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Rationalizability in Games with a Continuum of Players

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Abstract

The concept of Rationalizability has been used in the last fifteen years to study stability of equilibria on models with continuum of players such as standard competitive markets, macroeconomic dynamics and currency attacks. However, Rationalizability has been formally defined in a general setting only for games with a finite number of players and there is no general definition for Rationalizability in the case of games with continuum of players. In this work, we propose a definition for Point-Rationalizable Strategies in the context of *Non-Atomic Noncooperative Games with a Continuum of Players*. In a special class of this games where the payoff of a player depends only on his own strategy and an aggregate value that represents the *state* of the game, state that is obtained from the actions of all the players, we define the set of *Point-Rationalizable States*. This last set is characterized and some of its' properties are explored. We study as well *standard* Rationalizability in a subclass of these games. We present an exploratory framework that encompasses the previously mentioned models, over which we can link the established theory and its' macroeconomic applications on stability properties of equilibria.

KEYWORDS: Rationalizable Strategies, Non-atomic Games, Expectational Coordination, Rational Expectations, Eductive Stability, Strong Rationality.

JEL CLASSIFICATION: D84, C72, C62.

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1 Introduction

The concept of *Strong Rationality* was first introduced by Guesnerie (1992) in a model of a standard market with a continuum of producers. An equilibrium of the market is there said to be *Strongly Rational*, or *Eductively Stable*, if it is the only *Rationalizable Solution* of the economic system. Inspired in the work of Muth (1961), the purpose of such an exercise was to give a rationale for the Rational Expectations Hypothesis. Strong rationality has been studied as well in macroeconomic models in terms of stability of equilibria. See for instance Evans and Guesnerie (1993), where they study Eductive Stability in a general linear model of Rational Expectations or Evans and Guesnerie (2003, 2005) for dynamics in macroeconomics. More examples of applications of Strong Rationality can be found in the recent book by Chamley (2004) where he presents models of Stag Hunts in the context of coordination in games with strategic complementarities.

The *Rationalizable Solution* of the economic system assessed by Guesnerie in the definition of *Strong Rationality*, refers to the concept of *Rationalizable Strategies* as defined by Pearce (1984) in the context of games with a finite number of players and finite sets of strategies. *Rationalizable Strategies* were formally introduced by Bernheim (1984) and Pearce (1984) as the “adequate” solution concept when agents are modeled as rational utility maximizers that take decisions independently. Adequate because Rationalizable Strategy Profiles are outcomes of a game that cannot be discarded based on agents’ rationality and common knowledge. The work of Pearce focused mainly in refinements of equilibria of extensive form finite games, while Bernheim gave a definition and characterization in the context of general normal form games, along with comparison between the set of Nash Equilibria and the set of Rationalizable Strategy Profiles. In both papers and later treatments, however, the definition and characterization of rationalizable “solutions” were developed for games with a finite number of players.

On the other hand, each one of the works that are mentioned in the first paragraph of this introduction including the seminal work by Guesnerie (1992), feature an ad-hoc definition of the concept of *Rationalizable Solution*¹. In each work, based on the original definition and characterization of Rationalizable Strategies, and the intuitions behind them, such definitions are adapted to models with a continuum of players.

It is this gap between the established theory and its’ economic applications that motivate this work. Since there is no established definition for Rationalizable Strategies, or Rationalizability for what matters, in a general framework with a continuum of agents, in this paper then we link the game-theoretical concept of Rationalizability to its’ applications in macroeconomics and economic models. We will propose a treatment for Rationalizability in a general class of models with continuum of agents.

These applied models have two common features: (i) there is a continuum of agents that interact and (ii) payoffs of agents depend on an aggregate value that cannot be affected unilaterally by any agent. These features are captured in a now standard line of research: Non-Atomic Games. A particular class of such games are games with a continuum of players.

We present then a framework of a general class of noncooperative games with a continuum of players, in which we explore the ideas of Rationalizability. We will begin by defining the concept of *Point-Rationalizable Strategies*² in a general setting. Then we will turn to the special case where payoffs depend on players’ own actions and the average of the actions taken by all the players. We will call this average the *state* of the game, and we will define the set of *Point-Rationalizable States*. This last approach is not evident nor a generalization of finite player games, since in “small” games, and as opposed to what we do here, players can actually affect directly and unilaterally the payoff of other players. Consequently we will get a setting in which it is possible to study Rationalizability and general properties of (local) strongly rational equilibria as in the economic applications.

The remainder of the paper is as follows: in section 2 we motivate our discussion with an example; in section 3 we introduce *games with a continuum of players* ; in section 4 we define

¹They are not necessarily defined in the space of strategy profiles.

²Following the terminology of Bernheim.

Point-Rationalizable Strategies in the context of this games and, for the special class of games that we consider, we define as well Point-Rationalizable States. Our main result is the study of the set of Point-Rationalizable States, for which we give a characterization and show its' convexity and compactness. In 4.4 we introduce the concept of *Strongly Rational Equilibrium* and explore the relation between Point-Rationalizable Strategies and States. We argue in favor of the use of this last approach, states instead of strategies, in the context of this games. Before concluding, we explore in section 5 the concept of (standard) Rationalizability in the context of such games and we study the particular case in which (pure) strategies are chosen from finite sets and payoffs depend on the integral of the profile of mixed strategies (Schmeidler 1973).

2 Rationalizability in an Economic Example

We begin by motivating our presentation and the introduction of non-atomic games as a general framework with an example that can be found in Guesnerie (1992). In this example, a variation of the Rational Expectations model due to Muth (1961), we consider a market with a continuum of firms that produce an amount of good before the price at which it will be sold is known. We will describe the model and illustrate how the Rationalizability concept is presented in Guesnerie's work.

Example 1. Consider that we have a group of farmers, represented by the $[0, 1] \equiv I$ interval, that participate in a market in which production decisions are taken one period before production is sold. Each farmer $i \in I$ has a cost function $c_i : \mathbb{R}_+ \rightarrow \mathbb{R}$. The price p at which the good is sold is obtained from the (given) inverse demand function $P : \mathbb{R}_+ \rightarrow [0, p_{\max}]$ evaluated in total aggregate production $p = P(\int q(i) di)$ where $q(i)$ is farmer i 's production. Since an individual change in production does not change the value of the price, the product is sold at price taking behavior, so farmers maximize $u(i, \cdot, \cdot) : \mathbb{R}_+ \times [0, p_{\max}] \rightarrow \mathbb{R}$ defined by $u(i, q(i), p) \equiv pq(i) - c_i(q(i))$. An equilibrium of this system is a price p^* such that $p^* = P(\int q^*(i) di)$ and $u(i, q^*(i), p^*) \geq u(i, q, p^*) \forall q \in \mathbb{R}, \forall i \in I$.

At the moment of taking the production decision, farmers do not actually know the value of the price at which their production will be sold. Consequently they have to rely on forecasts of the price or of the production decision of the other farmers. The concept in scrutiny in our work is related to *how this (these) forecast(s) is (are) generated*.

Forecasts of farmers should be rational in the sense that no *unreasonable* price should be given positive probability of being achieved. It is in this setting that Guesnerie introduces the concept of *strong rationality* or *eductive stability*³ as the uniqueness of *rationalizable prices* which are obtained from the elimination of the unreasonable forecasts of possible outcomes. To obtain these rationalizable prices, Guesnerie describes, in what he calls the *eductive procedure*, how the unreasonable prices can be eliminated using Pearce's (1984) iterative process of elimination of non-best-response strategies.

The particular structure of this example actually allows us to look at outcomes on the set of prices (or aggregate production), instead of the set of strategy profiles, as is done in Pearce or Bernheim. This allows for a special characterization of the *rationalizable set* as the limit of an iterative process of elimination of unreasonable prices, and not necessarily production profiles. The eductive procedure consists in eliminating the prices that do not emerge as a consequence of farmers taking productions decisions that are best responses to the remaining strategy profiles, or equivalently, remaining values of aggregate production or prices. A second characteristic of this setting is that the eductive procedure can be done by simply eliminating prices that are beyond the upper and lower bounds that are obtained in each iteration. This comes from the monotonicity properties of the aggregation operator (the integral) and the supply function of the farmers.

In Figure 1 we can see how the eductive process works in this setting. From the farmers problem we can obtain for each farmer his supply function $s(i, \cdot) : [0, p_{\max}] \rightarrow \mathbb{R}_+$ and under

³An equilibrium of the system is *strongly rational* or *eductively stable* if it is the only Rationalizable outcome of the system. We will refer equivalently to outcomes as begin strongly rational or eductively stable.

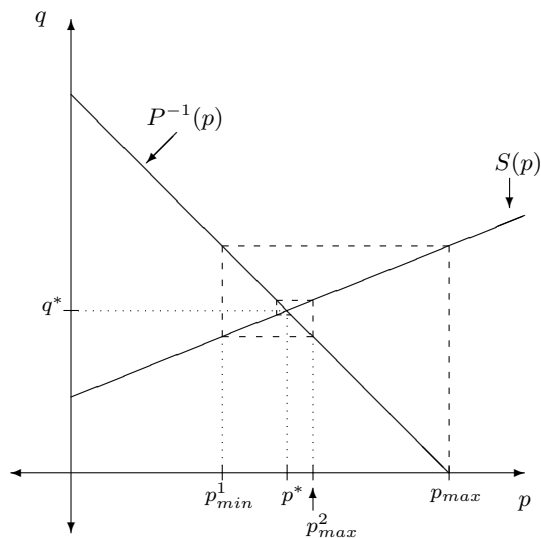


Figure 1: The *eductive process*

standard measurability hypothesis we can obtain an aggregate supply function $S(p) = \int_0^1 s(i, p) di$ that is shown in Figure 1 along with the demand function. Farmers know that a price higher than p_{max} gives no demand and so prices higher than those are unreasonable. Since all farmers can obtain this conclusion, all farmers know that the other farmers should not have forecasts that give positive weight to prices that are greater than p_{max} . Then, they know that aggregate supply can not be greater than $S(p_{max})$. Since the price is obtained through the inverse demand function, farmers then know that prices smaller than p_{min}^1 , in the Figure, will not emerge. All farmers know then that forecasts are constrained by the interval $[p_{min}^1, p_{max}]$. The process, as described in Figure 1, continues until they eliminate all the prices except the equilibrium price. Note that the eductive process could “fail”, in the sense that it could give more than only the equilibrium point, and the *rationalizable set* could well be the whole interval $[0, p_{max}]$.

For more details on the example see the paper of Guesnerie (1992). The procedure of eliminating higher or lower prices comes from the previously mentioned monotonicity property of the example and is not always the case that the eductive process works like that. This second feature is closely related to the ordered structure of the games studied in Milgrom and Roberts (1990) and can actually be obtained from a general point of view in more abstract models that encompass the market described above and other examples (Guesnerie and Jara-Moroni 2007).

What we have to keep in mind is that in this example we have a continuum of farmers and an aggregate variable that has all the relevant information for the farmers to take a decision ⁴. We are interested in defining *Rationalizability* in a general setting considering these two issues.

3 Model

Since the concept of Strong Rationality introduced by Guesnerie in his paper, relies on a concept that comes from the game-theory literature, our interest is to look at the setting described in the example as a strategic interaction situation. This idea of strategic interaction is then: payoffs of

⁴A similar structure and exercise can be found in Chapter 11 of Chamley (2004), in Stag Hunt models (see also Morris and Shin (1998)) and in the general linear Rational Expectations setting of Evans and Guesnerie (1993) in which they compare two stability concepts in macroeconomics.

agents depend on the actions of other agents. This interaction would occur through the aggregation of the production and the evaluation in the price function. The payoff of a single farmer depends on the production of all the farmers through P and $u(i, \cdot, \cdot)$, as follows: each farmer $i \in I$ chooses a production $q(i)$ in the positive interval. The price is determined by evaluating the price function in the value of total production, that is on the integral of the production profile. Each agent $i \in I$ obtains payoff $u(i, q(i), p)$. The second feature we need in the mathematical formulation, is that it allows to model the inability of single agents to influence the state of the system, in this case the price, or for what matters, total production, which calls for a mathematical formulation where the weight of single agent is small compared to the whole set or the remaining agents. These two features are captured in the mathematical model presented below.

We consider then games with a continuum of players. Schmeidler (1973) introduced a concept of equilibrium and gave existence results in games where a strategy profile is an equivalence class of measurable functions from the set of players into a strategy set, and the payoff function of a player depends on his own strategy and the strategy profile played. A different approach was presented later by Mas-Colell (1984)⁵ and more general frameworks can be found in Khan and Papageorgiou (1987) and Khan, Rath, and Sun (1997) as well. For a comprehensive review of games with many players see Khan and Sun (2002). We will focus mainly in Schmeidler's general setting and specially in games where payoffs depend on an "average" of the actions taken by all the players (Rath 1992).

In a Non-Atomic Game the set of players is a non-atomic measure space $(I, \mathcal{I}, \lambda)$ where I is the set of interacting agents $i \in I$ and λ is a non atomic measure on \mathcal{I} . This is, $\forall E \in \mathcal{I}$ such that $\lambda(E) > 0$, $\exists F \in \mathcal{I}$ such that $0 < \lambda(F) < \lambda(E)$. We will consider the set of players I as the unit interval in \mathbb{R} and the non-atomic measure λ as the Lebesgue measure.

Given a set $X \subseteq \mathbb{R}^n$ we will denote the set of equivalence classes of measurable functions from I to X as X^I . We identify then, for a general set X of available actions, X^I with the set of strategy profiles. So a strategy profile is a measurable function from I to X , the set of strategies. By doing this we are assuming that all players have the same strategy set (pure or mixed strategies). We will denote S the set of strategies and we will not make a difference a priori between pure or mixed strategies. However, since we assume that S is in \mathbb{R}^n it is better to think of this set as a set of pure strategies. We will come back to this issue on Section 5.

For each agent $i \in I$, we will denote by $\pi(i, \cdot, \cdot) : S \times S^I \rightarrow \mathbb{R}$ the general payoff functions of a game, that depend on the action of each player as an element of the set S and the profile of strategies as an element of the set S^I described as above. To specify how the functions $\pi(i, \cdot, \cdot)$ depend on these variables, we will use functions $u(i, \cdot, \cdot) : S \times X \rightarrow \mathbb{R}$, that depend on the action taken by the player in his strategy set S and some vector taken from the set $X \subset \mathbb{R}^K$, that is obtained from the strategy profile \mathbf{s} . Usually then $\pi(i, \cdot, \cdot)$ will be defined by an operation between the function $u(i, \cdot, \cdot)$ and some other mathematical objects⁶.

3.1 Payoff Functions that Depend on the Integral of the Strategy Profile

The aim is to capture the relevant features of a wide variety of models that are similar to the example described in Section 2. Consider then the class of models where there is a set $\mathcal{A} \subseteq \mathbb{R}^K$ and a variable $a \in \mathcal{A}$ that represent, respectively, the *set of states* and the *state* of an economic system. For each agent $i \in I$, the payoff function is now defined on the product of S and \mathcal{A} , $u(i, \cdot, \cdot) : S \times \mathcal{A} \rightarrow \mathbb{R}$ and depends on his own action $s(i) \in S$ and the state of the economy $a \in \mathcal{A}$. Finally, we have an aggregation operator: $A : S^I \rightarrow \mathcal{A}$ that gives the state of the system $a = A(\mathbf{s})$ when agents take the action (or strategy) profile \mathbf{s} .

In the example, the state of the system could have been identified with aggregate production or the price, and the aggregation operator would have been the integral of the production profile or the evaluation of the price function on such a quantity (respectively).

⁵In Mas-Colell (1984) what matters is not strategy profiles but a distribution on the product set of payoff functions and strategies.

⁶See equations 5.5 and 3.1.

Agents' impossibility of affecting unilaterally the state of the system is formalized by the following property of A :

$$A(\mathbf{s}) = A(\mathbf{s}') \quad \forall \mathbf{s}, \mathbf{s}' \in \text{dom } A \text{ such that } \lambda(\{l \in I : s(l) \neq s'(l)\}) = 0$$

That is, since A is defined on S^I , for all strategy profiles that are in the same equivalence class of S^I , the value of the mapping A is the same.

To capture this setting, let S be now a compact subset of \mathbb{R}^n . The aggregation operator is chosen for convenience to be the integral with respect to the Lebesgue measure:

$$A(\mathbf{s}) \equiv \int_I \mathbf{s}(i) \, di$$

so that S^I , the set of measurable functions from I to S , is contained in $\text{dom } A$, the set of integrable functions from I to \mathbb{R}^n , and the set $\mathcal{A} \equiv \text{co}\{S\}$ ⁷.

The payoff functions $\pi(i, \cdot, \cdot)$ mentioned above in the description of a game are calculated by composing the functions $u(i, \cdot, \cdot)$ and A of the economic system, that is

$$\begin{aligned} \pi(i, s(i), \mathbf{s}) &:= u(i, s(i), A(\mathbf{s})) \\ &\equiv u\left(i, s(i), \int_I s(i) \, di\right). \end{aligned} \tag{3.1}$$

In this way we are in Rath's extension of Schmeidler's formulation of games with a continuum of players, where, in a particular class of these games, agents' utility functions depend on their own actions, that are elements of a general compact set, and an "average" of all agents' actions. The description of a game will be given then by a mapping that associates each player $i \in I$ with a real valued continuous function $u(i, \cdot, \cdot)$ defined on $S \times \mathcal{A}$.

We denote the set of real valued bounded continuous functions defined on a space X by $C_b(X)$. Let $\mathcal{U}_{S \times \mathcal{A}} := C_b(S \times \mathcal{A})$ denote the set of real valued continuous functions defined on $S \times \mathcal{A}$ endowed with the sup norm topology.

Definition 3.1. A game is a measurable function $\mathbf{u} : I \rightarrow \mathcal{U}_{S \times \mathcal{A}}$.

To refer to games with the aggregate variable as in Definition 3.1 we will use the notation \mathbf{u} , as opposed to more general games related to the function π that to each player $i \in I$ associates a payoff function $\pi(i) : S \times S^I \rightarrow \mathbb{R}$. We will note then equivalently $\mathbf{u}(i)$ and $u(i, \cdot, \cdot)$. Since the set S is compact, so is \mathcal{A} and so the payoff functions $\mathbf{u}(i)$ are as well bounded. Under this description of the game, the fact that payoffs depend on the strategy profiles is given by the rules of the game, and not the payoff function, i.e. the fact that the state of the game is calculated with the integral of the strategy profile.

Definition 3.2. A Nash Equilibrium of a game \mathbf{u} is a strategy profile $\mathbf{s}^* \in S^I$ such that λ -almost-everywhere in I :

$$u\left(i, \mathbf{s}^*(i), \int_I \mathbf{s}^* \, di\right) \geq u\left(i, y, \int_I \mathbf{s}^* \, di\right) \quad \forall y \in S,$$

In this framework Rath shows that for every game there exists a Nash Equilibrium.

⁷The aggregation operator can as well be the integral of the strategy profile with respect to any measure that is absolutely continuous with respect to the lebesgue measure, or the composition of this result with a continuous function. That is,

$$A(\mathbf{s}) \equiv G\left(\int_I s(i) \, d\bar{\lambda}(i)\right)$$

where $\bar{\lambda}$ is absolutely continuous with respect to the lebesgue measure and $G : \text{co}\{S\} \rightarrow \mathcal{A}$ is a continuous function; the results in this work could well be extended to this setting. For instance Theorem 4.7 holds and if G is linear, Corollary 4.8 holds.

Theorem 3.3 (Rath 1992). *Every game \mathbf{u} as in Definition 3.1, has a (pure strategy) Nash Equilibrium.*

We will present a proof for this Theorem below. The proof in Rath's paper uses Kakutani's fixed point theorem on the mapping Γ that maps a state $a \in \mathcal{A}$ into all the possible states that rise as the consequence of agents taking best response actions to this state. This mapping goes from \mathcal{A} into itself and is proved to have a closed graph with non-empty, convex values. The only step where one should be careful is on the proof for non-emptiness of $\Gamma(a)$ in which a measurable selection argument is needed. However this is a consequence of the assumption on measurability of the mapping that defines the game⁸. As Rath mentions in his paper, the assumption on continuity and measurability of the payoff functions are both hidden in the definition of a game. The relevant fact is that \mathcal{A} is a convex, compact subset of \mathbb{R}^n .

4 Point-Rationalizability

In the example, each agent acts in ignorance of the actions taken by the other players. Thus, these agents must rely on forecasts or subjective priors over the possible outcomes of the system. We assume of course that agents are rational not only in the sense that they act by maximizing their payoff, but also considering that the subjective priors that they form do not contradict any information that they may have.

These two hypothesis have been already listed in the works by Bernheim (1984), Pearce (1984), and Tan and da Costa Werlang (1988). In each one of these documents, the main assumptions on agent's behavior that justify *Rationalizable Strategies* as solution concepts can be summed up to two basic principles: rationality of agents and common knowledge (structural and of rationality of agents). The implications of these assumptions can be exhausted, as is done in Pearce (1984) and Guesnerie (1992), by considering sequential and independent reasoning by the agents, where they rule out certain outcomes of the system as impossible.

Since agents are rational, they only use strategies that are best responses to some forecast over the possible strategy profiles that can actually be played by the others. Hence, the assumption of rationality implies that strategies that are not best responses will never be played. Following the assumption of common knowledge, each agent knows that all other agents are rational. They can then reach the same conclusion: that only best responses can be played; and taking that into account, each agent may discover that some of his (remaining) strategies are no longer best responses and so he will eliminate them. Then rationality implies that forecasts will be restricted to strategy profiles that are not eliminated. Since all agents are rational and know this second conclusion, they can continue this process of elimination of strategies. This generates a sequence of elimination of non-best-responses that under suitable hypothesis will converge in a sense to be formalized to some (hopefully strict) subset of the original strategy profile set. This is the so called *eductive process* of Guesnerie.

Following the terminology of Bernheim we will make a difference between *Rationalizability*, understood as forecasts being probability measures on the set of outcomes, and *Point-Rationalizability*, understood as forecasts being points on the set of outcomes. We will continue now by giving a formal definition of the concept of Point-Rationalizability for the case of games with a continuum of agents. Further-on we will address the issue of standard Rationalizability.

4.1 Point-Rationalizable Strategies

We will present the previous ideas in a formal mathematical context. The key assumption of this approach is that we suppose that players make forecasts over the set of strategy profiles. These forecasts are in the form of points in this set.

⁸See Lemma 4.2 below.

Consider the following line of reasoning. Given the strategy profile set S^I , each agent will only play a strategy that is a best response to some strategy profile $\mathbf{s} \in S^I$. For each agent we define the best response mapping $\text{Br}(i, \cdot) : S^I \rightrightarrows S$:

$$\text{Br}(i, \mathbf{s}) := \operatorname{argmax} \{ \pi(i, y, \mathbf{s}) : y \in S \}. \quad (4.1)$$

The mapping $\text{Br}(i, \cdot)$ gives the optimal set for player $i \in I$ facing a strategy profile \mathbf{s} . We use the function $\pi(i, \cdot, \cdot)$ that associates strategy profiles to payoffs in a general way. As we said before, rationality of players implies that they will only use strategies that are optimal to some forecast. So, in a first order basis, the strategies of agent $i \in I$ are restricted to the set

$$\text{Br}(i, S^I) \equiv \bigcup_{\mathbf{s} \in S^I} \text{Br}(i, \mathbf{s}),$$

so strategy profiles can be actually secluded into the set:

$$S_1^I \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the} \\ \text{correspondence } i \rightrightarrows \text{Br}(i, S^I) \end{array} \right\}.$$

That is, in a first order basis, agents will not play a strategy that is not a best response to some strategy profile. This is captured by selections of the mapping $i \rightrightarrows \text{Br}(i, S^I)$. Taking this into account, agents can deduce, at a step t of this process, that strategy profiles must actually be in the set S_t^I ,

$$S_t^I \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the} \\ \text{correspondence } i \rightrightarrows \text{Br}(i, S_{t-1}^I) \end{array} \right\}.$$

This exercise motivates the definition of the recursive process of elimination of non best responses. For this, we define the mapping⁹

$$Pr(H) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the} \\ \text{correspondence } i \rightrightarrows \text{Br}(i, H) \end{array} \right\}. \quad (4.2)$$

The set $Pr(H)$ could well be empty. A sufficient condition for non-emptiness of $Pr(H)$ is the measurability of the correspondence $i \rightrightarrows \text{Br}(i, H)$. The mapping Pr represents strategies that are obtained as the reactions of agents to strategy profiles contained in the set $H \subseteq S^I$. It has to be kept in mind that strategies of different agents in a strategy profile in $Pr(H)$ can be the reactions to (possibly) different strategy profiles in H .

The line of reasoning developed in the previous sections implies that a strategy profile that is point rationalizable should never be eliminated during the process generated by the iterations of Pr . Let us note $Pr^t(S^I) \equiv Pr(Pr^{t-1}(S^I))$ and $Pr^0(S^I) \equiv S^I$. The set $Pr^t(S^I)$ is the one obtained in the t^{th} step of the process of elimination of non-best-response strategy profiles. It is direct to see that $Pr^1(S^I) \equiv S_1^I$ and $Pr^t(S^I) \equiv S_t^I$. Note that this process gives a nested family of subsets of S^I and so a point that is never eliminated should be in the intersection of all of them. This means that the set of Point-Rationalizable Strategies, \mathbb{P}_S , must satisfy:

$$\mathbb{P}_S \subseteq \bigcap_{t=0}^{+\infty} Pr^t(S^I). \quad (4.3)$$

However, it is not enough to ask for this property, since rationality of players implies that a strategy should only be played if it is justified by a rationalizable strategy profile. We capture this second feature by asking condition (4.4) below,

$$\mathbb{P}_S \equiv Pr(\mathbb{P}_S). \quad (4.4)$$

A strategy profile in \mathbb{P}_S must be obtained as a profile of strategies that are, each one of them, best reply to some (possibly different) strategy profile in the set \mathbb{P}_S . Note that condition (4.4) implies (4.3), since a set that satisfies (4.4) would never be eliminated. One could wonder whether these two conditions are equivalent. In Theorem 4.7 we address this issue. With the concepts displayed so far, we can finally give a definition for the Point-Rationalizable Strategy Profiles set.

Definition 4.1. The set of *Point-Rationalizable Strategy Profiles* is the maximal subset $H \subseteq S^I$ that satisfies condition (4.4) and we note it \mathbb{P}_S . For each player, $i \in I$, there will be a set of Point-Rationalizable Strategies, namely the union, over all the Point-Rationalizable Strategy Profiles in \mathbb{P}_S , of the best response set of the considered player. That is, the set of Point-Rationalizable Strategies for player $i \in I$ is,

$$\mathbb{P}_S^i := \bigcup_{\mathbf{s} \in \mathbb{P}_S} \text{Br}(i, \mathbf{s})$$

A well known result for the case of games with a finite number of players is that all Nash Equilibria of the game are elements of the Point-Rationalizable Strategies set (Bernheim 1984). The same is true for our definition, since if \mathbf{s}^* is a Nash Equilibrium, then it is a selection taken from $i \rightrightarrows \text{Br}(i, \mathbf{s}^*)$ and so it satisfies $\{\mathbf{s}^*\} \subseteq \text{Pr}(\{\mathbf{s}^*\})$ which implies the property.

We now turn to a different approach to Rationalizability. In the context that interests us, players form expectations not on the space of strategy profiles, but on the set of states of the economic system. Thus Rationalizability should also be stated in terms of forecasts on this set of states. This is what we present in the next subsection.

4.2 Point-Rationalizable States

We turn to the particular class of games with a continuum of players, that follow from Definition 3.1, where payoffs depend explicitly on the average of the actions of all the players, which we call the state of the game. In this framework it is natural to model agents as having forecasts on the set of states, rather than on the set of strategy profiles, since the relevant information that agents need to take a decision is the value of the state a ¹⁰.

In what follows, we will define Point-Rationalizability on the set of states. So now instead of using the correspondence $\text{Br}(i, \cdot)$ defined in (4.1), we use the mapping $B(i, \cdot) : \mathcal{A} \rightrightarrows S$ that gives the optimal strategy set given a state of the system,

$$B(i, a) := \text{argmax} \{u(i, y, a) : y \in S\}.$$

There are two main differences between this approach and the one presented in the previous subsection. First, here we use the specific function \mathbf{u} that defines a game following Definition 3.1 instead of the general function π as in (4.1), and second, the mapping $B(i, \cdot)$ goes from $\mathcal{A} \subset \mathbb{R}^n$, instead of S^I , to $S \subset \mathbb{R}^n$. For each $i \in I$ and a set $X \subseteq \mathcal{A}$, consider the image through $B(i, \cdot)$ of the set X

$$B(i, X) := \bigcup_{a \in X} B(i, a).$$

Before continuing, we state a relevant property associated to the mapping B .

Lemma 4.2. *In a game \mathbf{u} , for a closed set $X \subseteq \mathcal{A}$ the correspondence $i \rightrightarrows B(i, X)$ is measurable and has compact values.*

The proof is relegated to the Appendix. With Lemma 4.2 we can now prove Theorem 3.3.

¹⁰See as well Guesnerie (2002) for a discussion on this matter.

Proof of Theorem 3.3:

Consider the correspondence $\Gamma : \mathcal{A} \rightrightarrows \mathcal{A}$ defined by ¹¹

$$\Gamma(a) := \int_I B(i, a) \, di.$$

Note that a fixed point of Γ defines an equilibrium of the game \mathbf{u} . Lemma 4.2 implies that for all $a \in \mathcal{A}$, $\Gamma(a) \neq \emptyset$. By definition, for all $a \in \mathcal{A}$, $\Gamma(a)$ is convex. Under our assumptions, the correspondences $B(i, \cdot) : \mathcal{A} \rightrightarrows S$ are u.s.c. and from Aumann (1976) so is Γ . This last assertion implies as well that $\Gamma(a)$ is compact $\forall a \in \mathcal{A}$. Applying Kakutani's fixed point Theorem we get that there exists $a^* \in \mathcal{A}$ such that $a^* \in \Gamma(a^*)$. ■

Let us now look at the process of elimination of non reachable or non generated states. Suppose that initially agents' common knowledge about the actual state of the model is a subset $X \subseteq \mathcal{A}$. Then, in a first order basis, an agent can assume that any of the states $a \in X$ can be the actual state, but point expectations are actually constrained by X , so the possible actions of a player $i \in I$ are constrained to the set $B(i, X)$. Since all players know this, each one of them can discard any strategy profiles $\mathbf{s} \in S^I$ that are not selections of the set valued mapping $i \rightrightarrows B(i, X)$ introduced above. Then, if the players know that forecasts are restricted to $X \subseteq \mathcal{A}$, they will know that the actual outcome has to be a state associated through the aggregation operator to some measurable selection of that mapping.

Therefore, given $X \subseteq \mathcal{A}$ consider the set of all the measurable selections taken from the correspondence $i \rightrightarrows B(i, X)$ that to each agent $i \in I$ associates the set $B(i, X)$. Then, take all the possible images through the aggregation mapping of such functions. We define then the mapping $\tilde{P}r : \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{A})$ that to each set $X \subseteq \mathcal{A}$ associates the set $\tilde{P}r(X) \subseteq \mathcal{A}$ defined by:

$$\tilde{P}r(X) := \left\{ a \in \mathcal{A} : a = A(\mathbf{s}), \quad \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the} \\ \text{correspondence } i \rightrightarrows B(i, X) \end{array} \right\}. \quad (4.5)$$

Our assumptions on the aggregation operator A allow us to re-write definition (4.5) as the integral of a set valued mapping:

$$\tilde{P}r(X) \equiv \int_I B(i, X) \, di$$

Lemma 4.2 above and Theorem 2 in Aumann (1965) assure that this integral is non empty and closed whenever X is closed. With this set to set mapping we can define a set of point rationalizable states.

As we did in the previous section, consider the process given by iterations of $\tilde{P}r$. That is,

$$\begin{aligned} \tilde{P}r^0(\mathcal{A}) &:= \mathcal{A} \\ \tilde{P}r^{t+1}(\mathcal{A}) &:= \tilde{P}r\left(\tilde{P}r^t(\mathcal{A})\right) \quad \text{for } t \geq 1. \end{aligned}$$

Observe that $\tilde{P}r^{t+1}(\mathcal{A}) \subseteq \tilde{P}r^t(\mathcal{A})$, this is not necessarily true for any subset $X \subseteq \mathcal{A}$. The set of Point-Rationalizable States, $\mathbb{P}_{\mathcal{A}}$, must then satisfy:

$$\mathbb{P}_{\mathcal{A}} \subseteq \bigcap_{t=0}^{\infty} \tilde{P}r^t(\mathcal{A}). \quad (4.6)$$

¹¹The integral of a correspondence $F : I \rightrightarrows \mathbb{R}^n$ is calculated, following Aumann (1965), as the set of integrals of all the integrable selections of F . This is,

$$\int_I F(i) \, di \equiv \left\{ \int_I f(i) \, di : f \text{ is an integrable selection of } F \right\}$$

where $\int f \, di := (\int f_1(i) \, di, \dots, \int f_n(i) \, di)$.

The right hand side of (4.6) represents the iterative elimination of non reachable states, as illustrated in Figure 2. At each step of this process, players only keep in mind the states that could be reached by rational actions based on point expectations given by the set of the previous step. If a state is not reached by actions following forecasts constrained at a certain step of the process, then it is not rationalizable. Since the family of sets $\{\tilde{P}r^t(\mathcal{A})\}_{t \in \mathbb{N}}$ is a nested (decreasing) family of closed sets, the infinite intersection in expression (4.6) turns out to be the exact *Painlevé-Kuratowski* limit ¹² of the sequence. The second condition that the set of Point-Rationalizable States must satisfy is:

$$\mathbb{P}_{\mathcal{A}} \equiv \tilde{P}r(\mathbb{P}_{\mathcal{A}}). \quad (4.7)$$

Condition (4.7) stands for the fact that Point-Rationalizable States should only be justified by Point-Rationalizable States. This means that if a state is Point-Rationalizable, it should rise as the consequences of players taking actions as reactions to point forecasts in the set of Point-Rationalizable States. Analogously to the case where point forecasts are taken over strategy profiles, it is direct to see that condition (4.7) implies (4.6). That is, if a set $X \subseteq \mathcal{A}$ satisfies condition (4.7) then $X \subseteq \bigcap_{t=0}^{\infty} \tilde{P}r^t(\mathcal{A})$. So we define the set of Point-Rationalizable States as follows:

Definition 4.3. The set of Point-Rationalizable States is the maximal subset $X \subseteq \mathcal{A}$ that satisfies condition (4.7) and we note it $\mathbb{P}_{\mathcal{A}}$.

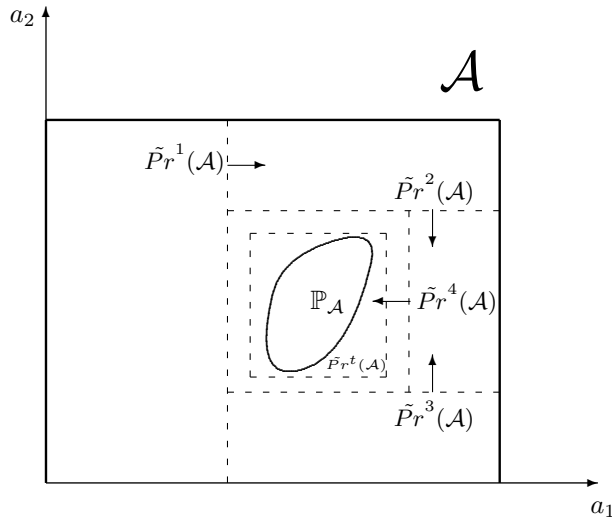


Figure 2: Iterative elimination of non reachable states.

Remark 4.4. Note that for the case of forecasts over the set of states, defining player-specific rationalizable states set makes no sense. This approach calls for different mathematical tools since now we are dealing with a set in a finite dimensional space as opposed to Definition 4.1. Moreover, the exercise of obtaining Point-Rationalizable States gives clear insights on properties of the Point-Rationalizable Strategy Profiles set, particularly for strongly rational equilibria, as can be seen in Proposition 4.10 below.

Remark 4.5. Conditions (4.4) and (4.7) are analogous to the definition of *Tight Sets Closed Under Rational Behavior* (Tight CURB Sets) given in Basu and Weibull (1991). Indeed Basu and Weibull make the observation that the set of rationalizable strategy profiles in a finite game with compact strategy sets and continuous payoff functions, is in fact the maximal tight curb set, which is analogous to our definitions of Point-Rationalizability.

4.3 Point-Rationalizable Strategies vs. Point-Rationalizable States

It is straightforward to ask how the concepts that we just defined are related. To address this issue, note that for the class of non-atomic games that we are considering, the iterations of the strategy-elimination and state-elimination mappings (Pr and \tilde{Pr}) are equivalent in the following sense. Consider the set to set mappings \bar{A} and \bar{B} defined below. Let $\bar{A} : \mathcal{P}(S^I) \rightarrow \mathcal{P}(\mathcal{A})$ be defined by:

$$\begin{aligned} \bar{A}(H) &:= \left\{ a \in \mathcal{A} : \begin{array}{l} a = \int_I s(i) di \text{ and } \mathbf{s} \text{ is a measur-} \\ \text{able function in } H \end{array} \right\} \\ &\equiv A(H) \end{aligned}$$

and let $\bar{B} : \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(S^I)$ be:

$$\bar{B}(X) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of} \\ i \rightrightarrows B(i, X) \end{array} \right\}.$$

These mappings satisfy

$$\begin{aligned} X_1 \subseteq X_2 \subseteq \mathcal{A} &\implies \bar{B}(X_1) \subseteq \bar{B}(X_2) \\ H_1 \subseteq H_2 \subseteq S^I &\implies \bar{A}(H_1) \subseteq \bar{A}(H_2). \end{aligned} \tag{4.8}$$

Then, in the context of a game \mathbf{u} , the mappings Pr and \tilde{Pr} satisfy:

$$\begin{aligned} Pr(H) &\equiv \bar{B}(\bar{A}(H)) \\ \tilde{Pr}(X) &\equiv \bar{A}(\bar{B}(X)) \end{aligned} \tag{4.9}$$

In particular, we get,

$$\tilde{Pr}^0(\mathcal{A}) \equiv \mathcal{A} \equiv \text{co}\{S\} \equiv \bar{A}(S^I)$$

which implies by induction that:

$$\begin{aligned} \tilde{Pr}^t(\mathcal{A}) &\equiv \bar{A}(Pr^t(S^I)) \\ Pr^t(S^I) &\equiv \bar{B}(\tilde{Pr}^{t-1}(\mathcal{A})) \end{aligned}$$

Theorem 4.6. *In a game \mathbf{u} we have:*

$$\mathbb{P}_S \equiv \bar{B}(\mathbb{P}_{\mathcal{A}}) \quad \text{and} \quad \mathbb{P}_{\mathcal{A}} \equiv \bar{A}(\mathbb{P}_S).$$

Proof.

Note that from (4.9) we have

$$\begin{aligned} Pr(\bar{B}(\mathbb{P}_{\mathcal{A}})) &\equiv \bar{B}(\tilde{Pr}(\mathbb{P}_{\mathcal{A}})) \equiv \bar{B}(\mathbb{P}_{\mathcal{A}}) \\ \tilde{Pr}(\bar{A}(\mathbb{P}_S)) &\equiv \bar{A}(Pr(\mathbb{P}_S)) \equiv \bar{A}(\mathbb{P}_S) \end{aligned}$$

That is, the sets $\bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq S^I$ and $\bar{A}(\mathbb{P}_S) \subseteq \mathcal{A}$ satisfy conditions (4.4) and (5.1) respectively, which implies that $\bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq \mathbb{P}_S$ and $\bar{A}(\mathbb{P}_S) \subseteq \mathbb{P}_{\mathcal{A}}$. Then

$$\mathbb{P}_S \equiv Pr(\mathbb{P}_S) \equiv \bar{B}(\bar{A}(\mathbb{P}_S)) \subseteq \bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq \mathbb{P}_S.$$

The second equality comes from (4.9) while the first inclusion comes from (4.8) and the previous observation. The proof for the second statement is analogous. ■

We see from Theorem 4.6 that in the context that we are considering, the set of Point-Rationalizable Strategies is paired with the set of Point-Rationalizable States. This implies that in the models that interest us, it is equivalent to study Point-Rationalizability in terms of strategies or states, an intuition quoted by Guesnerie and, of course, present in the example.

A second issue to be assessed, is the question of whether the two conditions that we asked Point-Rationalizable sets to satisfy, are equivalent. Our main result, Theorem 4.7, states that under the hypothesis of Rath's setting we can obtain their equivalence for Point-Rationalizable States, and so we obtain a first characterization of the Point-Rationalizable States set.

Theorem 4.7. *Let us write $\mathbb{P}'_{\mathcal{A}} := \bigcap_{t=0}^{\infty} \tilde{P}r^t(\mathcal{A})$. The set of Point-Rationalizable States of a game \mathbf{u} can be calculated as*

$$\begin{aligned} \mathbb{P}_{\mathcal{A}} &\equiv \mathbb{P}'_{\mathcal{A}} \\ &\equiv \bigcap_{t=0}^{\infty} \tilde{P}r^t(\mathcal{A}) \end{aligned}$$

Proof.

We will show that:

$$\tilde{P}r(\mathbb{P}'_{\mathcal{A}}) \equiv \mathbb{P}'_{\mathcal{A}}$$

Let us begin by showing that $\tilde{P}r(\mathbb{P}'_{\mathcal{A}}) \subseteq \mathbb{P}'_{\mathcal{A}}$. Indeed, if $a \in \tilde{P}r(\mathbb{P}'_{\mathcal{A}})$ then, by the definition of $\tilde{P}r$, there exists a measurable selection $\mathbf{s} : I \rightarrow S$ of $i \rightrightarrows B(i, \mathbb{P}'_{\mathcal{A}})$, such that $a = \int_I \mathbf{s}$. Since $\mathbb{P}'_{\mathcal{A}} \subseteq \tilde{P}r^t(\mathcal{A}) \forall t \geq 0$, we have that $B(i, \mathbb{P}'_{\mathcal{A}}) \subseteq B(i, \tilde{P}r^t(\mathcal{A})) \forall t \geq 0 \forall i \in I$. So \mathbf{s} is a selection of $i \rightrightarrows B(i, \tilde{P}r^t(\mathcal{A}))$ and then $a \in \tilde{P}r^{t+1}(\mathcal{A}) \forall t \geq 0$, which means that $a \in \mathbb{P}'_{\mathcal{A}}$.

Now we show that $\mathbb{P}'_{\mathcal{A}} \subseteq \tilde{P}r(\mathbb{P}'_{\mathcal{A}})$. For this inclusion, consider the following sequence $F^t : I \rightrightarrows S, t \geq 0$, of set valued mappings:

$$\begin{aligned} F^0(i) &:= S & \forall i \in I \\ \forall i \in I \quad F^t(i) &:= B(i, \tilde{P}r^{t-1}(\mathcal{A})) & t \geq 1 \end{aligned}$$

As we said before, we have that

$$\tilde{P}r^t(\mathcal{A}) \equiv \int_I F^t(i) \text{ di.}$$

Since $\mathbf{u}(i) \in \mathcal{U}_{S \times \mathcal{A}}$, then $\forall i \in I$ the mapping $B(i, \cdot) : \mathcal{A} \rightrightarrows S$ is u.s.c. and, as a consequence, the set $B(i, X)$ is compact for any compact subset $X \subseteq \mathcal{A}$ (Berge 1997). Since $\mathcal{A} \equiv \int_I F^0$, Aumann (1965) gives that \mathcal{A} is non empty and compact¹³. From Lemma 4.2 we get that F^1 is measurable and compact valued and by induction over t , we get that for all $t \geq 1$, $\tilde{P}r^{t-1}(\mathcal{A})$ is non empty, convex and compact, and F^t is measurable and compact valued.

Consider then the set valued mapping $F : I \rightrightarrows S$ defined as the point-wise lim sup of the sequence F^t , noted $\text{p-lim sup}_t F^t$, obtained as:

$$F(i) := (\text{p-lim sup}_t F^t)(i) \equiv \limsup_t F^t(i)$$

where the right hand side is the set¹⁴ of all cluster points of sequences $\{y^t\}_{t \in \mathbb{N}}$ such that $y^t \in F^t(i)$. From Rockafellar and Wets¹⁵ we get that F is measurable and compact valued.

So now let us take a point $a \in \mathbb{P}'_{\mathcal{A}}$. That is, $a \in \int_I F^t$ for all $t \geq 0$. This gives a sequence of measurable selections $\{\mathbf{s}^t\}_{t \in \mathbb{N}}$, such that $a = \int_I \mathbf{s}^t$. From the Lemma proved in Aumann (1976)

¹³We have already noted that in fact $\mathcal{A} \equiv \text{co}\{S\}$ and so in particular it is also convex, which is of no relevance in this proof, but is the key property in Corollary 4.8.

¹⁴See Rockafellar and Wets (1998) Ch. 4 and 5.

¹⁵Theorem 14.20.

we get that $a \in \int_I F$, since for each $i \in I$ the cluster points of $\{s^t(i)\}_{t \in \mathbb{N}}$ belong to $F(i)$ and a is the trivial limit of the constant sequence $\int s^t$.

Now it suffices to check that $F(i) \subseteq B(i, \mathbb{P}'_{\mathcal{A}})$, since then we would have

$$a \in \int_I F \text{ di} \subseteq \int_I B(i, \mathbb{P}'_{\mathcal{A}}) \text{ di} \equiv \tilde{P}r(\mathbb{P}'_{\mathcal{A}}).$$

This comes from the upper semi continuity of $B(i, \cdot)$. Take $y \in F(i)$. From the definition of F , y is a cluster point of a sequence $\{y^t\}_{t \in \mathbb{N}}$ such that $y^t \in F^t(i)$. That is, there is a sequence of elements of \mathcal{A} , $\{a^k\}_{k \in \mathbb{N}}$ such that $a^k \in \tilde{P}r^{t_k-1}(\mathcal{A})$, $y^{t_k} \in B(i, a^k)$ and $\lim_k y^{t_k} = y$. Through a subsequence of $\{a^k\}_{k \in \mathbb{N}}$ we get that the limit of $\{y^{t_k}\}_{k \in \mathbb{N}}$ must belong to $B(i, \mathbb{P}'_{\mathcal{A}})$, since all cluster points of $\{a^k\}_{k \in \mathbb{N}}$ are in $\mathbb{P}'_{\mathcal{A}}$, being the intersection of a nested family of compact sets. \blacksquare

The previous theorem gives a characterization of Point-Rationalizable States that is analogous the one given for Point-Rationalizable Strategies in Bernheim, in the case of finite player games with compact strategy sets and continuous utility functions. The difficulty of Theorem 4.7 is to identify the adequate convergence concept for the eductive process. In the case of finite player games there is no such a question, since in that case the technique is simply to take a convergent subsequence of points (in the finite dimensional strategy profile set) from the sequence of sets that participate in the eductive procedure and using (semi) continuity arguments of the best response mappings obtain the result ¹⁶. However, in our setting this arguments fail to prevail. From the proof of the Theorem, we see that the set of Point-Rationalizable States is obtained as the integral of the point-wise upper limit of a sequence of set valued mappings. So the relevant improvement in the proof (besides measurability requirements) is to give the adequate limit concept.

To see that the Theorem is not vacuous consider the following example.

Example 2. Consider the game where $I \equiv [0, 1]$, $S \equiv [0, 1]$ and $\mathbf{u}(i) \equiv u : [0, 1]^2 \rightarrow \mathbb{R}$ for all $i \in I$, such that it generates the following best reply correspondence, depicted in Figure 3a:

$$B(a) = \begin{cases} a^* & \text{if } a \leq \bar{a}, \\ \{0, \bar{a}(1 - \alpha) + a\alpha\} & \text{if } a > \bar{a}, \end{cases}$$

where a^* , \bar{a} and α are in $]0, 1[$ and $a^* < \bar{a}$.

The only equilibrium of the game is a^* .

Since all the players have the same best reply correspondence, the process of elimination of non-generated-states is obtained by:

$$\tilde{P}r(X) \equiv \text{co} \{B(X)\}$$

The image through the best reply correspondence of the state set is

$$B(\mathcal{A}) \equiv \{0, a^*\} \cup]\bar{a}, \bar{a}(1 - \alpha) + \alpha],$$

then $\tilde{P}r(\mathcal{A}) \equiv \text{co} \{B(\mathcal{A})\} \equiv [0, \bar{a}(1 - \alpha) + \alpha]$.

Then the second iteration is obtained by

$$\begin{aligned} \tilde{P}r^2(\mathcal{A}) &\equiv \text{co} \left\{ B(\tilde{P}r(\mathcal{A})) \right\} \equiv \text{co} \{B([0, \bar{a}(1 - \alpha) + \alpha])\} \\ &\equiv \text{co} \{ \{0, a^*\} \cup]\bar{a}, \bar{a}(1 - \alpha) + \alpha(\bar{a}(1 - \alpha) + \alpha)\} \\ &\equiv [0, \bar{a}(1 - \alpha^2) + \alpha^2]. \end{aligned}$$

We see from the form of the best reply correspondence that on each iteration of $\tilde{P}r$ we get an interval of the form $[0, a^t]$ where the sequence $\{a^t\}_{t=0}^{+\infty}$ satisfies:

$$a^{t+1} = \bar{a}(1 - \alpha) + a^t \alpha,$$

¹⁶See the proof of Proposition 3.1 in Bernheim (1984).

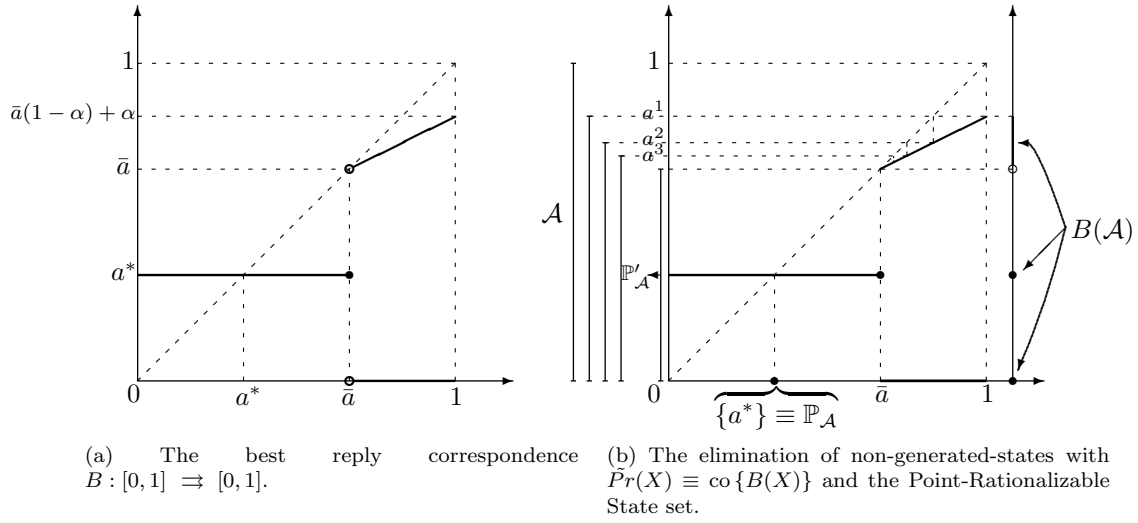


Figure 3: Example 2: The set of Point-Rationalizable States is not the set $\mathbb{P}'_{\mathcal{A}}$.

which gives,

$$a^t = \bar{a}(1 - \alpha^t) + \alpha^t,$$

with $a^0 = 1$, and so the sequence is decreasing and converges to \bar{a} (see Figure 3b). This allows for us to see that $\mathbb{P}'_{\mathcal{A}} \equiv [0, \bar{a}]$. However,

$$\begin{aligned} \tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) &\equiv \text{co}\{B(\mathbb{P}'_{\mathcal{A}})\} \equiv \text{co}\{B([0, \bar{a}])\} \equiv \text{co}\{\{a^*\}\} \equiv \{a^*\} \\ &\subsetneq \mathbb{P}'_{\mathcal{A}}. \end{aligned}$$

and so $\mathbb{P}'_{\mathcal{A}}$ is not equal to $\mathbb{P}_{\mathcal{A}}$, which is in fact equal to the set of equilibria: the singleton $\{a^*\}$.

Theorems 4.6 and 4.7 together imply that the set of Point-Rationalizable Strategies \mathbb{P}_S can be actually computed, in this setting, as the limit of the strategy elimination process governed by Pr (see condition (4.3)) answering a question that remained unanswered. In consequence, we have that in a game \mathbf{u} we can obtain the sets of Point-Rationalizable States and Strategies through the eductive process in the respective set (\mathcal{A} or S^I).

It is a common case where the equilibria or stable strategies of an infinite game are described with finite dimensional elements associated with an infinite set of infinite dimensional structure. That is the case for instance when what matters in the game is a distribution of types or of agents that choose between a finite number of alternatives. Here the distribution can be a unique one associated to a pure strategy choice (where agents are choosing the alternatives in the proportion dictated by the equilibrium distribution) or mixed strategy choices that give the same equilibrium distribution. So usually one can talk about a “unique” equilibrium in the set of states associated to an infinite set of functions that generate such an equilibrium. An illustration of this situation for rationalizable equilibria is Proposition 4.10 in the next subsection.

Theorem 3.3 implies that if the set of Point-Rationalizable Strategies (or States for what it matters) is well defined, then it is not empty, since as we already said, all the equilibria belong to this set. From Theorem 4.7 we get as a Corollary that in the games that we are considering, the set of Point-Rationalizable States is well behaved.

Corollary 4.8. *The set of Point-Rationalizable States of a game \mathbf{u} is well defined, non-empty, convex and compact.*

Proof.

From Theorem 4.7, $\mathbb{P}_{\mathcal{A}}$ is the intersection of a nested family of non-empty compact convex sets. From Theorem 3.3 we get that there is point $a^* \in \mathcal{A}$ such that $a^* \in \tilde{Pr}^t(\mathcal{A}) \forall t$. These two facts lead us to conclude that this intersection is compact, convex and non empty. ■

The properties stated in this Corollary are not trivial. In games with finite number of players we can find examples where the outcome of the iterative elimination of non-best-replies is an empty set. The same can be true in our context:

Example 3 (Based on Dufwenberg and Stegeman (2002)). Consider the game where $I \equiv [0, 1]$, $S \equiv [0, 1]$ and $\mathbf{u}(i) \equiv u : [0, 1]^2 \rightarrow \mathbb{R}$ for all $i \in I$, such that:

$$u(y, a) = \begin{cases} 1 - y & \text{if } 0 < \frac{a}{2} \leq y, \\ y & \text{if not.} \end{cases}$$

Then, the best reply correspondence is the same for all players:

$$B(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{a}{2} & \text{if } a > 0. \end{cases}$$

The mapping Γ turns out to be equal to the best reply correspondence:

$$\Gamma(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{a}{2} & \text{if } a > 0. \end{cases}$$

This mapping has no fixed point and so in this game there is no equilibrium.

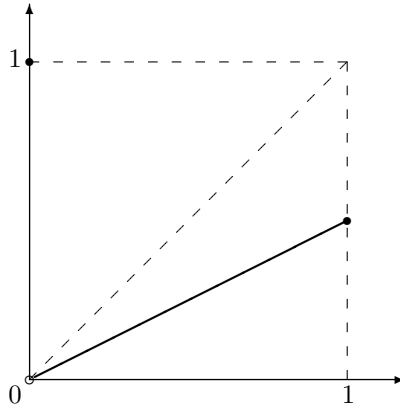


Figure 4: The best reply correspondence $B : [0, 1] \rightrightarrows [0, 1]$.

Let us study the Point-Rationalizable States set. The image through the best reply correspondence of the state set is $B(\mathcal{A}) \equiv \{1\} \cup]0, 1/2]$, then

$$\tilde{Pr}(\mathcal{A}) \equiv \text{co} \{B(\mathcal{A})\} \equiv]0, 1],$$

the second iteration gives

$$\tilde{Pr}^2(\mathcal{A}) \equiv \text{co} \left\{ B\left(\tilde{Pr}(\mathcal{A})\right) \right\} \equiv \text{co} \{B(]0, 1])\} \equiv]0, 1/2]$$

and the third

$$\tilde{Pr}^3(\mathcal{A}) \equiv \text{co} \left\{ B(\tilde{Pr}^2(\mathcal{A})) \right\} \equiv \text{co} \{ B([0, 1/2]) \} \equiv]0, 1/4], \dots, \text{etc.}$$

Then the set $\mathbb{P}'_{\mathcal{A}} \equiv \emptyset$ and so $\mathbb{P}_{\mathcal{A}} \equiv \emptyset$, that is, in this example there is no set $X \subseteq \mathcal{A}$ that satisfies $X \subseteq \tilde{Pr}(X)$.

Recent papers address the issue of the limit of processes of iterated elimination of strategies not being a fixed point of the iterated process in games with finite number of players (in our context we would consider what we call here the *eductive procedure*) and go beyond to explore more complex iterated processes of elimination of strategies (see for instance Dufwenberg and Stegeman (2002), Apt (2007), Chen, Long, and Luo (2007)). This problem may rise when we relax the assumptions on utility functions and strategy sets (namely the cases for unbounded strategy sets and/or discontinuous utility functions¹⁷). Although Theorem 4.7 and Corollary 4.8 assure that the eductive procedure achieves a non-empty Point-Rationalizable set of states, we see from Examples 2 and 3 that even in cases where the eductive procedure fails, in the sense that it does not deliver the Point-Rationalizable set, we may still identify a set as the correct Point-Rationalizable State set following our Definition 4.3, even more, the eductive procedure (obviously) can help to locate such set even in the case of failure and emptiness. Since our motivation is the plausibility of the Rational Expectations Hypothesis, we allow for an empty Rationalizable set, interpreting such as a pessimistic answer to the possibility of the coordination of expectations in economic contexts.

Another property stated in Corollary 4.8 and an important consequence of Theorem 4.7, is the convexity of the Point-Rationalizable States set, since in the case where we have multiple equilibria in the set of states, we know that the convex hull of this set of equilibria is also contained in the set of Point-Rationalizable States. This inclusion may be strict since if we have multiple equilibria in the set of strategies, even with uniqueness in the set of states, we may have multiple Point-Rationalizable States¹⁸. Convexity of the Point-Rationalizable States set is a relevant property since it has not been obtained (to our knowledge) for any other concept related to Rationalizability.

4.4 Strongly Rational Equilibrium

As we have already said, Guesnerie defines the concept of (local) Strongly Rational Equilibrium as an equilibrium that is the only Rationalizable State of an economic system. A particular feature of the work therein developed is that although the definition of Rationalizability is stressed in terms of strategy profiles, that is, on the profile of individual production quantities, the study of the (local) stability of the (unique) equilibrium can be developed in terms of aggregate production or even prices¹⁹. In our context, Strong Rationality would be defined in terms of the aggregate variable $a \in \mathcal{A}$.

Our purpose in this section then is to explore the relation between strategy profiles and aggregate states when we are interested in Strong Rationality and Point-Rationalizability.

Definition 4.9. An equilibrium state $a^* \in \mathcal{A}$ is a *Strongly Rational Equilibrium* if $\mathbb{P}_{\mathcal{A}} = \{a^*\}$.

Note that if a state a^* satisfies $\mathbb{P}_{\mathcal{A}} = \{a^*\}$, then it is the unique equilibrium of the system since (i) all equilibria are in $\mathbb{P}_{\mathcal{A}}$ and (ii) $Pr(\{a^*\}) \equiv \{a^*\}$ implies that a^* is the unique value obtained from the integral of the best response mapping $i \rightrightarrows B(i, a^*)$ and so is an equilibrium. Analogously if $\mathbb{P}_S \equiv \{\mathbf{s}^*\}$, then \mathbf{s}^* is the unique Nash Equilibrium of the game, since all Nash Equilibria are in \mathbb{P}_S and $Pr(\{\mathbf{s}^*\}) \equiv \{\mathbf{s}^*\}$ implies that \mathbf{s}^* is the (unique) measurable selection of $i \rightrightarrows Br(i, \mathbf{s}^*)$ and so it is a Nash Equilibrium. In particular this says that $Br(i, \mathbf{s}^*)$ is λ -a.e. single valued and hence can be associated to the concept of strict Nash Equilibrium.²⁰

¹⁷Dufwenberg and Stegeman (2002) and Chen, Long, and Luo (2007) put emphasis in Reny's (1999) better-reply secure games

¹⁸See Proposition 4.10.

¹⁹See note 10.

²⁰A situation in which any unilateral deviation incurs in a loss.

Proposition 4.10. *If \mathbf{s}^* is a Nash Equilibrium of \mathbf{u} and $A(\mathbf{s}^*) = a^*$, then:*

$$\begin{aligned} a^* \text{ is Strongly Rational} &\implies \mathbb{P}_S \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selec-} \\ \text{tion of } i \implies B(i, a^*) \end{array} \right\} \\ \mathbb{P}_S \equiv \{\mathbf{s}^*\} &\implies a^* \text{ is Strongly Rational} \end{aligned}$$

In particular, if $B(i, \cdot)$ is single valued at a^ λ -a.e. on I , then,*

$$a^* \text{ is Strongly Rational} \iff \mathbb{P}_S \equiv \{\mathbf{s}^*\}$$

Proof.

By the definition of \mathbb{P}_A and \mathbb{P}_S and the property of \mathbb{P}_A stated in Theorem 4.6 we have,

$$\bar{B}(\mathbb{P}_A) \equiv \mathbb{P}_S \tag{4.10}$$

Suppose that $\mathbb{P}_S \equiv \{\mathbf{s}^*\}$. Then, a^* is an equilibrium, so it satisfies $a^* \in \mathbb{P}_A$ which in turn implies that $\mathbb{P}_A \neq \emptyset$. Property (4.10) gives

$$\mathbb{P}_S \equiv \{\mathbf{s}^*\} \implies \tilde{P}r(\mathbb{P}_A) \equiv \{a^*\}$$

and by the definition of \mathbb{P}_A ,

$$\mathbb{P}_A \equiv \tilde{P}r(\mathbb{P}_A).$$

For the proof in the opposite sense, analogously we get:

$$\bar{A}(\mathbb{P}_S) \equiv \mathbb{P}_A \tag{4.11}$$

Since \mathbf{s}^* is a Nash Equilibrium, $\mathbb{P}_S \neq \emptyset$ and then from (4.11) we get:

$$\mathbb{P}_A = \{a^*\} \implies Pr(\mathbb{P}_S) \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selec-} \\ \text{tion of } i \implies B(i, a^*) \end{array} \right\}$$

And from the definition of \mathbb{P}_S ,

$$\mathbb{P}_S \equiv Pr(\mathbb{P}_S)$$

■

Proposition 4.10 shows that it is possible to study Eductive Stability of models with continuum of agents that fit our framework using the set of states as well as the set of strategies. Moreover, it can be even desirable to use the former approach since (local) uniqueness and stability are more pertinent in terms of the state of the system rather than in terms of strategy profiles, as is discussed in the previous section. For instance, the study of Strategic complementarities in coordination games or Strategic substitutability in general models as well, can be undertaken by looking at states of the system rather than strategy profiles (see the books by Cooper (1999) and Chamley (2004). See as well Guesnerie (2005) and Guesnerie and Jara-Moroni (2007)).

5 Rationalizability vs Point-Rationalizability

Rationalizability differs from Point-Rationalizability on the way we address forecasts. For Rationalizability, forecasts are no longer points in the corresponding sets but probability distributions whose supports are contained in these sets. Then, when a player has a subjective probability forecast over what may occur with the rest of the economic system, he maximizes his expected utility with respect to such a probability distribution to make a decision. Rationality implies that players should not give positive weight in their forecasts to strategies that are not best response to some rationally generated forecast.

Rationalizable Strategy Profiles, for instance, should be obtained from a similar exercise as done in Subsection 4.1, but considering forecasts as probability measures over the set of strategies of the opponents. Loosely speaking, each player should consider a profile of probability measures (his subjective forecasts over each of his opponents' play) and maximize some expected utility, expectation taken over an induced probability measure over the set of strategy profiles.

A difficulty in a context with continuum of players, relates with the continuity or measurability properties that must be attributed to subjective beliefs, as a function of the player's name. There is no straightforward solution in any case. However, in our framework it is possible to bypass this difficulty. Using the intuition just described for the strategy profiles case, we will reformulate the processes of elimination of strategies and states described by equations (4.2) and (4.5) by considering procedures where players eliminate strategies that are not best response to any possible (subjective probability) forecast (profile) over a given set of states or strategy profiles.

We present first, in the next Subsection, the concept of Rationalizable States, where forecasts and the process of elimination are taken over the set of states \mathcal{A} . Then, in Subsection 5.3 we will make use of Schmeidler's original framework of games with continuum of players, to formalize the idea of Rationalizable Strategy Profiles in that context.

5.1 Forecasts over the set of states

Before we enter into context we need to introduce some concepts and some notation. For a Borel set $X \subseteq \mathbb{R}^n$ we denote by $\mathcal{P}(X)$ the set of probability measures on the Borel sets of X . Equivalently this is the set of probability measures on the Borel sets of \mathbb{R}^n whose support is in X . We will endow the set $\mathcal{P}(X)$ with the weak* topology $w^* = \sigma(\mathcal{P}(X), C_b(X))$ ²¹. For a Borel subset Z of X in \mathbb{R}^n , $\mathcal{P}(Z)$ can be considered to be a subset of $\mathcal{P}(X)$ and the weak* topology on $\mathcal{P}(Z)$ is the relativization of the weak* topology on $\mathcal{P}(X)$ to $\mathcal{P}(Z)$. The set X can be topologically identified with a subspace of $\mathcal{P}(X)$ by the embedding $x \mapsto \delta_x$. An important property that we will use is that X is compact if and only if $\mathcal{P}(X)$ is compact (and metrizable, since we use the norm in \mathbb{R}^n)²². Finally, we will make use of the following result whose proof is relegated to the appendix.

Lemma 5.1. *Let Y and X be compact subsets of \mathbb{R}^n . Given a function $u \in C_b(Y \times X)$, the function $U : Y \times \mathcal{P}(X) \rightarrow \mathbb{R}$, which to each $(y, \mu) \in Y \times \mathcal{P}(X)$ associates the expectation*

$$U(y, \mu) \equiv \int_X u(y, x) d\mu(x),$$

is continuous when $\mathcal{P}(X)$ is endowed with the weak topology.*

As we said before, in the setting of Rath there is a simple way to get through the inconvenience of defining an induced probability measure over the set of strategy profiles, using the presence of the state variable of the game over which players have an infinitesimal influence.

We consider then games as defined in Definition 3.1. In this setting, we again consider players as having forecasts over the set of states rather than over each of the individual strategy sets. That is, forecasts are probability measures over the set of states rather than profiles of probabilities over the set of strategies. We define then for each player the set valued mapping that gives the actions that maximize expected utility given a probability measure μ over the set of states \mathcal{A} , $\mu \in \mathcal{P}(\mathcal{A})$, $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$:

$$\begin{aligned} \mathbb{B}(i, \mu) &:= \operatorname{argmax}_{y \in S} \mathbb{E}_\mu [u(i, y, a)] \\ &\equiv \operatorname{argmax}_{y \in S} \int_{\mathcal{A}} u(i, y, a) d\mu(a). \end{aligned}$$

As it has been along all this document, we can describe then, using $\mathbb{B}(i, \cdot)$, the process of elimination of unreasonable states, considering that players could now use probability forecasts

²¹Recall that $C_b(X)$ is the space of real valued bounded continuous functions on X .

²²See Aliprantis and Border (1999) for detailed treatments of this and other results.

over the set of states. If it is common knowledge that the actual state is restricted to a subset $X \subseteq \mathcal{A}$ then players will use strategies only in the set $\mathbb{B}(i, \mathcal{P}(X)) := \cup_{\mu \in \mathcal{P}(X)} \mathbb{B}(i, \mu)$. This is, each player $i \in I$ will behave optimally with respect to some subjective belief about the outcome of the game, whose support is contained in X . This means that rational strategy profiles have to be selections of the correspondence $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X))$ that maps the set of players on their set of optimal responses with respect to any possible forecast over X . The state of the game will then be the integral of one of these selections. This process is described with the mapping $\tilde{R} : \mathcal{B}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{A})$:

$$\begin{aligned} \tilde{R}(X) &:= \left\{ \int_I s(i) \, di : \begin{array}{l} \mathbf{s} \in S^I, \mathbf{s} \text{ is a measurable selection} \\ \text{of } i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X)) \end{array} \right\}. \\ &\equiv \int_I \mathbb{B}(i, \mathcal{P}(X)) \, di \end{aligned}$$

The set $\tilde{R}(X)$ gives the set of states that are obtained as consequence of optimal behavior when common knowledge about the outcome of the game is represented by X . As we said before, the difference between $\tilde{P}r$ and \tilde{R} is that the second process considers expected utility maximizers. For a given Borel set $X \subseteq \mathcal{A}$, X can be embedded into $\mathcal{P}(X)$. This means that $B(i, X) \subseteq \mathbb{B}(i, \mathcal{P}(X))$ and so we have that $\tilde{P}r(X) \subseteq \tilde{R}(X)$.

Proposition 5.2. *In a game \mathbf{u} , if $X \subseteq \mathcal{A}$ is nonempty and closed, then $\tilde{R}(X)$ is nonempty, convex and closed.*

Proof.

From Definition 3.1, $\mathbf{u}(i)$ belongs to $C_b(S \times \mathcal{A})$, and S and \mathcal{A} are compact sets in \mathbb{R}^n , so Lemma 5.1, along with Berges' Theorem, imply that for each $i \in I$ the correspondence $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ is upper semi continuous. Since X is closed, it is compact, which gives that $\mathcal{P}(X)$ is a compact subset of $\mathcal{P}(\mathcal{A})$ and so the set $\mathbb{B}(i, \mathcal{P}(X))$ is closed for every $i \in I$. From Theorem 4 in Aumann (1965) we get that $\int_I \mathbb{B}(i, \mathcal{P}(X)) \, di$ is closed.

On the other hand, Lemma 4.2 states that the correspondence $i \rightrightarrows B(i, X)$ is measurable, which means that it has a measurable selection \mathbf{s} . Since $B(i, X) \subseteq \mathbb{B}(i, \mathcal{P}(X))$, \mathbf{s} is also a selection of $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X))$. This implies that $\int_I \mathbb{B}(i, \mathcal{P}(X)) \, di$ is nonempty.

Convexity comes from the fact that $\tilde{R}(X)$ is obtained as an integral of a set valued mapping. ■

The previous Proposition allows us to define the Eductive Process in this case. As usual then we consider the iterative elimination of non generated states, but now allowing for probability forecasts of players. The iterative process begins with the whole set of outcomes, in this case \mathcal{A} .

$$\tilde{R}^0(\mathcal{A}) := \mathcal{A}$$

Then, on each iteration, the states that are not reached by the process \tilde{R} are eliminated:

$$\tilde{R}^{t+1}(\mathcal{A}) := \tilde{R}(\tilde{R}^t(\mathcal{A})).$$

Recall that since we start the process at \mathcal{A} , what we get is a nested family of sets that, following Proposition 5.2, are nonempty convex and closed. The Eductive Process gives then the set,

$$\mathbb{R}'_{\mathcal{A}} := \bigcap_{t=0}^{\infty} \tilde{R}^t(\mathcal{A}).$$

Theorem 5.3. *In a game \mathbf{u} , the set $\mathbb{R}'_{\mathcal{A}}$ is non empty, convex and closed.*

Proof.

$\mathbb{R}'_{\mathcal{A}}$ is the intersections of closed convex sets, so it is convex and closed. Theorem 3.3 assures that $\mathbb{R}'_{\mathcal{A}}$ is nonempty, since equilibria belong to every set $\tilde{R}^t(\mathcal{A})$. ■

As it was the case before, the assumptions of rationality and common knowledge of rationality imply that players must take into account that all their opponents construct their subjective forecasts rationally. This is formalized by asking that the set of Rationalizable States must be a subset of $\mathbb{R}'_{\mathcal{A}}$ (analogously to (4.6)), in the sense that states that are eliminated can not rise with positive probability and hence are not rationalizable. On the other hand, if a state is rationalizable then it must be an outcome associated to optimal reactions to forecasts with support in the set of Rationalizable States, this means that the set of Rationalizable States must satisfy an analogous condition to (4.7). This is, the set of Rationalizable States $\mathbb{R}_{\mathcal{A}}$ must satisfy

$$\mathbb{R}_{\mathcal{A}} \equiv \tilde{R}(\mathbb{R}_{\mathcal{A}}). \quad (5.1)$$

Note that if a set satisfies condition (5.1), then it is a subset of the set $\mathbb{R}'_{\mathcal{A}}$.

Definition 5.4. The set of Rationalizable States is the maximal subset $X \subseteq \mathcal{A}$ that satisfies:

$$\tilde{R}(X) \equiv X$$

and we note it $\mathbb{R}_{\mathcal{A}}$.

As we have already said, from the definition of the set of Rationalizable States, we have $\mathbb{R}_{\mathcal{A}} \subseteq \mathbb{R}'_{\mathcal{A}}$. The following Theorem, analogous of Theorem 4.7, shows that this two sets are actually the same.

Theorem 5.5. *The set of Rationalizable States of a game \mathbf{u} can be calculated as*

$$\mathbb{R}_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}}$$

Proof.

The proof follows the proof of Theorem 4.7. There are some aspects in which attention must be put.

We will show that:

$$\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \equiv \mathbb{R}'_{\mathcal{A}}$$

Theorem 5.3 assures that $\tilde{R}(\mathbb{R}'_{\mathcal{A}})$ is correctly defined.

Suppose that $a \in \tilde{R}(\mathbb{R}'_{\mathcal{A}})$. By the definition of \tilde{R} , there exists a measurable selection $\mathbf{s} : I \rightarrow S$ of $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(\mathbb{R}'_{\mathcal{A}}))$, such that $a = \int_I \mathbf{s}$. Since $\mathbb{R}'_{\mathcal{A}}$ is a Borel set and $\mathbb{R}'_{\mathcal{A}} \subseteq \tilde{R}^t(\mathcal{A})$, which are as well Borel sets $\forall t \geq 0$, we have that $\mathcal{P}(\mathbb{R}'_{\mathcal{A}}) \subseteq \mathcal{P}(\tilde{R}^t(\mathcal{A}))$ are well defined and, $\forall t \geq 0, \forall i \in I, \mathbb{B}(i, \mathcal{P}(\mathbb{R}'_{\mathcal{A}})) \subseteq \mathbb{B}(i, \mathcal{P}(\tilde{R}^t(\mathcal{A})))$. So \mathbf{s} is a selection of $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(\tilde{R}^t(\mathcal{A})))$ and then $a \in \tilde{R}^{t+1}(\mathcal{A}) \forall t \geq 0$, which means that $a \in \mathbb{R}'_{\mathcal{A}}$. This proves that $\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \subseteq \mathbb{R}'_{\mathcal{A}}$.

For the other inclusion, we consider again a sequence of set valued mappings $F^t : I \rightrightarrows S$, $t \geq 0$, whose p-lim sup limit will be again very helpful. Consider then $\forall i \in I$,

$$\begin{aligned} F^0(i) &:= S \\ F^t(i) &:= \mathbb{B}\left(i, \mathcal{P}(\tilde{R}^{t-1}(\mathcal{A}))\right) \quad t \geq 1 \end{aligned}$$

We have now $\forall t \geq 0$,

$$\tilde{R}^t(\mathcal{A}) \equiv \int_I F^t(i) \text{ di.}$$

We know that $\mathcal{A} \equiv \int_I F^0$ is non empty and compact. From Proposition 5.2 we get that so are the sets $\tilde{R}^t(\mathcal{A})$ for all $t \geq 1$.

From Lemma 5.1, we get that $\forall i \in I$ the mapping $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ is u.s.c. and, as a consequence, the set $\mathbb{B}(i, \mathcal{P}(X))$ is compact for any compact subset $X \subseteq \mathcal{A}$. So the correspondences F^t are compact valued. From the proof of Theorem 4.7 we get that they all have a measurable

selection, since for a closed set X , $B(i, X) \subseteq \mathbb{B}(i, \mathcal{P}(X))$ and from Lemma 4.2 the mappings $i \mapsto B(i, X)$ are measurable and hence have a measurable selection.

Consider then the set valued mapping $F : I \rightrightarrows S$ defined as the point-wise lim sup of the sequence F^t :

$$F(i) := (\text{p-lim sup}_t F^t)(i) \equiv \limsup_t F^t(i).$$

So now let us take a point $a \in \mathbb{R}'_{\mathcal{A}}$. That is, $a \in \int_I F^t$ for all $t \geq 0$. This gives a sequence of measurable selections $\{\mathbf{s}^t\}_{t \in \mathbb{N}}$, such that $a = \int_I \mathbf{s}^t$. From the Lemma proved in Aumann (1976) we get that $a \in \int_I F$, since for each $i \in I$ the cluster points of $\{s^t(i)\}_{t \in \mathbb{N}}$ belong to $F(i)$ and a is the trivial limit of the constant sequence $\int \mathbf{s}^t$.

To show that $F(i) \subseteq \mathbb{B}(i, \mathcal{P}(\mathbb{R}'_{\mathcal{A}}))$, we use that the weak* topology in $\mathcal{P}(\mathcal{A})$ is metrizable and the upper semi continuity of $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ to give an argument that follows the one at the end of the proof of Theorem 4.7. ■

We get directly that,

Proposition 5.6. *In a game \mathbf{u} we have:*

$$\mathbb{P}_{\mathcal{A}} \subseteq \mathbb{R}_{\mathcal{A}}$$

In Proposition 5.6 it is not possible to obtain the equality in a general context. We give a sufficient condition in our setting to have it.

Proposition 5.7. *If in a game \mathbf{u} , we have $\forall \mu \in \mathcal{P}(\mathcal{A})$:*

$$\mathbb{E}_{\mu} [u(i, y, a)] \equiv u(i, y, \mathbb{E}_{\mu} [a])$$

then

$$\mathbb{P}_{\mathcal{A}} \equiv \mathbb{R}_{\mathcal{A}}$$

Proposition 5.7 says that if the utility functions are linear in the state variable, then we have that the Point-Rationalizable States set is equal to the set of Rationalizable States. Below we will see that we do have a general setting where the set of Rationalizable Strategies is well defined and in which we can get a result of equivalence between Point and standard Rationalizability in the strategy sets, improving the statement of Proposition 5.6. We address this issue in Subsection 5.3.

Proof.

If $\mathbb{E}_{\mu} [u(i, y, a)] \equiv u(i, y, \mathbb{E}_{\mu} [a])$ then

$$\mathbb{B}(i, \mu) \equiv B(i, \mathbb{E}_{\mu} [a]),$$

which implies that

$$\mathbb{B}(i, \mathcal{P}(X)) \equiv \bigcup_{\mu \in \mathcal{P}(X)} B(i, \mathbb{E}_{\mu} [a]).$$

For a convex set $X \subseteq \mathcal{A}$ we have $\mathbb{E}_{\mu} [a] \in X$, $\forall \mu \in \mathcal{P}(X)$.

This implies that under the hypothesis of the Proposition if X is convex,

$$\bigcup_{\mu \in \mathcal{P}(X)} B(i, \mathbb{E}_{\mu} [a]) \subseteq \bigcup_{a \in X} B(i, a)$$

and consequently

$$B(i, X) \subseteq \mathbb{B}(i, \mathcal{P}(X)) \equiv \bigcup_{\mu \in \mathcal{P}(X)} B(i, \mathbb{E}_{\mu} [a]) \subseteq \bigcup_{a \in X} B(i, a) \equiv B(i, X).$$

This is, $\mathbb{B}(i, \mathcal{P}(X)) \equiv B(i, X)$ and so $\tilde{R}(X) \equiv \tilde{P}r(X)$.

Noting that \mathcal{A} is convex, we get $\tilde{R}(\mathcal{A}) \equiv \tilde{P}r(\mathcal{A})$ which are as well convex. By induction we get that $\tilde{P}r^t(\mathcal{A}) \equiv \tilde{R}^t(\mathcal{A}) \forall t$ which gives (using the previous notation)

$$\mathbb{R}'_{\mathcal{A}} \equiv \bigcap_{t=0}^{\infty} \tilde{R}^t(\mathcal{A}) \equiv \bigcap_{t=0}^{\infty} \tilde{P}r^t(\mathcal{A}) \equiv \mathbb{P}'_{\mathcal{A}} \quad (5.2)$$

where this intersections give closed convex sets.

Finally we get,

$$\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \equiv \tilde{R}(\mathbb{P}'_{\mathcal{A}}) \equiv \tilde{P}r(\mathbb{P}'_{\mathcal{A}}) \equiv \mathbb{P}'_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}}$$

which implies that $\mathbb{R}'_{\mathcal{A}} \equiv \mathbb{R}_{\mathcal{A}}$. The first inequality comes from (5.2), the second one is true because $\mathbb{P}'_{\mathcal{A}}$ is convex, the third one comes from Theorem 4.7 which states that $\mathbb{P}'_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}}$ and the last one comes again from (5.2).

We conclude that $\mathbb{R}_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}} \equiv \mathbb{P}'_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}}$. ■

5.2 Forecasts over the set of strategies

When we consider forecasts over the sets of strategies, players should have a prior over each of the other player's individual actions. A forecasts in this case then would be a profile of probability measures with support in the set of strategies S . The question is not trivial since what we would have in this case is a continuum of random variables indexed by the set of players and it is not clear how payoff should depend on this profile of probability measures. However, in the setting of the original paper by Schmeidler (1973) this technical issue can be overtaken since in this setting payoffs depend on the profiles of probability measures that represent mixed strategies, which are the same mathematical objects as forecasts. We will give first a loose description of how the educative process should work when agents use forecasts over the set of strategies, to continue with a formal description for the case where S is a finite pure strategy set.

Given a set of strategy profiles $H \subseteq S^I$, consider the set of strategies that a player $i \in I$ may use in strategy profiles in H and denote it $H(i)$:

$$H(i) := \{ y \in S : y = s(i), \text{ and } \mathbf{s} \in H \}.$$

Ideally we would like to have forecasts on this set, and use the set $\mathcal{P}(H(i))$. Since we do not know wether $H(i)$ is a Borel set, we may use for instance the closure of $H(i)$, $\text{cl}\{H(i)\}$ and consider then $\mathcal{P}(\text{cl}\{H(i)\})$.

We will say that a (measurable) mapping $\mathbf{m} : I \rightarrow \mathcal{P}(S)$ is a *forecast profile over H* if $m(i)$ has support on $\text{cl}\{H(i)\}$ λ -a.e.. The question is, how should rational forecasts be generated?

If originally any strategy can be used, when thinking about possible actions taken by their rivals players should consider any possible forecast profile in the set of actions. This should generate for each player a set of best-replies-to-forecasts. The issue is still on how players use their forecasts to generate this set, but suppose we have this. Once all the players have done this exercise, we will have a correspondence that maps players to the set of strategies that represents all the possible strategy profiles that could be played reacting optimally to some forecast (where different players could be using different forecasts, recall that this is an issue of forecasting the forecasts of the others). This correspondence would be the result of a first iteration and should be the point of depart for the second iteration. Forecasts now should be profiles of probability measures where the support of each probability should be in the closure of the set associated to each player by this correspondence.

We define the mapping of best-reply-to-forecasts as the set of strategies that maximize some “expected” utility given a forecast profile on S , $\text{Br}(i, \cdot) : \mathcal{P}(S)^I \rightrightarrows S$:

$$\text{Br}(i, \mathbf{m}) := \operatorname{argmax}_{y \in S} \mathbb{E}_{\mathbb{P}(\mathbf{m})} [\pi(i, y, \mathbf{s})].$$

We use the notation $\mathbb{E}_{\mathbb{P}(\mathbf{m})}$ to represent that the payoff that is being maximized is an expected payoff where the expectation comes from the fact that players are using non-degenerate forecasts over the set of strategies. We note $\mathbb{P}(\mathbf{m})$ to indicate that the profile \mathbf{m} induces in some sense a probability measure over the set of strategy profiles. We do not give an answer here on to how this is done.

Given a measurable set valued mapping $F : I \rightrightarrows S$, we can obtain, for each agent $i \in I$, the set of best-replies to forecasts over this mapping as:

$$\text{Br}(i, F) := \left\{ y \in S : \begin{array}{l} y \in \text{Br}(i, \mathbf{m}), \mathbf{m} \text{ is a forecast} \\ \text{profile over } F \end{array} \right\}. \quad (5.3)$$

Finally, we can define the process of elimination of non best-reply-to-forecasts, described in the previous paragraphs, with the mapping R that takes a set valued mapping $F : I \rightrightarrows S$ and returns a subset $R(F) \subseteq S^I$,

$$R(F) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection} \\ \text{of } i \rightrightarrows \text{Br}(i, F) \end{array} \right\}. \quad (5.4)$$

The process²³ described by equation (5.4) considers that strategy profiles that are “kept” are those that can be constructed from best replies of agents taking decisions considering any of the possible forecast profiles over F . Of course, as was the case before, on a same strategy profile \mathbf{s} of $R(F)$ the strategies of two different agents can be best-responses to two different forecast profiles over F .

Definition 5.8. The set of Rationalizable Strategy Profiles is the maximal subset $H \subseteq S^I$ that satisfies:

$$R(H) \equiv H$$

and we note it \mathbb{R}_S . For each player, $i \in I$, there will be a set of Rationalizable Strategies, namely the union, over all the rationalizable strategy profiles in \mathbb{R}_S , of the best response set of the considered player. That is, the set of Rationalizable Strategies for player $i \in I$ is,

$$\mathbb{R}_S^i := \text{Br}(i, \mathbb{R}_S)$$

So now that we have presented the main ideas of rationalizability in terms of strategies, let’s turn to a setting where the best-reply-to-forecast mapping has a concrete sense.

5.3 Games with a continuum of players and finite strategy set

In Schmeidler’s formulation of a game with a continuum of players, payoff functions $\pi(i, \cdot, \cdot)$ are defined on the product set $\Delta \times \Delta^I$, where $\{e^1, \dots, e^L\} \subseteq \mathbb{R}^L$ is a finite set of pure strategies that we identify with the canonical base of \mathbb{R}^L and $\Delta \equiv \operatorname{co} \{ \{e^1, \dots, e^L\} \}$ is the set of mixed strategies, the convex hull of $\{e^1, \dots, e^L\}$ and the simplex in \mathbb{R}^L . The functions $\pi(i, \cdot, \cdot) : \Delta \times \Delta^I \rightarrow \mathbb{R}$ take in this setting the form:

$$\pi(i, y, \mathbf{m}) := y \cdot h(i)(\mathbf{m}) \quad (5.5)$$

where $h(i) : \Delta^I \rightarrow \mathbb{R}^L$ is an auxiliary vector utility function whose coordinate l gives the utility of player i when he chooses action e^l , \mathbf{m} is a (mixed) strategy profile and $y \in \Delta$ is a (mixed) strategy

²³The set $R(F)$ can itself be regarded as a set valued mapping from I to S . This has to be taken in account to have the set of Rationalizable Strategies well defined.

of player i . If payoffs of players depend on the integral of the (mixed) strategy profile \mathbf{m} , $\int_I \mathbf{m}$, then we can say that Schmeidler's setting is ours with $S \equiv \mathcal{A} \equiv \Delta$. In this case the functions $h(i)$ can be regarded as depending only on the values of the integrals (as in our setting)²⁴. As quoted by Schmeidler himself, the central result of his paper is the existence and purification theorem, Theorem 5.9 below, where the main assumption is precisely this last one. We state this theorem in the context of our framework.

Theorem 5.9 (Schmeidler 1973). *If the following conditions are satisfied:*

1. *The functions $h(i)$ depend only on the integral of the mixed strategy profile \mathbf{m} ,*
2. *The functions $\hat{h}(i) : \Delta \rightarrow \mathbb{R}^L$ such that $h(i)(\mathbf{m}) \equiv \hat{h}(i)(\int \mathbf{m})$, are continuous,*
3. *For all $\mathbf{m} \in \Delta^I$ and all $l, k \in \{1, \dots, L\}$ the set*

$$\left\{ i \in I : h(i)^l(\mathbf{m}) > h(i)^k(\mathbf{m}) \right\}$$

is measurable,

then there exists a Pure Strategy Nash Equilibrium of the game π .

This theorem motivates that we look at Schmeidler's formulation from a slightly different point of view. As we have already said, one possibility is to consider this setting in the context of a game as in Definition 3.1 where the set of strategies S , is $S \equiv \Delta$. The implicit properties imposed on payoff functions in this definition imply the hypothesis of Theorem 5.9 and so we know that the results stated so far are true for the set of mixed strategies. However, if we keep focusing on the set of pure strategies, we can benefit from the structure of Schmeidler's formulation to have a well defined best-reply-to-forecast mapping. The payoff functions depend on the integral of the mixed strategy profile, in particular when we consider only pure strategies they also depend on the integral of pure strategy profiles. We can make then a difference between (Point-)Rationalizability in pure or mixed strategies. At this point we have to introduce some more notation: we will represent by a subscript on the corresponding set or operator whether we are considering pure or mixed strategies²⁵. In Schmeidler's formulation we have $S_p \equiv \{e^1, \dots, e^L\}$ and $S_m \equiv \Delta$. We continue to consider the game where $S \equiv \mathcal{A} \equiv \Delta$ and so \mathcal{A} is the same as the mixed strategy set Δ which in turn is equal to the set of probability measures over the set of pure strategies $\mathcal{P}(S_p)$, and so we get:

$$S \equiv S_m \equiv \mathcal{A} \equiv \Delta \equiv \mathcal{P}(\{e^1, \dots, e^L\}) \equiv \mathcal{P}(S_p).$$

If S_p is a finite pure strategy set, then any forecast in the form of a probability distribution over a subset $Y \subseteq S_p$ can be considered as a point in $\mathcal{P}(Y) \equiv \text{co}\{Y\} \subseteq \Delta \equiv \text{co}\{S_p\}$. So a forecast profile would be a function $\mathbf{m} : I \rightarrow \text{co}\{Y\}$. Since mixed strategies are the same mathematical objects as probability forecasts over the set of actions of each player, in the current setting we get to identify the expected utility $\mathbb{E}_{\mathbb{P}(\mathbf{m})}[\pi(i, y, \mathbf{s})]$ mentioned above, with the following expression:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}(\mathbf{m})}[\pi(i, y, \mathbf{s})] &\equiv y \cdot h(i)(\mathbf{m}) \\ &\equiv u\left(i, y, \int \mathbf{m}\right) \end{aligned} \tag{5.6}$$

²⁴See Rath (1992) for a discussion on this matter.

²⁵For instance, the Pointwise eductive procedure Pr can be defined on pure strategies:

$$Pr_p(H) \equiv \left\{ \mathbf{s} \in S_p^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the cor-} \\ \text{respondence } i \Rightarrow \text{Br}_p(i, H) \end{array} \right\}$$

where $H \subseteq S_p^I$ and $\text{Br}_p(i, \cdot) : S_p^I \Rightarrow S_p$ is defined by:

$$\text{Br}_p(i, \mathbf{s}) \equiv \text{argmax} \{ \pi(i, y, \mathbf{s}) : y \in S_p \}$$

Where now $\mathbb{E}_{\mathbb{P}(\cdot)}[\pi(i, \cdot, \mathbf{s})] : S_p \times \Delta^I \rightarrow \mathbb{R}$ is a function that depends on the *pure* strategy $y \in S_p$ and we interpret the profile of probability distributions $\mathbf{m} \in \Delta^I \equiv \mathcal{P}(S)^I$ as a forecast, under the hypothesis that $h(i)$ depends on \mathbf{m} through the integral.

In this setting then we get that the set of forecast profiles on a mapping $F : I \rightrightarrows S_p$ can be described by the mapping $\text{co}\{F\} : I \rightrightarrows \Delta$ defined as $\text{co}\{F\}(i) := \text{co}\{F(i)\}$. With this, we can now consider six different rationalizable sets:

1. The set of Point-Rationalizable Pure Strategies \mathbb{P}_{S_p}
2. The set of Point-Rationalizable Mixed Strategies \mathbb{P}_{S_m}
3. The set of Rationalizable Pure Strategies \mathbb{R}_{S_p}
4. The set of Rationalizable Mixed Strategies \mathbb{R}_{S_m}
5. The set of Point-Rationalizable States $\mathbb{P}_{\mathcal{A}}$
6. The set of Rationalizable States $\mathbb{R}_{\mathcal{A}}$

Where the first four are obtained from the educative processes defined by optimal strategies and forecasts on the corresponding sets.

Theorem 5.10. *In a game where $S \equiv \Delta$ we have,*

$$\forall t, \quad Pr_p^t(S_p^I) \equiv R_p^t(S_p^I).$$

Proof.

From the relation in (5.6) we can see that to look at all forecast profiles is equivalent to look at all the integrals of such profiles. Moreover, we see that in equation (5.3), and considering F to be the constant mapping $F(i) \equiv S_p$, the set of forecast profiles over F is the set valued mapping $\text{co}\{F\}$ as defined above. That is, to obtain $\text{Br}_p(i, S_p^I)$ we are interested in the integral of F while the calculus of $\mathbb{B}r_p(i, F)$ considers the integral of $\text{co}\{F\}$. From Aumann (1965) the integral of the convex hull mapping is equal to the integral of the mapping itself and so we have:

$$\int_I F(i) \, di \equiv \int_I \text{co}\{F\}(i) \, di.$$

So no matter whether we are considering point or standard forecasts we obtain the same set of states. In consequence we get that the set of maximizers is the same:

$$\text{Br}_p(i, S_p^I) \equiv \mathbb{B}r_p(i, F)$$

Thus we take measurable selections from the same mapping and so,

$$Pr_p(S_p^I) \equiv R_p(S_p^I).$$

By induction over t we get that

$$Pr_p^t(S_p^I) \equiv R_p^t(S_p^I).$$

■

Corollary 5.11. *In a game where $S \equiv \Delta$ we have,*

$$\mathbb{R}_{S_p} \equiv \mathbb{P}_{S_p}$$

Corollary 5.11 says that in the setting where payoff functions depend on the integral of the mixed strategy profile, and we consider S to be the set of mixed strategies associated to a finite pure strategy set, then we get that Point-Rationalizability is equivalent to Rationalizability in terms of pure strategy profiles.

In this context we identify the set \mathbb{P}_S with \mathbb{P}_{S_m} and so from Theorem 4.6 we know that $\mathbb{P}_A \equiv \bar{A}(\mathbb{P}_{S_m})$. What can we say about the relation between \mathbb{P}_A and the (Point-)Rationalizable sets in pure strategies? An answer is given in Corollary 5.12.

Corollary 5.12. *In a game where $S \equiv \Delta$ we have,*

$$\begin{aligned} \mathbb{P}_A &\equiv \bar{A}(\mathbb{P}_{S_m}) \quad \text{and} \quad \mathbb{P}_{S_m} \equiv \bar{B}(\mathbb{P}_A) \\ &\equiv \left\{ \mathbf{m} \in S_m^I : \begin{array}{l} \mathbf{m} \text{ is a measurable selection of} \\ i \rightrightarrows B_m(i, \mathbb{P}_A) \end{array} \right\}; \\ \mathbb{P}_A &\equiv \bar{A}(\mathbb{P}_{S_p}) \quad \text{and} \quad \mathbb{P}_{S_p} \equiv \left\{ \mathbf{s} \in S_p^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of} \\ i \rightrightarrows B_p(i, \mathbb{P}_A) \end{array} \right\}. \end{aligned}$$

Proof.

Noting that the best response mappings $B_p(i, \cdot) : \mathcal{A} \rightrightarrows S_p$ and $B_m(i, \cdot) : \mathcal{A} \rightrightarrows S_m$ satisfy for $X \subseteq \mathcal{A}$:

$$\cup_{a \in X} B_p(i, a) \subseteq \cup_{a \in X} B_m(i, a) \equiv \cup_{a \in X} \text{co} \{B_p(i, a)\} \subseteq \text{co} \{\cup_{a \in X} B_p(i, a)\},$$

we get

$$\tilde{P}r_p(X) \equiv \int_I B_p(i, X) \, \text{di} \subseteq \int_I B_m(i, X) \, \text{di} \subseteq \int_I \text{co} \{B_p(i, X)\} \, \text{di} \equiv \tilde{P}r_p(X),$$

where the integral in the middle is $\tilde{P}r_m(X)$. The Corollary is then consequence of Theorem 4.6 and Corollary 5.11. ■

This is, in a game as in Definition 3.1 with $S \equiv \Delta \equiv S_m$ we have that the set of Rationalizable Pure Strategies is equal to the set of Point-Rationalizable Pure Strategies, and these sets are paired with the set of Point-Rationalizable States which in turn is paired with the set of Point-Rationalizable Mixed Strategies.

Finally let's note that the hypothesis of Theorem 5.9 are implied by Definition 3.1 if we consider the set S to be the set of mixed strategies of a finite strategy set game. Moreover, since we want to deal with rationalizability in terms of pure strategies, it is not enough to use Definition 3.1 with S being the finite set of pure strategies, since in that case we would not be asking that the utility functions depended on the mixed strategy profiles through their integral which is crucial for our results.

6 More of the Example

Let us now illustrate our results and definitions with the example that motivates this presentation, introduced in Section 2. In this example the strategy set was \mathbb{R}_+ . Without loss of generality we can assume it to be a compact interval $S \equiv [0, q_{\max}]$, where q_{\max} could be the quantity that makes the price equal to 0 : $q_{\max} := \inf \{P^{-1}(0)\}$. Now we identify the state set. As we have already said, we could choose the state set to be the set of production quantities or prices. This depends on the aggregation operator that we are considering. In our case, we will consider A to be the operator that gives prices. This is, $A : S^I \rightarrow [0, p_{\max}]$

$$A(\mathbf{s}) \equiv P \left(\int_I s(i) \, \text{di} \right).$$

We are using then a variation, as in footnote 7, of the aggregation operator that we used to obtain our results. This implies that in this example we will identify \mathcal{A} with the set:

$$\begin{aligned}\mathcal{A} &\equiv \{p \in \mathbb{R} : \exists q \in [0, q_{\max}], p = P(q)\} \\ &\equiv P([0, q_{\max}]) \\ &\equiv [0, p_{\max}].\end{aligned}$$

This set is convex (a useful property, as we have seen, of the state set), since P is a continuous function that goes from one-dimensional aggregate-production set \mathbb{R}_+ to the set of prices $[0, p_{\max}] \subset \mathbb{R}$. The utility function $u(i, \cdot, \cdot) : [0, q_{\max}] \times [0, p_{\max}] \rightarrow \mathbb{R}$ is then:

$$u(i, s(i), p) \equiv ps(i) - c_i(s(i)).$$

Although we are using a variation of the aggregation operator, since P is continuous, Theorem 4.7 holds and so we can compute the Point-Rationalizable State set using the eductive procedure described by the right hand side of (4.6). Since the strategy and states set are unidimensional²⁶ we get the result of Corollary 4.8 and we know then that this set is a compact interval. Furthermore, in this setting we have that the utility function is linear in the state variable and so Proposition 5.7 holds and we then see that what we are actually calculating is in fact the set of Rationalizable States.

7 Comments and Conclusions

In this work we have formally defined the concept of Point-Rationalizable Strategies for models that use a continuum of agents. The concept was adapted to games with a continuum of players with compact strategy sets, from the original definitions for games with finite set of players. When such models have the particularity that payoffs depend on other players' actions through an aggregate variable that cannot be unilaterally affected, we have defined as well the set of Point-Rationalizable States. This last setting is an important generalization of several models that explore Rational Expectations in economics such as models of currency attacks, stag hunts, standard markets, macroeconomic dynamics and global games.

The set of Point-Rationalizable Strategies is paired with the set of Point-Rationalizable States. We have shown that the set of Point-rationalizable States can be obtained, as in the case of finite player games with (Point) Rationalizable Strategies, by eliminating unreasonable states. Moreover, this set is non-empty, convex and compact.

We have seen as well that for the most important application of Rationalizability in economic models, namely Strong Rationality (or Eductive Stability), it is equivalent in terms of properties and more desirable in terms of tractability to use the state approach rather than the strategy profile approach.

To incorporate standard Rationalizability to our framework, we have formally defined Rationalizable States. We give a sufficient (but not at all general) condition on payoff functions, in order to have equivalence between standard and point Rationalizability.

In the particular case where the strategy sets are finite and payoff functions depend on the integral of the mixed strategy profile, we were able to formally define Rationalizable strategies and we have extended an equivalence result to Rationalizability vs. Point-Rationalizability in terms of strategy profiles, which in turn implies that in this setting the three concepts: Rationalizable Strategies, Point-Rationalizable Strategies and Point-Rationalizable States; give the same outcomes.

We have defined a key concept in a unified exploratory framework that encompasses all the previously mentioned models. With this, we have a general framework on which we can study general properties of equilibria such as (local) eductive stability of equilibria and applications to

²⁶Otherwise we would have it if we had used only the integral and so aggregate production instead of prices.

models with continuum of agents that feature strategic complementarities or substitutes (Cooper (1999), Chamley (2004), Guesnerie (2005), Guesnerie and Jara-Moroni (2007)).

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Proof of Lemma 4.2.

We show first that the mapping $G : I \rightrightarrows \mathcal{A} \times S$, that associates with each agent $i \in I$ the graph of the best response mapping $B(i, \cdot)$, $G(i) := \text{gph } B(i, \cdot)$, is measurable.

Take a closed set $C \subseteq \mathcal{A} \times S$. We need to prove that the set

$$G^{-1}(C) \equiv \{i \in I : C \cap \text{gph } B(i, \cdot) \neq \emptyset\}$$

is measurable. Consider the subset $U \subseteq \mathcal{U}_{S \times \mathcal{A}}$ defined by:

$$U := \{g \in \mathcal{U}_{S \times \mathcal{A}} : \exists (a, s) \in C \text{ such that } g(s, a) \geq g(y, a) \forall y \in S\}$$

note that $\mathbf{u}^{-1}(U) \equiv G^{-1}(C)$ and so, from the measurability assumption over \mathbf{u} , it suffices to prove that U is closed. That is, we have to show that for any sequence $\{g^\nu\}_{\nu \in \mathbb{N}} \subset U$, such that $g^\nu \rightarrow g^*$ uniformly $g^* \in U$.

Since the functions g^ν are finite and continuous in $S \times \mathcal{A}$, from Weierstrass’ Theorem g^* is continuous and so it belongs to $\mathcal{U}_{S \times \mathcal{A}}$. Moreover, g^ν converges continuously to g^* , that is, for any convergent sequence (a^ν, s^ν) with limit (a^*, s^*) , the sequence $g^\nu(s^\nu, a^\nu)$ converges to $g^*(s^*, a^*)$. Indeed, consider any $\varepsilon > 0$. By the continuity of g^* there exists $\bar{\nu}_1 \in \mathbb{N}$ such that $\forall \nu > \bar{\nu}_1$,

$$2|g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \frac{\varepsilon}{2}.$$

From the uniform convergence of g^ν we get that there exists $\bar{\nu}_2 \in \mathbb{N}$ such that,

$$|g^\nu(s, a) - g^*(s, a)| < \frac{\varepsilon}{2} \quad \text{for all } \nu \geq \bar{\nu}_2 \text{ and } \forall (s, a) \in S \times \mathcal{A},$$

in particular this is true for all the elements of the sequence of points. We get then that $\forall \nu \geq \max\{\bar{\nu}_1, \bar{\nu}_2\}$,

$$\begin{aligned} |g^\nu(s^\nu, a^\nu) - g^*(s^*, a^*)| &\leq |g^\nu(s^\nu, a^\nu) - g^*(s^\nu, a^\nu)| \\ &\quad + |g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \varepsilon. \end{aligned}$$

We have to show then that there exists a point $(a, s) \in C$ such that $g^*(s, a) \geq g^*(y, a) \forall y \in S$. Since $g^\nu \in U$, we have for each $\nu \in \mathbb{N}$, points $(a^\nu, s^\nu) \in C$ such that $g^\nu(s^\nu, a^\nu) \geq g^\nu(y, a^\nu) \forall y \in S$. Let $(a^*, s^*) \in C$ be the limit of a convergent subsequence of $\{(a^\nu, s^\nu)\}_{\nu \in \mathbb{N}}$, which without loss of generality we can take to be the same sequence. We see that (a^*, s^*) is the point we are looking for since for a fixed $y \in S$, continuous convergence implies that in the limit

$$g^*(s^*, a^*) \geq g^*(y, a^*).$$

We conclude then that $g^* \in U$. Thus, U is closed and since \mathbf{u} is a measurable mapping, $\mathbf{u}^{-1}(U)$ is measurable.

With this in mind, consider a closed set $X \subseteq \mathcal{A}$ and the mapping $i \rightrightarrows B(i, X)$. Applying Theorem 14.13 in Rockafellar and Wets (1998) to the constant mapping $i \rightrightarrows X$ along with G above, we get that the correspondence $i \rightrightarrows B(i, X)$ is measurable and has closed values (hence compact since S is compact). ■

Proof of Lemma 5.1.

We write U as the composition of two functions:

$$\begin{aligned} (y, \mu) \in Y \times \mathcal{P}(X) &\rightarrow (u(y, \cdot), \mu) \in C_b(X) \times \mathcal{P}(X) \\ \text{and } (f, \mu) \in C_b(X) \times \mathcal{P}(X) &\rightarrow \int_X f(x) d\mu(x) \end{aligned}$$

If we endow $C_b(X)$ with the sup norm topology and $\mathcal{P}(X)$ with the weak* topology, from Corollary 15.7 in Aliprantis and Border (1999) we get that $(f, \mu) \rightarrow \int f d\mu$ is continuous on $C_b(X) \times \mathcal{P}(X)$.

Therefore, the result will follow from the continuity of the function

$$(y, \mu) \rightarrow \mathcal{F}(y, \mu) = (u(y, \cdot), \mu).$$

Note first that this function is defined component to component by functions that depend each only on one variable, this is $\mathcal{F}(y, \mu) = (\mathcal{F}_1(y), \mathcal{F}_2(\mu))$, and second that \mathcal{F}_2 is the identity. Thus, we only need to prove that $\mathcal{F}_1 : Y \rightarrow C_b(X)$ is continuous for the sup norm topology in $C_b(X)$.

Let $y^\nu \rightarrow y$ and take $\varepsilon > 0$.

Since $Y \times X$ is compact and u is in $C_b(Y \times X)$, this function is as well uniformly continuous. Thus, $\exists \delta > 0$ (that depends only on ε) such that

$$\|(y, x) - (y', x')\| < \delta \implies |u(y, x) - u(y', x')| < \frac{\varepsilon}{3}$$

Since X is compact $\exists \{x_1, \dots, x_N\} \subset X$ such that $X \subseteq \cup_{i=1}^N B(x_i, \delta)$. This is, for any $x \in X$ there exists x_i in the previous set such that $x \in B(x_i, \delta)$.

Finally, since $y^\nu \rightarrow y$, there exist for each x_i numbers $\bar{\nu}_i$ such that

$$\|(y^\nu, x_i) - (y, x_i)\| < \delta$$

for all $\nu \geq \bar{\nu}_i$.

All together gives, for $\nu \geq \max \{\bar{\nu}_i : i \in \{1, \dots, N\}\}$ and $x \in X$:

$$\begin{aligned} |u(y^\nu, x) - u(y, x)| &< |u(y^\nu, x) - u(y^\nu, x_i)| + |u(y^\nu, x_i) - u(y, x_i)| \\ &\quad + |u(y, x_i) - u(y, x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

We conclude that $u(y^\nu, \cdot)$ converges to $u(y, \cdot)$ for the sup norm topology, which completes the proof. ■