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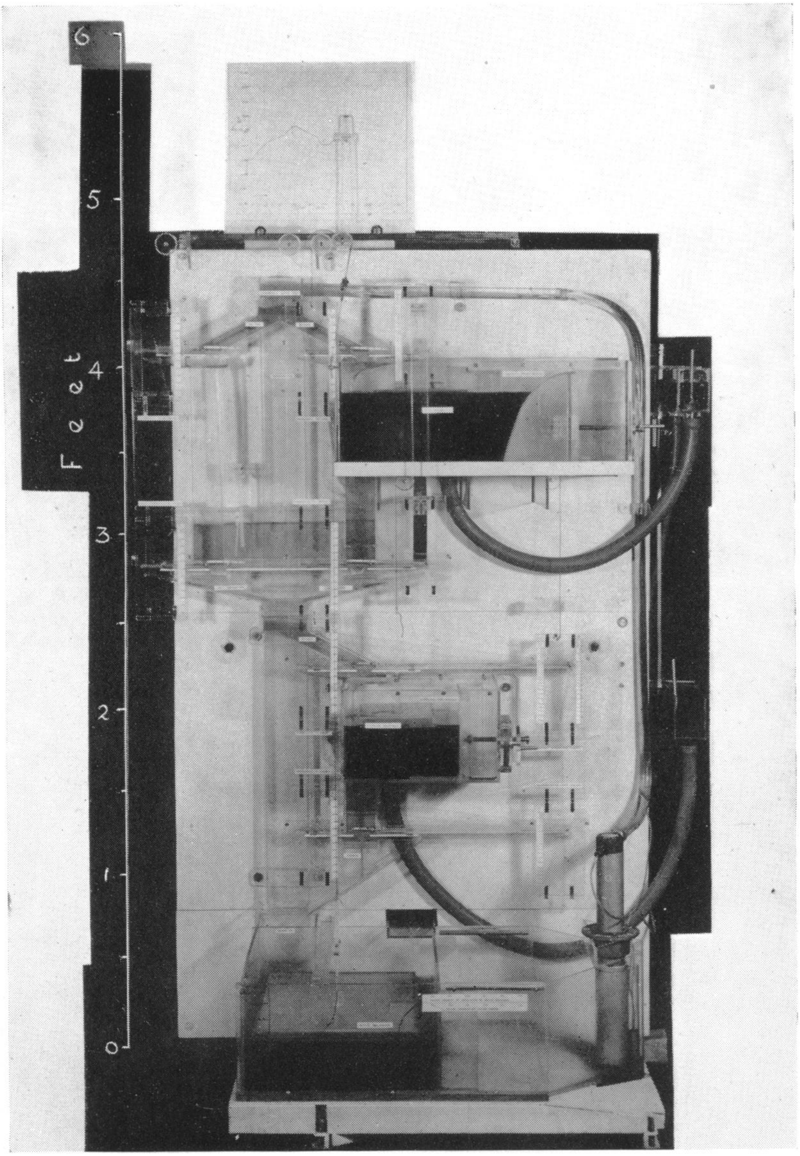

FIG 1.

Mechanical Models in Economic Dynamics[1]

By A. W. Phillips

There has been an increasing use in economic theory of mathematical models, usually in the form of difference equations, sometimes of differential equations, for investigating the implications of systems of hypotheses. However, those students of economics who, like the present writer, are not expert mathematicians, often find some difficulty in handling these models effectively. This article describes an attempt to develop some mechanical models which may help non-mathematicians by enabling them to see the quantitative changes that occur in an inter-related system of variables following initial changes in one or more of them. One model (see photograph in Fig. 1) has been made for the University of Leeds, a second and improved version is now being made for the London School of Economics.

I

Fundamentally, the problem is to design and build a machine the operations of which can be described by a particular system of equations which it may be found useful to set up as the hypotheses of a mathematical model, in other words, a calculating machine for solving differential equations. Since, however, the machines are intended for exposition rather than accurate calculation, a second requirement is that the whole of the operations should be clearly visible and comprehensible to an onlooker. For this reason hydraulic methods have been used in preference to electronic ones which might have given greater accuracy and flexibility,

[1] I wish to thank Mr. W. T. Newlyn, in co-operation with whom the original model was made, for his valuable assistance in design and construction. An article by Mr. Newlyn, based on his experience in using the model at the University of Leeds, will be published in the *Yorkshire Bulletin* in September, 1950. I also wish to thank Professor J. E. Meade, who suggested a number of improvements in the theoretical model and methods by which they might be included in the mechanical one. I am grateful to the London School of Economics for financial assistance which has enabled me to devote the last six months to developing the model.

I am greatly indebted to Mr. and Mrs. R. W. Langley for their willing and generous help in the construction of the first machine.

the machines being made of transparent plastic (" Perspex ") tanks and tubes, through which is pumped coloured water. The accuracy obtained depends on the precision with which the machines are constructed, but there is no difficulty in keeping it within about ± 4 per cent.

Both of the models mentioned above deal with macro-economic theory in terms of money flows; but they are based on an analogy given by Professor Boulding[1] to show how the production flow, consumption flow, stocks and price of a commodity may react on one another. Before describing them it will be convenient to show how Boulding's qualitative model can be developed to make the values of the variables and the relationships assumed to hold between them quantitatively precise and to enable shifts in these relationships to be introduced.

In the model shown diagrammatically in Fig. 2, the production flow of a commodity is represented by the flow of water into a tank. This flow is controlled by a valve, consisting of a flat plate sliding horizontally over a narrow parallel slot. The head of water over the valve is kept approximately constant by an overflow weir at a fixed height above it, so that the rate of flow of water through the valve is proportional to the length of slot uncovered, and can be measured by a linear scale attached to the valve. The production flow goes into the tank containing stocks, from which is drawn the consumption flow, controlled and measured by a second valve similar to the first. The small float in the consumption tube maintains a constant head of water over the valve, irrespective of the level of water in the tank and of the consumption flow.[2]

Price is assumed to be determined at any instant by the quantity of stocks, represented by the quantity of liquid in the tank, and the demand schedule for them, represented by the capacity of the tank at different levels, and is therefore shown inversely by the height of water in the tank. The tank is rectangular except for one end, the shape and position of which are derived from the demand curve for stocks so as to reproduce on the machine the relationship assumed or found to hold between stocks and price of the commodity. This end of the tank, though water-tight,

[1] *Economic Analysis*, Revised Edition, 1948, p. 117.

[2] The actual constant-level float mechanism used, though similar in principle, differs in construction from that shown.

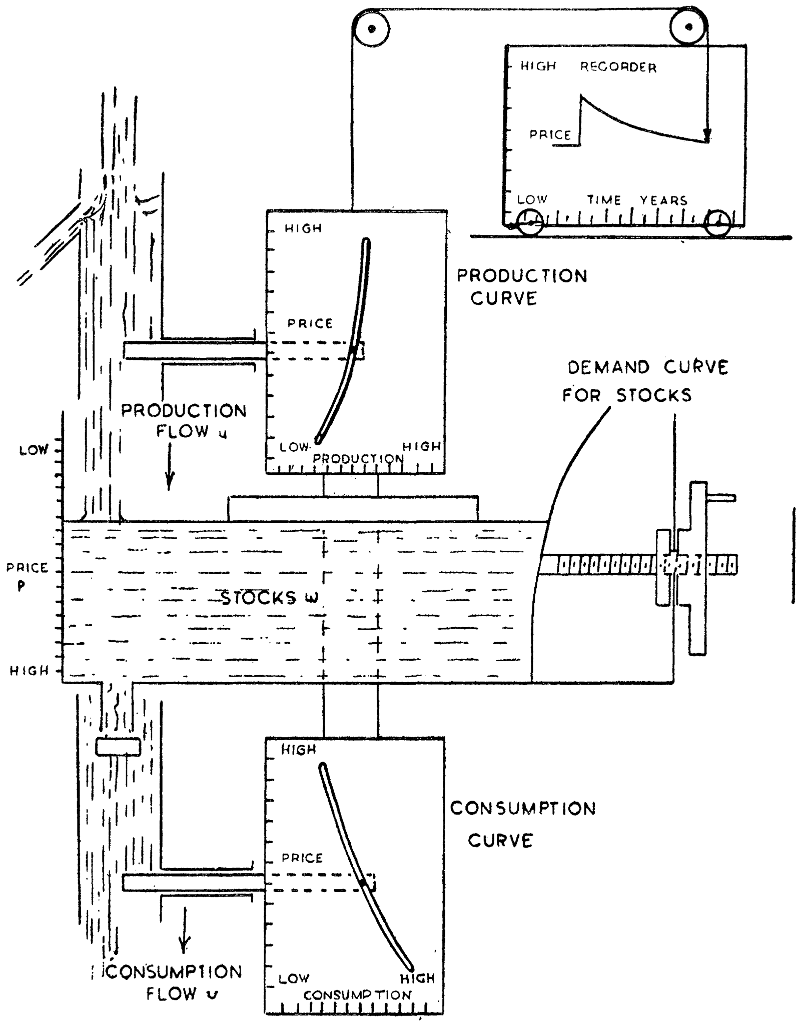
HIGH
RECORDER
PRICE
LOW
TIME YEARS
HIGH
PRICE
LOW
HIGH
PRODUCTION
PRODUCTION CURVE
DEMAND CURVE FOR STOCKS
PRODUCTION FLOW u
LOW
PRICE p
HIGH
STOCKS ω
HIGH
CONSUMPTION CURVE
PRICE
CONSUMPTION FLOW v
LOW
HIGH
CONSUMPTION

FIG 2.

slides freely when the hand-wheel is turned, enabling a shift in the demand curve for stocks to be introduced.

Attached to a float on the tank is a bar, free to move vertically between guides, and carrying two graphs, a production and a consumption curve, which move in front of their respective valves. Each graph is made by cutting a narrow slot in a thin sheet of plastic; a pin projecting from the end of the valve engages in this slot so that when the float moves the graph vertically, the graph moves the valve horizontally, opening or closing it according to the shape of the curve. The graphs are attached to the bar by spring clips and can be moved in any direction to enable shifts of the production and consumption functions to be introduced. Price can be read off a linear scale (the correct way up) marked along the ordinate of each graph, against a cursor line engraved along the side of the valve.

Given the dimensions of the tank and the valves, the choice of units for the scales determines a time constant for the model. This is perhaps most easily shown by an example. Assume that the price scale is so chosen that the required relationship between stocks and price of a commodity is reproduced on the model when one cubic inch of water is made equivalent to one hundred tons of the commodity. Assume also that the valves are so designed that for every inch of valve opening there is a flow of one hundred cubic inches of water per minute, equivalent to ten thousand tons of the commodity per minute of time on the model. If now scales marked ten thousand tons per year per inch of valve opening are chosen as being most suitable for the actual magnitudes of the production and consumption flows, a time constant has been fixed making one minute of time on the model equivalent to one year in reality. Having determined a time constant it is possible to record the path of the price change induced by a shift in one of the curves. The recorder consists of a clockwork mechanism, carrying a plate to which a chart can be attached, and moving towards the left along rails, at the rate of one inch per minute. With a time constant of one minute equalling one year, the abscissa of the chart can be marked off with a scale of one inch equal to one year. Against the chart rests a recorder pen connected to the graph bar by a thread so that it moves vertically with price, which may be marked along the ordinate of the chart. The pen then traces out a graph of price against time in years. Similar recorders could of course be fitted to the other variables.

It is easy to see from the diagram the results of a shift in one of the functions, from an equilibrium position. A spontaneous increase in consumption (shift of the consumption curve to the right) will result in a gradual decrease in stocks and rise in price. The price rise will induce a contraction of consumption and extension of production (movements along the curves) until a new equilibrium is established with higher values of consumption, production and price than at the previous equilibrium position. A spontaneous increase in production (shift of the production curve to the right) will induce a gradual change to a new equilibrium position, with higher values of production and consumption and a lower price than before the change. A spontaneous increase in the demand for stocks (a shift of the demand curve for stocks, or the end of the tank, to the right) will cause an immediate rise in price and production flow, and a fall in consumption flow. Production now being higher than consumption, stocks will gradually rise, inducing a gradual fall in price and production and a rise in consumption until production and consumption are equal again. If there has been no shift in the consumption and production curves this equality will only be reached when stocks have risen sufficiently to bring the price, and also the production and consumption flows, back to their original values. The shape of the graph of price against time drawn by the recorder will be something like that shown in Fig. 2.

This illustrates an interesting point. Although, on the assumptions of this model, it is correct to say that at every instant of time price is determined only by the quantity of stocks and the demand for them, yet it is also correct to say that the long-term trend of the price is completely unaffected by shifts in the demand curve for stocks, being determined only by the position of the production and consumption curves, i.e., the " flow " functions.

The hydraulic model will give solutions for non-linear systems as easily as for linear ones. It is not even necessary for the relationships to be in analytic form: so long as the curves can be drawn the machine will record the correct solutions, within the limits of its accuracy. In giving the equivalent mathematical model, however, the usual linearity assumption will be made, in view of the difficulty of working with non-linear differential or difference equations.

Let u be the production flow, v the consumption flow, w the stocks and p the price, all measured, for simplicity, from a base at which the system is in equilibrium. We have then, by definition, the identity,

$$u - v = \frac{dw}{dt},$$

and three hypotheses,

$$u = lp$$
$$v = mp$$
$$\text{and } w = np,$$

where l, m and n are parameters.

If at time $t = 0$, from equilibrium, there is a spontaneous change, Δv, in consumption, then

$$\frac{dw}{dt} = u - (v + \Delta v) = (l - m)\,p - \Delta v.$$

Also $$\frac{dp}{dw} = \frac{1}{n}.$$

Therefore $$\frac{dp}{dt} = \frac{dp}{dw} \cdot \frac{dw}{dt} = \frac{l - m}{n} \cdot p - \frac{\Delta v}{n}$$

or $$n\frac{dp}{dt} - (l - m)\,p + \Delta v = 0.$$

The solution of this equation,

$$p = \frac{\Delta v}{l - m}\left(1 - e^{\frac{l-m}{n} \cdot t}\right),$$

gives the path of the induced price change. The stability condition is that $\frac{l - m}{n} < 0$. If it is assumed that price falls as stocks increase, then $n < 0$ and the stability condition is that $l - m > 0$ or that $l > m$. In this case the price converges exponentially towards the equilibrium value $\frac{\Delta v}{l - m}$ as $t \rightarrow \infty$. If $l < m$ the system is explosive.

For a spontaneous change, Δu, in the production flow the induced price change is

$$p = -\frac{\Delta u}{l - m}\left(1 - e^{\frac{l-m}{n} \cdot t}\right).$$

And for a spontaneous change in the demand for stocks, causing an immediate change in price of $\triangle p$, the path of the subsequent induced price change is

$$p = -\triangle p\left(1 - e^{\frac{l-m}{n}\cdot t}\right),$$

i.e., the price returns to its original equilibrium value.

This simple model could be further developed, in particular by making a distinction between working and liquid stocks[1], introducing lags into the production and consumption functions[2], and linking the demand curve for liquid stocks to the rate of change of price through a co-efficient of expectations.[3] Each of these developments would result in an oscillatory system. They will not be considered further here, however ; the simple model has been described to show the main principles of the mechanism, which will now be applied to macroeconomic models.

II

1. Description of a Simple Model

In the model shown in Fig. 3, an economy without foreign trade or government operations is assumed. These simplifying assumptions will be relaxed later. The water in the bottom tank, M_1, represents active or transactions money balances, defined as the minimum working balances needed to carry on a given level of economic activity, and assumed to be a function of income.[4] Income, Y, flows from M_1 through a slot, shown in the inset in Fig. 3. The slot is of such a shape that the rate of flow through it is proportional to the height of water in the tank[5] ; the height can therefore be used to measure the income flow, on a linear scale. An

[1] See J. M. Keynes, *Treatise on Money*, 1930, Ch. 29, also J. R. Hicks, *A Contribution to the Theory of the Trade Cycle*, 1950, pp. 47–55.

[2] In the form appropriate to a continuous analysis, i.e., the actual value approaches the value that would obtain if there were no lag at a rate which is a function of the difference between the actual and the unlagged values.

[3] As defined by Lloyd A. Metzler in "The Nature and Stability of Inventory Cycles", *Review of Economic Statistics*, 1941 ; but with slight changes to make the concept applicable to a continuous instead of a discontinuous model.

[4] This hypothesis has been used in a mathematical model by Richard M. Goodwin. See "Secular and Cyclical Aspects of the Multiplier and the Accelerator", *Essays in Honor of Alvin Hansen*, pp. 112–118. Expressed in a slightly different form, it is also made the basis of a model by Paul A. Samuelson : *Foundations of Economic Analysis*, pp. 276–280.

[5] The shape is given by the formula $W = c.\frac{1}{\sqrt{h}}$ where W is the width of the slot at any height h, and c is a constant depending on the viscosity of the liquid.

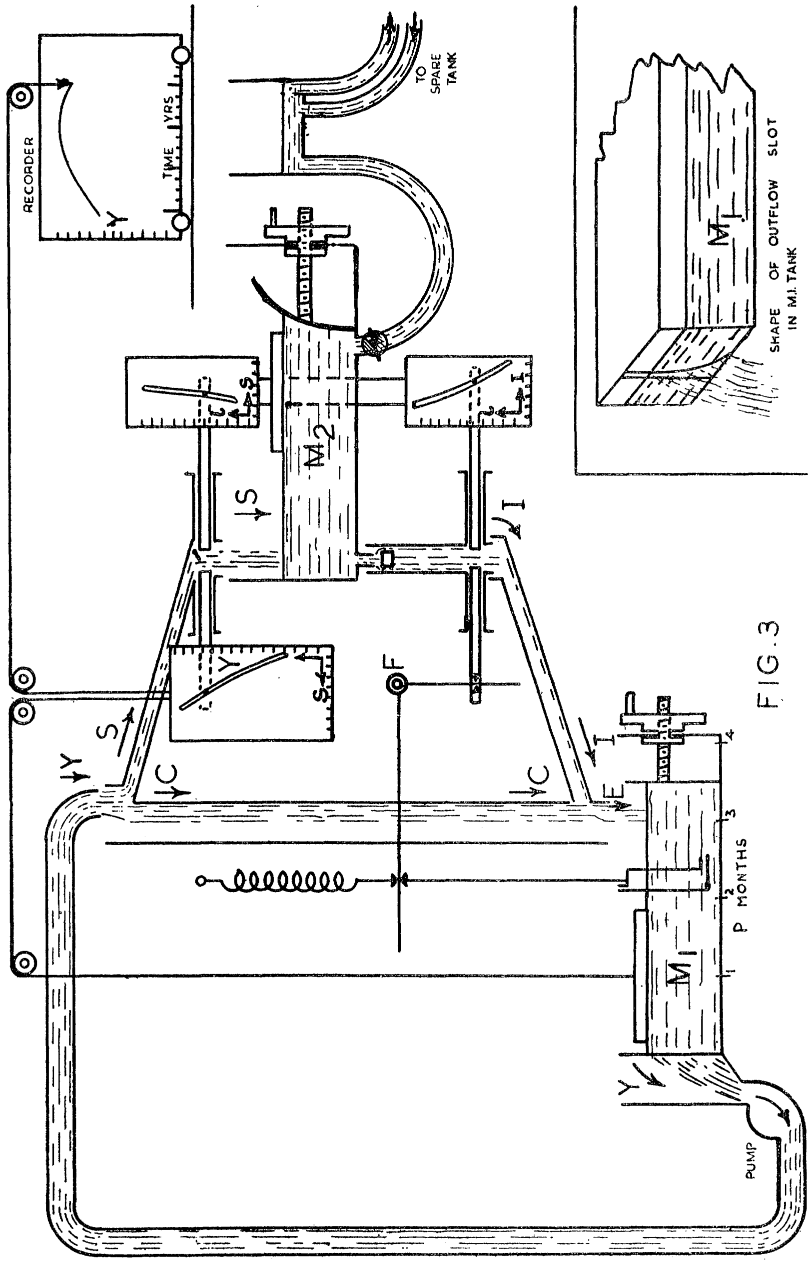
RECORDER
TIME YRS
TO SPARE TANK
SHAPE OF OUTFLOW SLOT IN M.I. TANK
PUMP
P MONTHS
F

FIG. 3

electric pump carries the income flow to the top of the model, where it divides into savings, S, and consumption, C.[1]

Savings are controlled by two opposed valves, and measured by the opening between them, consumption being left as a residual. (In a later model consumption is controlled and savings made the residual.) Consumption expenditure flows directly back into M_1. Savings flow into the tank containing idle or surplus balances, M_2, defined as all money in excess of minimum working balances. Investment expenditure, I, controlled and measured by two valves, is drawn from M_2 and, combining with consumption to form total expenditure, E, flows into M_1.

To allow for extension or contraction of the supply of money, water is pumped from a separate tank (not shown in the diagram) into the small box on the extreme right of the model, the overflow from the box returning to the tank. The box is connected by a flexible tube to the M_2 tank. The height of the box can be varied to make the level of water in it higher or lower than that in the M_2 tank, causing a flow to or from M_2 roughly proportional to the difference between the two levels. A tap in the tube can be closed when it is desired to operate with a constant quantity of money.

The time taken for the active balances to circulate once round the system, equal to $\frac{M_1}{Y}$, is the reciprocal of the income velocity of circulation of active balances and will be called the circulation period, P. The actual length of the circulation period seems to be somewhere between three and four months in England and the United States[2]. On the model, if the adjustable end of the M_1 tank is vertical, P will be proportional to the operative length of the tank irrespective of the value of Y, since both M_1 and Y are then proportional to the height of water in the tank. A linear scale may therefore be fitted along the tank, giving the circulating period in months. The position of this scale determines a time constant for the model. If, when the adjustable end of the tank is set at the mark three months on the scale, the time taken for the quantity of water in the tank to flow

[1] The pump motor is fed through two electrodes partially immersed in the water over the inflow to the pump, providing automatic speed regulation of the pump and keeping the level of water in the inflow tube constant. Since the quantity of water in the outflow tube is also constant, the flow into C and S at the top of the model always equals Y.

[2] See F. Machlup, "Period Analysis and Multiplier Theory", *Quarterly Journal of Economics*, November, 1939.

once round the circuit is half a minute, the time constant is two minutes on the model to one year in reality. A recorder may then be fitted to trace a chart of income against time in years.

When the system is in equilibrium $I = S$, $E = Y$, and the rate of change of M_1 and M_2 is zero. If, from this position there is an increase in investment and therefore in E, Y does not increase immediately by an equal amount, but only gradually as the excess of E over Y gradually increases M_1.

This lag between expenditure and income (and their real counterparts sales and output) occurs because an increase in expenditure and sales at first leads chiefly to a reduction in stocks, and must be transmitted through complex chains of intermediate transactions, some short, others very long, before it produces an equivalent increase in output and income.[1] In the model, there is no lag between income and expenditure, i.e., no Robertsonian lag. If, however, we redefine Y to be identical with E, the inflow into M_1, and call the outflow from M_1 " disposable income ", we no longer have a sales-output lag but have instead a distributed equivalent of the Robertsonian income-expenditure lag. The operation of the model is in no way affected by the choice between these sets of definitions, which lead to similar results providing the periods of the lags are the same. From empirical evidence[2] it seems that the income-expenditure lag is small in relation to the expenditure-income (or sales-output) lag, so the latter interpretation is used here.

There is, of course, no reason why a model including both lags should not be made. In the mechanical model this would require the inclusion of a small tank, the outflow from which would be proportional to the height of water in it, in the income flow. The inflow to this tank would be income, and the outflow would be the equivalent in continuous analysis terms of Robertsonian disposable income. Water

[1] The sales-output lag used by Lundberg in his dynamic sequence analysis is in some ways more realistic than the one used here. He relates the length of this lag to the " production-planning period ", not to the circulation period of active balances, and also allows for attempts by producers to keep their stocks constant, or at a constant proportion of sales. Any change is then accompanied by a series of damped oscillations, as is shown by Lloyd A. Metzler in " The Nature and Stability of Inventory Cycles ", *Review of Economic Statistics*, August, 1941. In the model described here the accelerator must be used to deal with the effects of changes in stocks, investment being interpreted to include expenditure by producers to maintain stocks. When total expenditure is rising and stocks are being depleted, there will be additional investment by producers attempting to maintain stocks, this additional induced investment being made a lagged function of the rate of change of income.

[2] See Lloyd A. Metzler : " Three Lags in the Circular Flow of Income ", *Essays in Honor of Alvin H. Hansen*, pp. 11–32.

in the first tank would represent business working balances and in the second would represent personal working balances. Business and personal savings would have to be treated separately, business savings being assumed to be a function of output (income), and personal savings a function of disposable income. It can be shown that a change in the system would then be made through a series of damped oscillations.[1]

The relationship between the variables can be seen from the diagram. The shape and position of the curved end of the M_2 tank are derived from the liquidity preference function, so that the height of the water in the M_2 tank measures (inversely) the rate of interest, i. The rate of interest operates the "classical" savings curve and the marginal efficiency of capital curve to control savings and investment. Savings are also controlled through the left-hand valve by the propensity to save curve, which is operated by a float on the M_1 tank so that it moves vertically with income. A "propensity to invest" curve could be used similarly to operate the left-hand valve controlling investment; but it has been thought preferable to introduce an accelerator relationship between income and investment.[2] The accelerator mechanism consists of a narrow, deep float with a small hole in the bottom, hanging from a spring, and connected by a bent lever to the investment valve. In equilibrium the level of water inside the float will be the same as that in the M_1 tank, and the position of the valve is that of zero induced investment. If Y is decreasing, the level of M_1 will be falling and will be temporarily below the level of water in the float, since the water inside the float can leak out only slowly. Extra weight will be placed on the spring, which will extend, closing the investment valve a little, and so causing negative induced investment. Conversely, when Y is increasing induced investment will be positive, being always a lagged function of the rate of change of income.

It will be noticed that in this model $S=f(Y)+\phi(i)$, and similarly for investment, though a better assumption would be that $S=\psi(Y, i)$. The latter assumption could be used if the two savings valves were replaced by a single

[1] For a geometrical treatment of a similar model on the assumption that the two lags are equal, see a forthcoming article by Ralph Turvey and Dr. Hans Brems.

For an account of how the macroeconomic concepts can be further broken down to give complex systems of lags and functions see R. M. Goodwin: "The Multiplier as Matrix" *Economic Journal*, December, 1949.

[2] The model shown in Fig. 1 has a propensity to save and no accelerator.

spring-loaded one, bearing against a block with surface contours given by $S = \psi(Y, i)$, the block moving vertically with income and horizontally, at right angles to the valve movement, with the rate of interest.

In models of this type it is usual to assume either that the values are given in some kind of real units, or that they are in money units but that prices are constant. In the mechanical model we can, however, introduce prices indirectly into the system, though in a rough and cumbersome way, by making use of the fact that the machine will deal with non-linear relationships. The general price level is assumed to be a function of money income, Y. This function can be drawn on the side of the M_1 tank, with Y along the ordinate and the price level along the abscissa, the values being read off at the intersection of the curve and the level of the water. Another curve of real income against money income can also be drawn, derived from the price curve. Any relationship given in real units can then be converted into an equivalent relationship in money units, and the graph of the latter used on the machine.

2. The Multiplier with Constant Rate of Interest

In the mathematical treatment of the model the functions must be assumed to be linear and prices to be constant. If the rate of interest is held constant by appropriate variations in the supply of money, and the accelerator neglected so that investment is made an independent variable, we have the following identities by definition, the variables being measured, for simplicity, from a base at which the system is in equilibrium.

$$Y = C + S,$$

$$E = C + I,$$

$$\text{and } E - Y = \frac{dM_1}{dt}.$$

Therefore

$$I - S = \frac{dM_1}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The hypotheses are:

$$M_1 = PY, \text{ or } \frac{dM_1}{dt} = P\frac{dY}{dt}, \quad \ldots\ldots \quad (2)$$

$$\text{and } S = \sigma Y, \quad \ldots\ldots \quad (3)$$

where σ is the marginal propensity to save.

If, from equilibrium at time $t=0$, investment increases by $\triangle I$, then, from equations (1), (2) and (3),

$$P\frac{dY}{dt}+\sigma Y-\triangle I=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

The solution is

$$Y_t=\frac{\triangle I}{\sigma}\left(1-e^{-\frac{\sigma}{P}.t}\right), \quad \ldots\ldots\ldots\ldots \quad (5)$$

where t is time in years and P, the circulation period, is also given as a fraction of a year.

Also,

$$E_t=\frac{\triangle I}{\sigma}\left(1-e^{-\frac{\sigma}{P}.t}\right)+\triangle I e^{-\frac{\sigma}{P}t}. \quad \ldots\ldots\ldots\ldots \quad (6)$$

Equations (5) and (6) give the same final value for the multiplier as is obtained in the usual period analysis, and the equilibrium values are approached along a similar path, though of course it is continuous instead of stepped. If the curves are drawn however, it will be found that the process of adjustment in this model is slower than that obtained in a period analysis model in which the lag is made equal to the circulation period. This results from the form of distributed lag used here. This lag, L, may be defined as the time interval by which Y lags behind E, i.e., the value of L which makes $Y_{t+L}=E_t$ for all values of t. For instance, if E increases by 100 units, it is the time which elapses before Y has also increased by 100 units. By substituting equations (5) and (6) in this expression, we obtain the relation

$$L=P\left[-\frac{\log_e(1-\sigma)}{\sigma}\right].$$ [1]

Some values of L with different marginal propensities to save are worked out in the table below.

σ	L
0	P
.2	$1.12P$
.5	$1.39P$
.8	$2.01P$
1.0	∞

[1] I am indebted to Mr. W. T. Newlyn and Mr. J. D. Sargan, of Leeds University, for pointing out this relation.

The speed of the adjustment process is determined not by the circulating period alone, but by the distributed lag L, which is longer than the circulating period except when the marginal propensity to save is zero. Though both the continuous analysis and the period anaylsis are only crude approximations to reality, I think there is no doubt that the continuous analysis is more realistic than one in which adjustments are assumed to occur in steps at intervals of three or four months. If this is so, it must be concluded that the time taken for a multiplier process to work itself out is longer than that shown by a period analysis sequence in which the lag is made equal to the circulation period of active balances.

3. The Multiplier with Constant Quantity of Money

If the quantity of money is constant, the rate of interest being free to vary, we must make use of the part of the mechanical model which deals with the determination of the rate of interest. It is obvious from Fig. 3, that in this case the multiplier process following a spontaneous increase in investment $\triangle I$ will be to some extent checked if the transfer of money from M_2 to M_1 induces a rise in the rate of interest, since this will reduce the excess of investment over savings (unless the curves are such that a rise in the rate of interest decreases savings more than it does investment, which is highly improbable).

Measuring the variables again from a base at which the system is in equilibrium, and assuming linearity, we have another identity,

$$M_2 = -M_1, \qquad (7)$$

and two additional hypotheses,

$$M_2 = \lambda i \qquad (8)$$

$$\text{and } I = \eta i, \qquad (9)$$

where λ is the liquidity preference function "proper" (Keynes's L_2 function) and η is the marginal efficiency of capital function. Equation (3) must also be changed to

$$S = \sigma Y + \xi i, \qquad (10)$$

where ξ is the slope of the interest-savings curve.

From (1), (9) and (10), and introducing the spontaneous change $\triangle I$, we obtain

$$\frac{dM_1}{dt} = \triangle I + (\eta - \xi)i - \sigma Y,$$

and substituting (2), (7) and (8) in this and rearranging gives

$$P\frac{dY}{dt}+\left[\sigma+\frac{P}{\lambda}(\eta-\xi)\right]Y-\Delta I=0 \quad \ldots\ldots \quad (11)$$

The solution is :

$$Y_t=\frac{\Delta I}{\sigma+\frac{P}{\lambda}(\eta-\xi)}\left[1-e^{-\left(\frac{\sigma}{P}+\frac{\eta-\xi}{\lambda}\right)t}\right], \quad \ldots\ldots \quad (12)$$

which gives the path of the multiplier process.

P is always >0, and since we may be confident that $\lambda<0$, $\eta<0$ and $\xi>\eta$, we know that $\frac{\eta-\xi}{\lambda}\geqq 0$. Equation (12) therefore shows that when the quantity of money is constant the multiplier is smaller, and the process of adjustment more rapid, than when the rate of interest is constant. The stability condition, that $\frac{\sigma}{P}+\frac{\eta-\xi}{\lambda}>0$, also shows that the system is more stable; a marginal propensity to consume of more than unity ($\sigma<0$) does not necessarily make the system explosive. If the elasticity of the liquidity preference curve is infinite, $\frac{\eta-\xi}{\lambda}=0$, and we have the Keynesian special case, in which equation (12) reduces to equation (5) to give the usual multiplier process and stability condition.

4. The Accelerator

In this model the hypothesis is made that induced investment depends on the rate of change of income. It is not necessary to assume that all investment is induced investment, but as the equations will show only variations from an equilibrium position the constant part of investment need not appear. The relationship between income and induced investment must be lagged in some way if the system is to be oscillatory. We therefore set up the following hypotheses :

$$\bar{I}=\beta\frac{dY}{dt}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

where $\bar{I}$ is the value that induced investment would have if there were no lag, and β is the acceleration co-efficient, and

$$\gamma\frac{dI}{dt}=\bar{I}-I, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

where γ is a lag constant. In words, induced investment approaches the value it would have if there were no lag at a rate proportional to the difference between the actual and the unlagged values.

Referring to Fig. 3, for a given construction of float and valve mechanism, β will be increased if the bent lever is raised, by raising both the fulcrum F and the point of attachment to the cord carrying the float, since this will increase the valve movement for a given extension of the spring. The lag constant will be increased by decreasing the size of the leakage hole in the float. This will also alter β, however, so the position of the lever must be adjusted whenever the lag constant is changed, if β is to be kept constant.

If it is assumed again that the rate of interest is held constant, we obtain by combining equations (13) and (14),

$$\frac{dI}{dt} = \frac{\beta}{\gamma} \cdot \frac{dY}{dt} - \frac{I}{\gamma}.$$

Combining equations (1), (2) and (3) gives

$$I = P\frac{dY}{dt} + \sigma Y,$$

and, differentiating with respect to time,

$$\frac{dI}{dt} = P\frac{d^2Y}{dt^2} + \sigma\frac{dY}{dt}.$$

Substituting these expressions in the previous one, and rearranging, we obtain

$$\gamma P\frac{d^2Y}{dt^2} + (\gamma\sigma + P - \beta)\frac{dY}{dt} + \sigma Y = 0 \quad \text{........(15)}$$

for the homogeneous part of the accelerator-multiplier equation. The roots of the characteristic equation become complex when

$$(\gamma\sigma + P - \beta)^2 < 4\gamma P\sigma,$$

and the system then becomes oscillatory.[1]

The accelerator will, of course, give different results if the rate of interest is allowed to vary; but there is not space to develop them here.

[1] The accelerator used here thus gives results identical with those obtained from a mathematical model by Richard M. Goodwin: "Secular and Cyclical aspects of the Multiplier and the Accelerator", *Essays in Honor of Alvin Hansen*, pp. 118–124.

5. The Determination of the Rate of Interest

Discussions about the rate of interest seem often to have suffered through lack of a suitable technique for showing the process of change through time of the inter-related factors. The model described here may help to make it clear that the liquidity preference and loanable funds theories are neither inconsistent with each other, as might be thought from some of the controversies between their exponents, nor merely different ways of saying the same thing, as is sometimes implied,[1] but are complementary parts of a wider system.

Referring to Fig. 3, we see that in the model the rate of interest is determined at any instant only by the supply of and demand for " stocks " of idle balances. A decrease in liquidity preference " proper ", or an increase in M_2 if the liquidity preference curve is not infinitely elastic, will cause an immediate fall in the rate of interest. But if S and I are at all interest-elastic they will be changed both directly by the change in the rate of interest, and indirectly by the subsequent change in income, working through the propensity to save and the income-investment relation. Any difference between S and I causes a gradual change in M_2 and so reacts back on the rate of interest, so that the full effects of the change depend on the shapes of the " flow " functions.

We may assume that owing to the initial fall in the rate of interest I will increase, while S may decrease and in any case will not increase as much as I. The resulting excess of I over S causes a gradual fall in M_2, so that the rate of interest tends to rise again ; but it also causes a gradual rise in M_1 and Y. If, as Y increases, it induces a greater increase in S than in I, this tendency will be checked before the rate of interest has returned to its original value. If, however, owing to the effect of the accelerator, or to a spread of optimism leading to a shift in the schedule of the marginal efficiency of capital, the increase in Y causes a greater increase in I than in S, the process of expansion will go further, and the rate of interest may temporarily rise above its original value. On the other hand, the rise in Y may also cause a further decrease in liquidity preference, so checking the rise in the rate of interest and enabling the expansionary process to continue even longer.

[1] See J. R. Hicks : *Value and Capital*, 2nd edition, 1946, pp. 153–162.

Differences of opinion concerning the main determinants of the rate of interest, and the effects of a change in the rate of interest, thus depend on different assumptions made as to the shapes and stability of the relations involved, and it would seem more useful to attempt the difficult task of testing these relations empirically than to engage in arguments based on assumptions about them. However, one argument may be suggested here. If the liquidity preference curve is not infinitely elastic, and if a change in the rate of interest does not directly change savings in the same direction as, and by an equal or greater amount than, it changes investment, then the rate of interest has an equilibrating function in the system. Its effect is weaker and slower in action than was thought before the work of Keynes, and, moreover, in times of wide fluctuations in income it may be almost completely swamped by the effects of the accelerator and shifts in the liquidity preference and marginal efficiency of capital functions. But if sudden changes in the level of income are avoided by fiscal or other policies, the equilibrating influence of the rate of interest becomes relatively stronger, so that monetary policy becomes a necessary supplement to fiscal policy.

The paths followed by income and the rate of interest after initial changes in the system can be worked out mathematically. For instance, if from equilibrium at time $t=0$, M_2 is increased by $\triangle M_2$, the path of the change in the rate of interest (neglecting the accelerator) is given by:

$$i_t = \triangle M_2 \left\{ \frac{1}{\lambda} - \frac{\eta - \xi}{\sigma P + \lambda(\eta - \xi)} \left[1 - e^{-\left(\frac{\sigma}{P} + \frac{\eta - \xi}{\lambda}\right)t} \right] \right\},$$

and the path of the change in income is given by

$$Y_t = \triangle M_2 \left\{ \frac{\eta - \xi}{\sigma\lambda + P(\eta - \xi)} \left[1 - e^{-\left(\frac{\sigma}{P} + \frac{\eta - \xi}{\lambda}\right)t} \right] \right\}.$$

The final effect on the rate of interest is therefore greater when $|\lambda|$ is small, $|\eta - \xi|$ is small and σ is large, while the final effect on income is greater when $|\lambda|$ is small, $|\eta - \xi|$ is large and σ is small. As in the case of the multiplier-accelerator process, such exercises have their uses, providing the limitations imposed by the assumptions are kept in mind.

Similar processes could be worked out on the mechanical model if the rate of interest were recorded. Non-linear relations could be used, and the effects of superposing shifts in different functions at different times could be observed, giving a sort of simplified picture of consecutive events in economic history.

III

So far we have assumed a closed economy with no government operations. In the model shown in Fig. 4, these assumptions are relaxed.[1] The income flow, after being pumped to the top of the model, divides into taxation, T, and income after taxation. The taxation flow is controlled and measured by a valve operated from the float on the M_1 tank through an income-taxation curve. Income after taxation flows into a small measuring box, the outflow slot of which is identical with that of the M_1 tank, so that the flow is measured by the height of liquid in the box, on the same scale as the measurement of income. The box is small so the error caused by water being trapped in it is negligible[2]. After flowing through the measuring box, income after taxation divides into consumption and savings, consumption being controlled by a propensity to consume curve and an interest-rate-consumption curve, savings being a residual.

The position of the consumption function is controlled by the level of water in the measuring box. As this box is too small for a float-operated control to work satisfactorily, it is necessary to use a small servo-motor mechanism. This consists of an electric motor driving the pulley over which the connecting thread passes. The speed and direction of the motor are controlled by two small electrodes partially immersed in the water. When the level rises, the motor turns the pulley in an anti-clockwise direction, lifting the electrodes and so maintaining their position relative to the water level. Conversely, when the level of the water falls, the motor turns the pulley in a clockwise direction. The operation is therefore similar to that which would occur if the electrodes were a float large enough to operate the consumption function directly.

The servo-mechanism also makes it possible to lag consumption behind income after taxation. The electrodes are immersed in the water in a small tube connected to the

[1] This model is basically similar to that shown in the photograph in Fig. 1, but includes a number of improvements.

[2] This box could be made larger, and the liquid in it interpreted as personal working balances, so introducing a Robertsonian lag into the model.

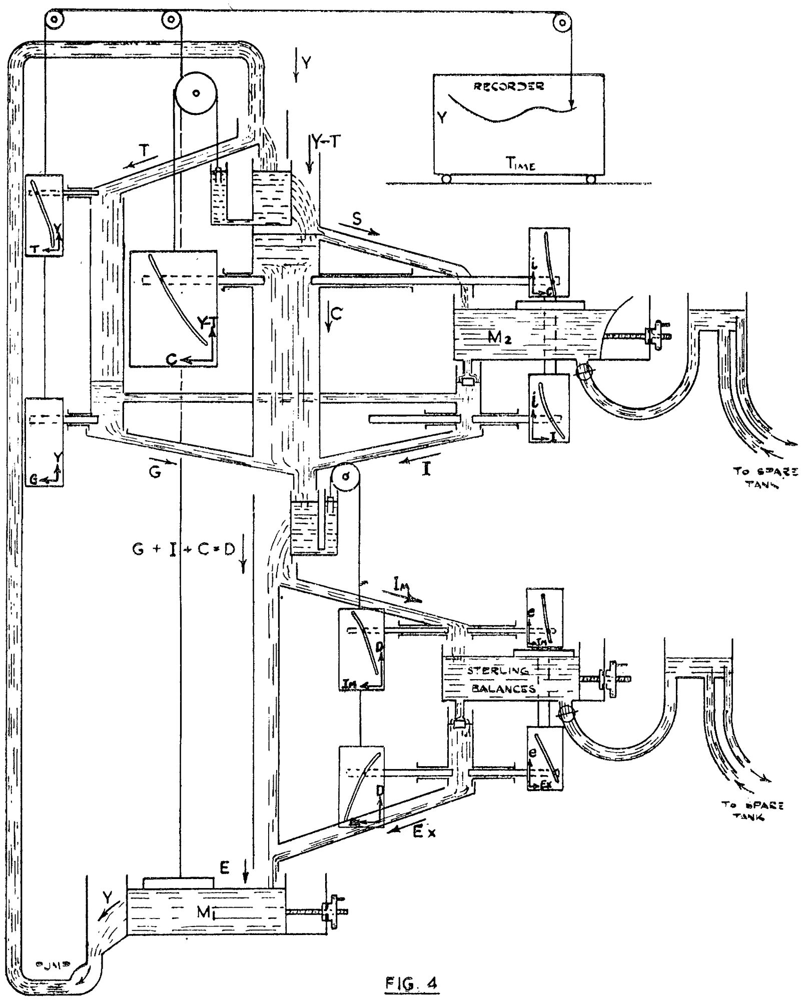
Y
RECORDER
Y
TIME
T
Y-T
S
C
M2
G
I
G + I + C = D
To SPARE TANK
STERLING
BALANCES
Ex
E
Y
M1
PUMP
To SPARE TANK

FIG. 4

measuring box, not in the box itself. The size of the hole connecting the tube to the box is adjustable, and when it is small the level in the tube lags behind that in the box, so that the consumption flow becomes a lagged function of income after taxation. A distinction may be made between this type of lag and that which occurs in, say, the Robertsonian model. In the latter, the lag occurs as a result of a time interval between income and expenditure. This might be called a " structural " lag, and is represented in a continuous analysis by the time taken, after an increase in income, to build up working balances to a level commensurate with the higher expenditure. The type of lag just introduced acts in addition to the " structural " lag ; it might be called a " psychological " lag, caused by inertia in the changing of habits.

Government expenditure, G, is controlled and measured by a valve, which may also be operated as a function of income if it is desired to illustrate the stabilising effects of an automatic compensatory fiscal programme. The level of water over this valve is kept constant by connecting it through the horizontal tube to the investment tube. This also causes any budget deficit to be met automatically by drawing from idle balances, M_2, and any budget surplus to contribute to investment expenditure, the drain on idle balances being diminished by an equivalent amount.

Consumption, investment, and government expenditure combine to form domestic expenditure, D. Expenditure abroad for imports, Im, is taken from domestic expenditure, and receipts from abroad for exports, Ex, are added to give total expenditure on the goods of the home country, E, which flows back to the M_1 tank. Domestic expenditure is measured by taking it through a small box similar to that described above, and a second servo-motor mechanism operates a propensity to import curve, making imports a function (lagged if desired) of domestic expenditure.[1]

Payments for imports flow into, and receipts from exports flow from, the small tank marked " sterling balances ", the water in which represents foreign holdings of the money of the home country.[2] The quantity of these balances,

[1] For the use of this relationship in the analysis of the multiplier effects of international trade see J. E. Meade : " National Income, National Expenditure and the Balance of Payments ", *Economic Journal*, December, 1948, and March, 1949.

[2] There is probably some error involved in separating these balances from idle balances, since some part of them may be related to the rate of interest.

together with the demand schedule for them, represented by the capacity of the tank at different levels, determine at any instant the rate of exchange, in the same way as in the case of price and of the rate of interest. This demand schedule might appropriately be called the sterling preference function.

The rate of exchange is thus represented by the level of liquid in the tank, and a float can be used to operate exchange-rate-imports and exchange-rate-exports graphs controlling imports and exports through the right-hand pair of valves. A small box, fed from a spare water tank and connected to the sterling balances tank by a flexible tube, can be used to represent the operations of an exchange equalisation account, putting funds on to, or taking them from, the foreign exchange market in order to control the rate of exchange. When the tap in the tube is turned off, a system with freely fluctuating exchange rates is represented.

National income is recorded as in the previous model, and in the version which is now being constructed small measuring boxes and floats are being inserted in the imports and exports flows, enabling imports and exports to be recorded on a single chart, so that the trade balance will be seen directly from the gap between them. Assuming there are no other items in the balance of payments, it will then be possible to demonstrate the operation of the gold standard by adjusting credit expansion or contraction to a multiple of the trade balance at any time.[1]

It seems possible that these small boxes, included in this model admittedly for a purely mechanical purpose, may have some economic significance. For instance, those in the imports and exports flows, which will contain a volume of water proportional to the arithmetic average of imports and exports, might be interpreted as containing working balances necessary for financing actual foreign trade transactions, the sterling balances tank containing balances surplus to those required for current transactions. Similarly a box inserted in the investment flow would contain what might be called investment working balances, a circulating fund covering the time interval between money being taken off the money market and its actual expenditure on investment goods. This would seem to be the equivalent of Keynes's

[1] I am indebted to Professor J. E. Meade for suggesting this device, as well as a number of others described in this section.

"Finance" balances. The additional relationships introduced into the system by such a division of working balances would considerably complicate the mathematical analysis in any particular case, though it can easily be shown that they would result in adjustments taking place through a series of oscillations.

It is possible to connect together two of the models shown in Fig. 4, to deal with the multiplier relationships between the incomes of two countries, or of one country and the rest of the world. To connect more than two would be difficult, since each country would have to have a propensity to import function for each other country. The easiest method of interconnection would be to assume a fixed rate of exchange, and run the imports flow of one model into the exports tube of the other. Another though rather more difficult way would be to use a servo-mechanism to operate the left-hand exports valve of each model, keeping exports always equal to imports on the other model multiplied by the rate of exchange, which could be held stable by the "equalisation account" operations.

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