$</td>
</tr>
</tbody>
</table>

Note: The errors refer to the acceptable range of values consistent with a linear $p_c(L)$ vs. $L^{-1/\eta}$ fit of our data points.
Fig. 5. Log plot of \( P(p) \) vs. \( p - p_c(L) \) for (a) \( L = 704 \) giving \( \beta \simeq 0.42 \); (b) \( L = 3840 \) giving \( \beta \simeq 0.17 \).

Table 2
Calculated critical exponents for corresponding BP and DP models \((k = 7 - \ell)\) in the triangular lattice

<table>
<thead>
<tr>
<th>Bootstrap ( \ell )</th>
<th>Diffusion ( k )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell = 2 )</td>
<td>( k = 5 )</td>
<td>( 2.05 \pm 0.04 )</td>
</tr>
<tr>
<td>( \ell = 3 )</td>
<td>( k = 4 )</td>
<td>( 2.06 \pm 0.04 )</td>
</tr>
</tbody>
</table>

Note: The errors refer to the fitting accuracy of our data in \( n_s \) vs. \( s \) plots.

3. Conclusions

Our results for the percolation thresholds and the exponent \( \tau \) are consistent with universal exponent \( v, \tau \) and \( \beta \) for both BP and DP.

Non-universal \( \beta \) values have emerged only from approximate (RG) calculations or numerical work based on simulations performed on relatively small cells. Our results for intermediate values of \( L \) in \( \ell = 3 \) BP also gave apparent values of \( \beta \) larger than that of RP (and the same was true for the SQ and HC lattices [2]). In Fig. 5(a), a log plot of \( P(p) \) versus \((p - p_c(L))\), for \( L = 704 \), gives \( \beta \simeq 0.42 \pm 0.04 \) which is comparable with the ones quoted in Table II of Ref. [4].

Regarding DP and BP models we find that, within numerical errors, (1) is obeyed as an equality (see Table 1).

In summary, we have performed simulations in two dimensions for all possible BP and DP models in the triangular lattice (this work) and in the SQ and HC lattices [2]. Our cell sizes are at least two orders of magnitude larger than those previously studied.
The main conclusion is that bootstrap and diffusion constraints do not lead to different percolation exponents in \( d = 2 \).

Acknowledgements

We want to thank B. Koiller from the Instituto de Física, Universidade Federal do Rio de Janeiro for useful discussions on BP and DP and to Paulo Mello from the Chemistry Department of our University for computational assistance.

Work partially supported by CNPq.

References