EQUILIBRIA IN ECONOMIES WITH A MEASURE SPACE OF AGENTS AND NON-ORDERED PREFERENCES

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Abstract. A new approach is proposed to prove the existence of a Walrasian equilibrium for production economies with a measure space of agents and finitely many commodities. The new approach, based on the discretization of measurable correspondences, allows us to provide an existence result for economies with non-ordered but convex preferences as well as for economies with partially ordered (possibly incomplete) but non-convex preferences. This paper generalizes results of Aumann [3], Schmeidler [24] and Hildenbrand [15].

Keywords. Measure space of agents, non-ordered but convex preferences, partially ordered but non-convex preferences, discretization of measurable correspondences.

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1. Introduction

Aumann [3] and Hildenbrand [15] provide existence results of Walrasian equilibria for exchange and production economies with a measure space of agents and ordered preferences. In the framework of strictly monotone preferences, Main Theorem in Schmeidler [24] dispense with completeness of preferences. In the recent years attempts (e.g. in [19]) were made to generalize these results to economies with externalities in consumption. In Balder [6], it is shown that the usual conditions used for these attempts force the *preferred to* correspondence to be empty-valued almost everywhere on the non-atomic part of the measure space of agents, rendering these attempts pointless.

Following a *discretization* approach, we provide in this paper an existence result for both non-ordered (but without externalities) and partially ordered (possibly incomplete) preferences. For economies with non-ordered preferences, we can not dispense with a convex assumption on preferences. Indeed, we provide a simple counterexample of a continuum economy with non-transitive preferences, satisfying all usual assumptions except the convex one, and for which no Walrasian equilibrium exists. For economies with partially ordered preferences, our result generalizes Main Theorem in Aumann [3], Main Theorem in Schmeidler [24] and Theorems 1 and 2 in Hildenbrand [15].

The *discretization* approach proposed in this paper, consists on considering an economy with a measure space of agents as the *limit* of a sequence of economies with a finite, but larger and larger, set of agents. We construct a sequence of partitions of the measure space depending on the measurable characteristics of the economy. To each partition we define a *subordinated simple* economy. Each *simple* economy will be identified as an economy with a finite set of agents, and applying a classical equilibria existence result for economies with finitely many agents, we get a sequence of equilibria which will converge to a quasi-equilibrium for the initial economy.

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The paper is organized as follows. In Section 2, we set the main definitions and notations. In Section 3 we define the model of production economies with a measure space of agents, we introduce the concepts of equilibria, we give the list of assumptions that economies will be required to satisfy and finally, we present an existence result (Theorem 3.1) for free-disposal economies and an existence result (Corollary 3.1) for economies with strictly monotone preferences. The Section 4 is devoted to the mathematical *discretization* of measurable correspondences. The proof of the main existence result (Theorem 3.1) is given in Section 5. The existence result for economies with finitely many agents is provided in Appendix A and Appendix B is devoted to mathematical auxiliary results about measurability and integration of correspondences.

2. Notations and definitions

Let $\mathbb{L}$ be a finite dimensional vector space induced with its natural topology. The dual of $\mathbb{L}$ is noted $\mathbb{L}^*$ and the natural dual pairing $\langle \mathbb{L}^*, \mathbb{L}\rangle$ is defined by $\langle p, x\rangle = p(x)$ for each $(p,x) \in \mathbb{L}^* \times \mathbb{L}$. Let $C \subset \mathbb{L}$ be a pointed convex cone[1]. The partial order induced[2] by C is noted $\geqslant$. We note $\mathbb{L}_+$ the positive cone $\{x \in \mathbb{L} \mid x \geqslant 0\}$. If $x \in \mathbb{L}$ then we note $x > 0$ ($x \gg 0$) if $x \geqslant 0$ and $x \neq 0$ (resp. x is an interior point of C). In the dual space $\mathbb{L}^*$ we let $\mathbb{L}^*{}_+ = \{p \in \mathbb{L}^* \mid \forall c \in C \quad p(c) \geqslant 0\}$ and we note $p \geqslant 0$ ($p > 0$) if $p \in \mathbb{L}^*{}_+$ (resp. $p \in \mathbb{L}^*{}_+$ and $p \neq 0$). A strictly positive functional, written $p \gg 0$ is a positive functional satisfying $p(x) > 0$ for all $0 < x \in \mathbb{L}$. If $X \subset \mathbb{L}$ is a subset, then the interior of X is noted $\operatorname{int} X$, the closure of X is noted $\operatorname{cl} X$. If $p \in \mathbb{L}^*$ then we let $p(X) = \{p(x) \mid x \in X\}$ and if $Y \subset \mathbb{L}$ then $p(X) \geqslant p(Y)$ means [if $(x,y) \in X \times Y$ then $p(x) \geqslant p(y)$]. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{L}$, the *sequential upper limit* of $(C_n)_{n\in\mathbb{N}}$, noted $\operatorname{Ls} C_n$, is defined by

$$\operatorname{Ls} C_n := \left\{ x \in \mathbb{L} \;\middle|\; x = \lim_{k\to\infty} x_k\,, \quad x_k \in C_{n(k)} \right\}.$$

The convex hull of X is noted $\operatorname{co} X$ and the closed convex hull of X is noted $\overline{\operatorname{co}}\, X$. We let $A(X) = \{v \in \mathbb{L} \mid X + \{v\} \subset X\}$ be the asymptotic cone of X. Note that if X is closed convex, then $A(X)$ is the set of vectors $v \in \mathbb{L}$ such that $v = \lim_{n\to\infty} \lambda_n u_n$ where $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence decreasing to 0 and $(u_n)_{n\in\mathbb{N}}$ is a sequence in X.

We consider $(A, \mathcal{A}, \mu)$ a finite measure space, that is, A is a set, $\mathcal{A}$ is a σ-algebra of subsets of A and μ is a finite measure on $\mathcal{A}$. The measure space $(A, \mathcal{A}, \mu)$ is complete if $\mathcal{A}$ contains all μ-negligible[3] subsets of A.

Let (D, d) be a separable metric space. The σ-algebra of Borel subsets of D is noted $\mathcal{B}(D)$. A correspondence (or a multifunction) $F : A \twoheadrightarrow D$ is *measurable* if for all open set $G \subset D$, $F^-(G) = \{a \in A \mid F(a) \cap G \neq \emptyset\} \in \mathcal{A}$. The correspondence F is said to be *graph measurable* if $\{(a,x) \in A \times D \mid x \in F(a)\} \in \mathcal{A} \otimes \mathcal{B}(D)$. A function $f : A \to D$ is a *measurable selection* of F if f is measurable and if, for almost every $a \in A$, $f(a) \in F(a)$. The set of measurable selections of F is noted $S(F)$. When $D \subset \mathbb{L}$ the set of integrable selections of F is noted $S^1(F)$ and we note F_Σ the following (possibly empty) set $F_\Sigma := \int_A F(a) d\mu(a) := \left\{ v \in D \;\middle|\; \exists x \in S^1(F) \quad v = \int_A x(a) d\mu(a) \right\}$.

Let X be a space and $P \subset X \times X$ be a binary relation on X. The relation P is irreflexive if $(x,x) \notin P$, for all $x \in X$. The relation P is transitive if $[(x,y) \in P$ and $(y,z) \in P]$ implies $(x,z) \in P$, for all $(x,y,z) \in X^3$. The relation P is negatively transitive if $[(x,y) \notin P$ and $(y,z) \notin P]$ implies $(x,z) \notin P$, for all $(x,y,z) \in X^3$. The relation P is a partial order if it is irreflexive and transitive. The relation P is an order if it is irreflexive, transitive and negatively transitive. When P is an order, it is usually noted $\succ$ and $X^2 \setminus P$ is noted $\preceq$. Note that when P is an order, then $\preceq$

[1] That is C is a cone: $\alpha C \subset C$ for all $\alpha \geqslant 0$, C is convex: $C + C \subset C$ and C is pointed: $C \cap (-C) = \{0\}$.

[2] That is for all $(x,y) \in \mathbb{L}^2$, $x \geqslant y$ whenever $x - y \in C$.

[3] A set N is μ-negligible if there exists $E \in A$ such that $N \subset E$ and $\mu(E) = 0$.

is transitive, reflexive ($x \preceq x$ for all $x \in X$) and complete (for all $(x,y) \in X^2$ either $x \preceq y$ or $y \preceq x$).

3. The model, the equilibrium concepts and the assumptions

3.1. **The Model.** We consider a finite dimensional vector space $\mathbb{L}$, a complete measure space $(A, \mathcal{A}, \mu)$, a function e from A to $\mathbb{L}$, two correspondences X and Y from A into $\mathbb{L}$ and a correspondence of preferences P in X, that is, P is a correspondence from A into $\mathbb{L} \times \mathbb{L}$ such that for all $a \in A$, $P(a) \subset X(a) \times X(a)$ and $P(a)$ is irreflexive.

An economy $\mathcal{E}$ is a list

$$\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X, Y, P, e)).$$

The commodity space is represented by $\mathbb{L}$ and the natural dual pairing $\langle \mathbb{L}^*, \mathbb{L} \rangle$ is interpreted as the *price-commodity* pairing.

The set of agents (or consumers) is represented by A, the set $\mathcal{A}$ represents the set of admissible coalitions, and the number $\mu(E)$ represents the fraction of consumers which are in the coalition $E \in \mathcal{A}$.

For each agent $a \in A$, the consumption set is represented by $X(a) \subset \mathbb{L}$ and the preference relation is represented by $P(a)$. We define the correspondence[4] $P_a : X(a) \twoheadrightarrow X(a)$ by $P_a(x) = \{x' \in X(a) \mid (x, x') \in P(a)\}$. In particular, if $x \in X(a)$ is a consumption bundle, the set $P_a(x)$ is the set of consumption bundles strictly preferred to x by the agent a. The set of consumption allocations (or plans) of the economy is the set $\mathrm{S}^1(X)$ of integrable selections of X. The aggregate consumption set X_Σ is defined by

$$X_\Sigma := \int_A X(a) d\mu(a) := \left\{ v \in \mathbb{L} \;\middle|\; \exists x \in \mathrm{S}^1(X) \quad v = \int_A x(a) d\mu(a) \right\}.$$

The initial endowment of the consumer $a \in A$ is represented by the commodity bundle $e(a) \in \mathbb{L}$. We assume that the function $e : A \to \mathbb{L}$ is an integrable function and we note $\omega := \int_A e(a) d\mu(a)$ the aggregate initial endowment. The production possibilities available to the consumer $a \in A$ are represented by the set $Y(a) \subset \mathbb{L}$. The set of production allocations (or plans) of the economy is the set $\mathrm{S}^1(Y)$ of integrable selections of Y. The aggregate production set Y_Σ is defined by

$$Y_\Sigma := \int_A Y(a) d\mu(a) = \left\{ u \in \mathbb{L} \;\middle|\; \exists y \in \mathrm{S}^1(Y) \quad u = \int_A y(a) d\mu(a) \right\}.$$

3.2. **The Equilibrium Concepts.**

Definition 3.1. A *Walrasian equilibrium* of an economy $\mathcal{E}$ is an element (x^*, y^*, p^*) of $\mathrm{S}^1(X) \times \mathrm{S}^1(Y) \times \mathbb{L}^*$ such that $p^* \neq 0$ and satisfying the following properties.

- **(a)** For a.e. $a \in A$, $p^*(x^*(a)) = p^*(e(a)) + p^*(y^*(a))$ and if $x \in P_a(x^*(a))$ then $p^*(x) > p^*(x^*(a))$.
- **(b)** For a.e. $a \in A$, if $y \in Y(a)$ then $p^*(y) \leqslant p^*(y^*(a))$.
- **(c)** $\int_A x^* d\mu = \int_A e d\mu + \int_A y^* d\mu$.

An element $(x^*, y^*, p^*) \in \mathrm{S}^1(X) \times \mathrm{S}^1(Y) \times \mathbb{L}^*$ with $p^* \neq 0$, is a *Walrasian quasi-equilibrium* of an economy $\mathcal{E}$ if the conditions (b) and (c) together with

- **(a')** for a.e. $a \in A$, $p^*(x^*(a)) = p^*(e(a)) + p^*(y^*(a))$ and if $x \in P_a(x^*(a))$ then $p^*(x) \geqslant p^*(x^*(a))$

are satisfied.

A Walrasian equilibrium of an economy $\mathcal{E}$ is clearly a Walrasian quasi-equilibrium of $\mathcal{E}$. We provide in the following remark, classical conditions on $\mathcal{E}$ under which a Walrasian quasi-equilibrium is in fact a Walrasian equilibrium.

[4]Note that the binary relation $P(a)$ coincide with the graph of the correspondence P_a.

Remark 3.1. Every quasi-equilibrium (x^*, y^*, p^*) of a production economy $\mathcal{E}$ is an equilibrium if we assume that, for almost every agent $a \in A$, $X(a)$ is convex, the strict-preferred set $P_a(x^*(a))$ is open in $X(a)$ and $\inf p^*(X(a)) < p^*(e(a)) + \sup p^*(Y(a))$. In particular, if $p^* > 0$ then the last condition is automatically valid if for a.e. agent $a \in A$, $(\{e(a)\} + Y(a) - X(a)) \cap \operatorname{int} \mathbb{L}_+ \neq \emptyset$.

The model of production economies defined above encompasses the two models : the private ownership economy and the coalitional economy, presented in Hildenbrand [15].

3.3. **The Assumptions.** We present the list of assumptions that economies will be required to satisfy. We suppose that $\mathbb{L}$ is endowed with a partial linear order defined by a pointed closed convex cone $\mathbb{L}_+$. On the consumption side we consider both non-ordered but convex preferences (Assumption C_n) and partially ordered (possibly incomplete) but non-convex preferences (Assumption C_p).

Assumption (C_n). ***[possibly non-ordered but convex]*** *For almost every agent $a \in A$,*

- **(i)** *the consumption set $X(a)$ is closed and P_a is continuous, that is, for all $x \in X(a)$, $P_a(x)$ and $P_a^{-1}(x)$*[5] *are open in $X(a)$,*
- **(ii)** *the preference relation $P(a)$ is convex, that is, the consumption set $X(a)$ is convex and for each bundle $x \in X(a)$, $x \notin \operatorname{co} P_a(x)$.*

Assumption (C_p). ***[partially ordered but non-convex]*** *For almost every agent $a \in A$,*

- **(i)** *the consumption set $X(a)$ is closed and P_a is continuous,*
- **(ii)** *if a belongs to the non-atomic*[6] *part of $(A, \mathcal{A}, \mu)$ then $P(a)$ is partially ordered, and if a belongs to an atom of $(A, \mathcal{A}, \mu)$, then the preference relation $P(a)$ is convex.*

Remark 3.2. In the frameworks of Aumann [3], Hildenbrand [15], Schmeidler [24] and Cornet and Topuzu [10], Assumption C_p is valid. In general, Assumptions C_n and C_p are not comparable but if for almost every agent $a \in A$, the preference relation $P(a)$ is convex, then Assumption C_p implies Assumption C_n.

Assumption (C). ***[Consumption side]*** *Assumption C_p or Assumption C_n is satisfied.*

Assumption (M). ***[Measurability]*** *The correspondences X, Y and P are graph measurable.*

Remark 3.3. Under Assumption C, if preferences are ordered, following Proposition A.5 (in Appendix A), we can replace in Assumption M, the graph measurability of P by the Aumann measurability of preferences. It follows that in the framework of Aumann [3] and Schmeidler [24], Assumption M is valid.

Assumption (P). ***[Production side]*** *The aggregate production set Y_Σ is a non-empty closed convex subset of $\mathbb{L}$.*

If we let for all $a \in A$, $\tilde{Y}(a) := \operatorname{cl}(\overline{\operatorname{co}}\, Y(a) + A(Y_\Sigma))$, then following Proposition A.7 (in Appendix A), $\tilde{Y}$ satisfies Assumption P, and the economy $\mathcal{E}$ has a Walrasian (quasi-) equilibrium if and only if the economy $\tilde{\mathcal{E}}$, defined by replacing Y by $\tilde{Y}$ in $\mathcal{E}$, has a Walrasian (resp. quasi-) equilibrium.

[5] For each $y \in X(a)$, $P_a^{-1}(y) = \{x \in X(a) \mid y \in P_a(x)\}$.

[6] An element $E \in \mathcal{A}$ is an atom of $(A, \mathcal{A}, \mu)$ if $\mu(E) \neq 0$ and $[B \in \mathcal{A}$ and $B \subset E]$ implies $\mu(B) = 0$ or $\mu(E \setminus B) = 0$.

Assumption (S). ***[Survival]*** *For a.e.* $a \in A$, $X(a) \cap \Big(\{e(a)\} + \tilde{Y}(a)\Big) \neq \emptyset$.

Remark 3.4. Assumption S means that we need compatibility between individual needs and resources. In [15], Hildenbrand supposed that for almost every agent $a \in A$, $0 \in Y(a)$ and $X(a) \cap (\{e(a)\} + A(Y_\Sigma)) \neq \emptyset$. Yamazaki in [28] proposed a different survival assumption.

Assumption (B). ***[Bounded]*** *The consumption set correspondence X is integrably bounded from below*[7] *and the set of free-production $Y_\Sigma \cap \mathbb{L}_+$ is bounded.*

Assumption (LNS). ***[Local Non Satiation]*** *For almost every agent $a \in A$, for all bundle $x \in X(a)$, $x \in \overline{co}\, P_a(x)$.*

3.4. **Existence of equilibria for free-disposal economies.**

Assumption (FD). ***[Free Disposal]*** *One of the two following properties holds.*

- **(a)** *The aggregate production set is free-disposal, that is, $Y_\Sigma - \mathbb{L}_+ \subset Y_\Sigma$.*
- **(b)** *The preferences are weakly monotone, that is, for almost every agent $a \in A$, $X(a) + \mathbb{L}_+ \subset X(a)$ and for all $(x, y) \in X(a) \times X(a)$, $y \geqslant x \Rightarrow P_a(y) \subset P_a(x)$.*

Remark 3.5. If preferences are supposed to be strictly monotone (Assumption MON in the next subsection) and transitive, then the condition (b) in Assumption FD is automatically valid.

In order to prove that a quasi-equilibrium of $\mathcal{E}$ is in fact an equilibrium, the economy will be required to satisfy the following assumption.

Assumption (SS). ***[Strong Survival]*** *For almost every agent $a \in A$, there exists $x^0(a) \in X(a)$ and $y^0(a) \in \tilde{Y}(a)$ such that $e(a) + y^0(a) - x^0(a) \in int\mathbb{L}_+$ and such that $X(a)$ is star-shaped*[8] *about $x^0(a)$.*

Remark 3.6. In [15], Hildenbrand assumed that for almost every $a \in A$, $X(a)$ is convex (and thus star-shaped about each point), $0 \in Y(a)$ and $(\{e(a)\} + \mathrm{int}\, A(Y_\Sigma)) \cap X(a) \neq \emptyset$. This assumption obviously implies Assumption SS. The consumption $X(a)$ need not to be convex in order to satisfy Assumption SS (see Example 3.2).

We are now ready to state the main existence result.

Theorem 3.1. *If an economy $\mathcal{E}$ satisfies Assumptions **C, M, P, S, B, LNS** and **FD**, then a Walrasian quasi-equilibrium (x^*, y^*, p^*) exits, with $p^* > 0$. If moreover $\mathcal{E}$ satisfies **SS** then (x^*, y^*, p^*) is a Walrasian equilibrium of $\mathcal{E}$.*

Remark 3.7. This equilibrium existence result improves Theorem 1 and 2 in Hildenbrand [15]. Indeed, Assumptions C_p, M, P, S, B, LNS, FD and SS of Theorem 3.1 are implied by those used in [15]. More precisely, we only require that preferences are partially ordered. We do not need to suppose, as in Hildenbrand [15], that preferences are ordered. Moreover, to prove the existence of a quasi-equilibrium, we do not assume that consumption sets are convex on the non-atomic part of $(A, \mathcal{A}, \mu)$. We do neither need to suppose that the aggregate production set Y_Σ satisfies an irreversibility property $Y_\Sigma \cap (-Y_\Sigma) = \{0\}$. Instead of supposing impossibility of free-production $Y_\Sigma \cap \mathbb{L}_+ = \{0\}$, we only suppose that the set of free-production is bounded. We replace possibility of inaction, that is, for almost every $a \in A$, $0 \in Y(a)$, by the weaker assumption that the aggregate production set is non-empty. Moreover Fatou's Lemma of Cornet and Topuzu [10] (Theorem A.2 in Appendix A) allows us to deal with a more general positive cone than $(\mathbb{R}_+)^\ell$ when $\mathbb{L} = \mathbb{R}^\ell$ for some $\ell \in \mathbb{N}$.

[7] That is there exists an integrable function $\underline{x}$ from A to $\mathbb{L}$ such that for a.e. $a \in A$, for all $x \in X(a)$, $x \geqslant \underline{x}(a)$.

[8] A subset X of $\mathbb{L}$ is star-shaped about $x^0 \in X$ if for all $x \in X$ the segment $[x^0, x]$ lie in X.

3.5. **Examples.** Aumann in [3] for exchange economies and Hildenbrand in [15] for production economies proved that for continuum economies, that is, economies with a non-atomic measure space of agents, the convex assumption on ordered preferences is not needed to prove the existence of a Walrasian equilibrium. But in Theorem 3.1, when preferences are possibly non-ordered (Assumption C_n) they are assumed to satisfy a convex property. We provide hereafter an example of a production economy satisfying all assumptions of Theorem 3.1, except the convex property, and for which no quasi-equilibrium exists. This shows that the "convexifying effect of aggregation" is no longer valid for production economies with non-transitive preferences.

Counterexample 3.1. We consider the following private ownership economy, with two commodities and one producer $\mathcal{E} = ((T, \mathcal{L}(T), \lambda), \langle \mathbb{R}^2, \mathbb{R}^2 \rangle, (X, P, e), (Y, \theta))$, where the continuum T is the unit interval equipped with Lebesgue measure. The production set is $Y := -\mathbb{R}^2_+$. For each $a \in T$, the consumption set is $X(a) := \mathbb{R}^2_+$, the initial endowment is $e(a) := (1, 1)$, the share is $\theta(a) = 1$ and the preferred sets are defined for all $x \in \mathbb{R}^2_+$ by $P_a(x) := \{x' \in \mathbb{R}^2_+ \mid x'_1 > x_1 \quad \text{or} \quad x'_2 > x_2\}$. The economy $\mathcal{E}$ satisfies Assumptions M, P, S, B, LNS, FD and C_n without the convex property. But $\mathcal{E}$ has no Walrasian quasi-equilibrium. Indeed, for each positive price $p \in \mathbb{L}_+ \setminus \{0\}$, we define the demand set $D(p) := \{x \in B(p) \mid P(x) \cap B(p) = \emptyset\}$, where $B(p) := \{x \in \mathbb{R}^2_+ \mid p\,(x) \leqslant p\,((1, 1))\}$ is the budget set. We then easily check that for all $p \in \mathbb{L}_+ \setminus \{0\}$, $D(p) = \emptyset$.

We provide hereafter two examples of production economies for which Theorem 3.1 applies but which are not covered by the existence results of Auman [3], Schmeidler [24] and Hildenbrand [15].

Example 3.1. We consider an economy with two goods, i.e., $\mathbb{L} = \mathbb{L}^* = \mathbb{R}^2$, one producer and the unit interval endowed with the Lebesgue measure $([0, 1], \mathcal{L}[0, 1], \lambda)$ as the measure set of agents. The production set correspondance Y is defined for all $a \in [0, 1]$ by $Y(a) := \{(y_1, y_2) \in \mathbb{R}^2 \mid \max(y_1, y_2) \leqslant 1\}$. For each agent $a \in [0, 1]$, the intial endowment is $e(a) := (2 - a, 2 - a)$, the consumption set is $X(a) := \{(x_1, x_2) \in \mathbb{R}^2 \mid \min(x_1, x_2) \geqslant 0\}$, and the preference correspondance P is defined for all $x = (x_1, x_2) \in X(a)$ by $P_a(x) := \{x' \in X \mid \langle (1, ax_2), x' - x \rangle > 0\}$.

The economy $\mathcal{E} = (([0, 1], \mathcal{L}[0, 1], \lambda), \langle \mathbb{R}^2, \mathbb{R}^2 \rangle, X, Y, P, e)$ satisfies the assumption of Theorem 3.1. But for each agent, the preference relation is not transitive, hence the existence of a Walrasian equilibrium for $\mathcal{E}$ is not covered by the existence results of Auman [3], Schmeidler [24] and Hildenbrand [15].

Example 3.2. We consider an economy with two goods, i.e., $\mathbb{L} = \mathbb{L}^* = \mathbb{R}^2$, one producer and the unit interval endowed with the Lebesgue measure $([0, 1], \mathcal{L}[0, 1], \lambda)$ as the measure set of agents. The production set correspondance Y is defined for all $a \in [0, 1]$, by $Y(a) := \{(y_1, y_2) \in \mathbb{R}^2 \mid \max(y_1, y_2) \leqslant 1\}$. Let $a \in [0, 1]$ be an agent. The intial endowment is $e(a) := (2 - a, 2 - a)$. For each $a \leqslant \lambda < 1$, we let $A_\lambda := [(\lambda, 0); (1, 1)] \bigcup [(0, \lambda); (1, 1)] \setminus \{(1, 1)\}$ and for each $1 \leqslant \lambda < +\infty$, we let $B_\lambda := [(\lambda, 0); (\lambda, \lambda)] \bigcup [(0, \lambda); (\lambda, \lambda)]$. The consumption set of agent $a \in [0, 1]$ is defined by $X(a) := \bigcup_{a \leqslant \lambda < 1} A_\lambda \cup \bigcup_{1 \leqslant \lambda < +\infty} B_\lambda$. Now we define the preference correspondance P_a, by for all $x \in X(a)$,

$$P_a(x) := \begin{cases} \bigcup_{\lambda < \lambda' < 1} A_{\lambda'} \cup \bigcup_{1 \leqslant \lambda' < +\infty} B_{\lambda'} \setminus \{(1, 1)\} & \text{if} \quad x \in A_\lambda \\ \bigcup_{\lambda < \lambda' < +\infty} B_{\lambda'} & \text{if} \quad x \in B_\lambda. \end{cases}$$

The economy $\mathcal{E} = (([0, 1], \mathcal{L}[0, 1], \lambda), \langle \mathbb{R}^2, \mathbb{R}^2 \rangle, X, Y, P, e)$ satisfies the assumption of Theorem 3.1. But for each agent, the preference relation is not negatively transitive, nor monotone, and the consumption sets are not convex, hence the existence of a Walrasian equilibrium for $\mathcal{E}$ is not covered by the existence results of Auman [3], Schmeidler [24] and Hildenbrand [15].

3.6. **Existence of equilibria for economies with monotone preferences.**

Assumption (MON). ***[Monotonicity]*** *For each agent $a \in A$, the consumption set $X(a)$ is convex comprehensive*[9] *and preferences are strictly monotone, that is, for each bundle $x \in X(a)$, for all $m > 0$, $x + m \in co\, P_a(x)$.*

Assumption (E). ***[Endowments]*** *There exists $(\bar{u}, \bar{v}) \in Y_\Sigma \times X_\Sigma$ such that $\omega + \bar{u} - \bar{v} \gg 0$.*

Remark 3.8. This assumption means that no commodity is totally absent from the market. In Aumann [3], it is supposed that $\omega \gg 0$. The Assumption E generalizes this assumption since in [3] the aggregate consumption set X_Σ coincide with $\mathbb{L}_+$ and the production sector is trivial.

In order to prove that a quasi-equilibrium of $\mathcal{E}$ is in fact an equilibrium, the economy will be required to satisfy the following assumption.

Assumption (Ss). *For almost every agent $a \in A$, one of the two following properties holds.*

- **(i)** *There exists $x^0(a) \in X(a)$ and $y^0(a) \in \tilde{Y}(a)$ such that $e(a)+y^0(a)-x^0(a) \in \mathbb{L}_+$ and $X(a)$ is star-shaped at $x^0(a)$.*
- **(ii)** $\{e(a)\} + Y(a) - X(a) \subset -\mathbb{L}_+$.

Remark 3.9. Survival Assumption S ensures that $0 \in \{e(a)\} + \tilde{Y}(a) - X(a)$. Assumption Ss(i) means that 0 is not the smallest non-negative vector in $\{e(a)\}+\tilde{Y}(a)-X(a)$. Assumption Ss will play the same role as Assumption SS introduced in the free-disposal on production framework, but SS is stronger than Ss. Indeed when preferences are strictly monotone, we prove the existence of a quasi-equilibrium with a price $p^* \gg 0$. This extra information allows us to lighten the Strong Survival Assumption SS.

Remark 3.10. In the framework of Aumann [3], the production sector is trivial, that is, for all $a \in A$, $Y(a) = 0$ and consumption sets coincide with the positive cone, that is, $X(a) = \mathbb{L}_+$. It follows that Assumption Ss is automatically valid. Indeed, Assumption S ensures that for almost every $a \in A$, $e(a) \in X(a) = \mathbb{L}_+$. If $e(a)$ is not zero, then Ss(i) is valid and if $e(a) = 0$, then it is Ss(ii) that is valid.

We present now, as a corollary of Theorem 3.1, a Walrasian equilibrium existence result for production economies with strictly monotone preferences.

Corollary 3.1. *If an economy satisfies Assumptions **C, M, P, S, B, MON** and **E**, then a Walrasian quasi-equilibrium (x^*, y^*, p^*) exists, with $p^* \gg 0$. If moreover $\mathcal{E}$ satisfies **Ss** then (x^*, y^*, p^*) is a Walrasian equilibrium.*

Remark 3.11. This equilibrium existence result improves Main Theorem in Aumann [3] and Main Theorem in Schmeidler [24]. Indeed, Assumptions C, M, P, S, B, MON, E and Ss of Corollary 3.1 are implied by those used in [3] and [24]. Moreover Corollary 3.1 deals with production economies and not only with pure exchange economies, and it provides the existence of a Walrasian equilibrium without assuming that consumption sets coincide with the positive cone.

Proof. Following Remark 3.1, to prove Corollary 3.1, it is sufficient to prove the existence of a Walrasian quasi-equilibrium. Let $\mathcal{E}$ be an economy satisfying Assumptions C, M, P, S, B, MON and E. Once again we can suppose without any loss of generality that for almost every $a \in A$, $Y(a) = \tilde{Y}(a)$. We propose to construct an auxiliary economy $\mathcal{E}'$ *close* to $\mathcal{E}$ and satisfying Assumption FD, in order to apply Theorem 3.1. We let $\mathcal{E}' := ((A', \mathcal{A}', \mu'), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X', Y', P', e'))$ be the production economy with the measure space of agents $A' = A \cup \{\infty\}$, the σ-algebra $\mathcal{A}' = \mathcal{A} \cup \{B \cup \{\infty\} \mid B \in \mathcal{A}\}$, the

[9] That is $X(a) + \mathbb{L}_+ \subset X(a)$.

measure μ' defined by $\mu'|_{\mathcal{A}} = \mu$, and for each $B \in \mathcal{A}$, $\mu'(B \cup \{\infty\}) = \mu(B) + 1$. The consumption sets correspondence X' is defined by $X'|_A = X$ and $X'(\infty) = \mathbb{L}_+$. The preference correspondence P' is defined by $P'|_A = P$ and $P'(\infty) := \{(x, y) \in \mathbb{L}_+^2 \mid y - x \in \operatorname{int} \mathbb{L}_+\}$[10]. The production sets correspondence Y' is defined by $Y'|_A = Y$ and $Y'(\infty) = -\mathbb{L}_+$. The initial endowment function e' is defined by $e'|_A = e$ and $e'(\infty) = 0$. It is straightforward to verify that $\mathcal{E}'$ satisfies Assumptions C, M, P, S, B, FD and LNS. Applying Theorem 3.1, there exists a Walrasian quasi-equilibrium (x^*, y^*, p^*) of $\mathcal{E}'$. Now we easily check that $(x^*|_X, y^*|_Y, p^*)$ is a quasi-equilibrium of $\mathcal{E}$. □

4. Discretization of measurable correspondences

4.1. **Notations and definitions.** We consider $(A, \mathcal{A}, \mu)$ a finite measure space and (D, d) a separable metric space. We recall that a function $f : A \to D$ is measurable if for all open subset $V \subset D$, $f^{-1}(V) := \{a \in A \mid f(a) \in V\} \in \mathcal{A}$, and a correspondence $F : A \twoheadrightarrow D$ is measurable if for all open subset $V \subset D$, $F^-(V) := \{a \in A \mid F(a) \cap V \neq \emptyset\} \in \mathcal{A}$.

Definition 4.1. A partition $\sigma = (A_i)_{i \in I}$ of A is a *measurable partition* if for all $i \in I$, the set A_i is non-empty and belongs to $\mathcal{A}$. A finite subset A^σ of A is *subordinated to the partition* σ if there exists a family $(a_i)_{i \in I} \in \prod_{i \in I} A_i$ such that $A^\sigma = \{a_i \mid i \in I\}$.

4.1.1. *Simple functions subordinated to a measurable partition.* Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of A, and $A^\sigma = \{a_i \mid i \in I\}$ is a finite set subordinated to σ, we consider $\phi(\sigma, A^\sigma)$ the application which maps each measurable function f to a simple measurable function $\phi(\sigma, A^\sigma)(f)$, defined by

$$\phi(\sigma, A^\sigma)(f) := \sum_{i \in I} f(a_i) \chi_{A_i} ,$$

where χ_{A_i} is the characteristic[11] function associated to A_i. Note that the sum is well defined since there exists at most one non zero factor.

4.1.2. *Simple correspondences subordinated to a measurable partition.* Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of A, and $A^\sigma = \{a_i \mid i \in I\}$ is a finite set subordinated to σ, we consider $\psi(\sigma, A^\sigma)$, the application which maps each measurable correspondence $F : A \twoheadrightarrow D$ to a simple measurable correspondence $\psi(\sigma, A^\sigma)(F)$, defined by

$$\psi(\sigma, A^\sigma)(F) := \sum_{i \in I} F(a_i) \chi_{A_i} .$$

Remark 4.1. If f is a function from A to D, let $\{f\}$ be the correspondence from A into D, defined for all $a \in A$ by $\{f\}(a) := \{f(a)\}$. We check that $\psi(\sigma, A^\sigma)(\{f\}) = \{\phi(\sigma, A^\sigma)(f)\}$.

4.1.3. *Hyperspace.*

Definition 4.2. The space of all non-empty subsets of D is noted $\mathcal{P}^*(D)$. We let τ_{W_d} be the Wijsman topology on $\mathcal{P}^*(D)$, that is the weak topology on $\mathcal{P}^*(D)$ generated by the family of distance functions $(d(x, .))_{x \in D}$. If $V \subset D$ is a subset of D, we note $V^- = \{Z \subset D \mid Z \cap V \neq \emptyset\}$, and we note $\mathcal{E}(D)$ the Effrös σ-algebra, that is the σ-algebra generated by all sets V^-, where V is open.

Hess proved in [13] that, restricted to the set of non-empty closed subsets of D, the Effrös σ-algebra $\mathcal{E}(D)$ and the Borel σ-algebra $\mathcal{B}(\mathcal{P}^*(D), \tau_{W_d})$ relative to the Wijsman topology coincide. In fact this result is still true if we do not restrict to closed subsets.

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[10] Following Assumption E, the positive cone $\mathbb{L}_+$ has an interior point.

[11] That is, for all $a \in A$, $\chi_{A_i}(a) = 1$ if $a \in A_i$ and $\chi_{A_i}(a) = 0$ elsewhere.

Theorem 4.1 (Hess).

$$\mathcal{E}(D) = \mathcal{B}(\mathcal{P}^*(D), \tau_{W_d}).$$

Proof. If $x \in D$, $\alpha > 0$ and $Z \subset D$, then we note $B(x,\alpha) = \{z \in D \mid d(x,z) < \alpha\}$ and $\delta_x(Z) := d(x,Z)$. We easily check that $\delta_x^{-1}([0,\alpha[) = [B(x,\alpha)]^-$. It follows that (we do not make use of separability) $\mathcal{B}(\mathcal{P}^*(D), \tau_{W_d}) \subset \mathcal{E}(D)$. Since D is separable, each open set in D is a countable union of open balls. It follows that $\mathcal{E}(D) \subset \mathcal{B}(\mathcal{P}^*(D), \tau_{W_d})$. □

Remark 4.2. A direct corollary of Theorem 4.1 is that a correspondence F from A into D is measurable if and only if for all $x \in D$, the real valued function $d(x, F(.))$ is measurable.

Definition 4.3. The Hausdorff semi-metric H_d on $\mathcal{P}^*(D)$ is defined for all $(A,B) \in \mathcal{P}^*(D)$ by $H_d(A,B) := \sup\{|d(x,A) - d(x,B)| \mid x \in D\}$. A subset C of D is the Hausdorff limit of a sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of D, if $\lim_{n\to\infty} H_d(C_n, C) = 0$.

4.2. **Approximation of measurable real valued functions.** We propose to prove that for a countable set of measurable real valued functions, there exists a sequence of measurable partitions *approximating* each function in the following sense.

Theorem 4.2. *Let $\mathcal{F}$ be a countable set of measurable real valued functions. There exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of finer and finer measurable partitions $\sigma^n = (A_i^n)_{i\in I^n}$ of A, satisfying the following properties.*

- **(i)** *Let $(A^n)_{n\in\mathbb{N}}$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n and let $f \in \mathcal{F}$. For all $n \in \mathbb{N}$, we define the simple function $f^n := \phi(\sigma^n, A^n)(f)$ subordinated to f.*
  1. *The function f is the pointwise limit of the sequence $(f^n)_{n\in\mathbb{N}}$.*
  2. *If $f(A)$ is bounded then f is the uniform limit of the sequence $(f^n)_{n\in\mathbb{N}}$.*
- **(ii)** *If $\mathcal{G} \subset \mathcal{F}$ is a finite subset of integrable functions, then there exists a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets A^n subordinated to the measurable partition σ^n, such that for each $f \in \mathcal{G}$, $\lim_{n\to\infty} \int_A |f^n - f| d\mu = 0$.*

Proof. Let $f : A \to \mathbb{R}_+$ be a measurable function. We will construct a sequence of measurable partitions depending on f. Let $n \in \mathbb{N}$, we pose $K^n = \{0, \dots, 2^{2n}\}$. We define the measurable partition $\pi^n(f) = (E_k^n(f))_{k\in K^n}$ by

$$E_k^n(f) = \begin{cases} f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[\right) & \text{if } k \in \{0, \dots, 2^{2n} - 1\}\ , \\ \\ f^{-1}([2^n, +\infty[) & \text{if } k = 2^{2n}. \end{cases}$$

Let $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ be a countable set of real valued measurable functions. Now for each $n \in \mathbb{N}$, we define $\mathcal{F}^n := \{f_k \mid 0 \leqslant k \leqslant n\}$ and σ^n as the following measurable partition

$$\sigma^n := (A_i^n)_{i\in I^n} \subset (A_i^n)_{i\in S^n} := \bigvee_{f\in\mathcal{F}^n} [\pi^n(f_+) \vee \pi^n(f_-)]\,,$$

where $I^n := \{i \in S^n \mid A_i^n \neq \emptyset\}$ and $\vee$ is the natural supremum operator on partitions.

We begin to prove part (i) of Theorem 4.2. Let $(A^n)_{n\in\mathbb{N}}$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n, let $f \in \mathcal{F}$ and $a \in A$. Following the construction of σ^n, we can suppose, without any loss of generality, that $f = f_+$. For all n large enough, $f \in \mathcal{F}^n$ and $f(a) < 2^n$, and following the construction of the partition σ^n, for all n large enough

$$\forall b \in A_i^n \quad |f(b) - f(a)| \leqslant \frac{1}{2^n}\,,$$

where $i \in I^n$ is such that $a \in A_i^n$. It follows that $\lim_{n\to\infty} f^n(a) = f(a)$, and this limit is uniform if $f(A)$ is bounded.

We now prove part (ii) of Theorem 4.2. Let $\mathcal{G} \subset \mathcal{F}$, be a finite set of integrable functions. Once again, we can suppose that all functions in $\mathcal{G}$ are positive. We let $h := \sum_{f\in\mathcal{G}} f$, this function defined from A to $\mathbb{R}_+$ is integrable. For each $n \in \mathbb{N}$, for each $i \in I^n$, A_i^n is non-empty and we can choose $a_i^n \in A_i^n$ such that $h(a_i^n) \leqslant 1+\inf\{h(b) \mid b \in A_i^n\}$. We have constructed a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets $A^n := \{a_i^n \mid i \in I^n\}$, subordinated to the measurable partition σ^n, such that for each $f \in \mathcal{G}$, for each $n \in \mathbb{N}$, $\forall a \in A \quad f^n(a) \leqslant 1 + h(a)$. Applying part (i) and the Lebesgue Dominated Convergence Theorem, for all $f \in \mathcal{G}$, $\lim_{n\to\infty} \int_A |f^n - f| d\mu = 0$ and the theorem follows. □

4.3. **Approximation of measurable correspondences.** As a corollary of Theorem 4.2, we propose to prove that for a countable set of measurable correspondences, there exists a sequence of measurable partitions *approximating* each correspondence in the following sense.

Corollary 4.1. *Let $\mathcal{F}$ be a countable set of measurable correspondences with non-empty values from A into D. There exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of finer and finer measurable partitions $\sigma^n = (A_i^n)_{i\in I^n}$ of A, such that, if $(A^n)_{n\in\mathbb{N}}$ is a sequence of finite sets A^n subordinated to the measurable partition σ^n, if $F \in \mathcal{F}$, then by defining F_n as the simple correspondence $F^n := \psi(\sigma^n, A^n)(F)$ subordinated to F, the following properties are satisfied.*

1. *For all $a \in A$, the set $F(a)$ is the Wijsman limit of the sequence $(F^n(a))_{n\in\mathbb{N}}$, i.e.,*
$$\forall a \in A \quad \forall x \in A \quad \lim_{n\to\infty} d(x, F^n(a)) = d(x, F(a)).$$
2. *If D is d-bounded then for all $x \in D$ the real valued function $d(x, F(.))$ is the uniform limit of the sequence $(d(x, F^n(.)))_{n\in\mathbb{N}}$.*
3. *If D is d-totally bounded*[12] *then F is the uniform Hausdorff limit of the sequence $(F^n)_{n\in\mathbb{N}}$.*

Remark 4.3. The property (a1) implies in particular that, if $(x^n)_{n\in\mathbb{N}}$ is a sequence of D, d-converging to $x \in D$, then for all $a \in A$, $\lim_{n\to\infty} d(x^n, F^n(a)) = d(x, F(a))$. It follows that if F is non-empty closed valued, then property (1) implies that for all $a \in A$, $\operatorname{Ls} F^n(a) \subset F(a)$.

Proof. If $F : A \twoheadrightarrow D$ is a correspondence, we consider the distance function associated to F, $\delta_F : A \times D \to \mathbb{R}_+$ defined by $\delta_F : (a, x) \mapsto \mathrm{d}(x, F(a))$. Let $F \in \mathcal{F}$, following Theorem 4.1, F is measurable if and only if, for all $x \in D$, $\delta_F(., x)$ is measurable. Since D is separable, there exist a sequence $(x_n)_{n\in\mathbb{N}}$ dense in D. We let, for each $n \in \mathbb{N}$, $\delta_n^F := \delta_F(., x_n)$. We define $\mathcal{F}_0 = \bigcup_{F\in\mathcal{F}} \{\delta_n^F \mid n \in \mathbb{N}\}$. Note that, if F is a correspondence from A into D, then for all measurable partition σ of A, for each subset A^σ subordinated to σ, and for all $x \in L$, $\phi(\sigma, A^\sigma)(d(x, F(.)) = d(x, \psi(\sigma, A^\sigma)(F)(.))$. We then apply Theorem 4.2 to the countable set $\mathcal{F}_0$ of measurable functions . Noting that, for each $a \in A$, for all $F \in \mathcal{F}$, the functions $\delta_F(a, .)$ are 1-Lipschitz, we easily get the desired result. □

5. Proof of the main Theorem

5.1. **Stronger existence results.** We will prove in fact stronger existence results than Theorem 3.1 and Corollary 3.1. Hereafter we present the Assumptions C'$_n$, C'$_p$, M' and SS' which are weaker than, respectively Assumptions C$_n$, C$_p$, M and SS.

Assumption (C'$_n$). *For almost every agent $a \in A$,*

(i) *the consumption set $X(a)$ is closed and P_a is lower semi-continuous*[13]*,*

[12] That is for each $\varepsilon > 0$ there exists a finite subset $\{x_1, \cdots, x_n\} \subset D$ such that the collection of balls $B(x_i, \varepsilon) = \{z \in D \mid d(z, x_i) < \varepsilon\}$ covers D.

[13] That is for all open set $V \subset \mathbb{L}$, $\{x \in X(a) \mid P_a(x) \cap V \neq \emptyset\}$ is open in X(a).

(ii) *the preference relation $P(a)$ is convex*[14]*.*

Remark 5.1. The properties required in Assumption C'$_n$, are the natural extension of those required in the finite agent's set-up to prove the existence of a quasi-equilibrium.

Assumption (C'$_p$). *For almost every agent $a \in A$,*

(i) *the consumption set $X(a)$ is closed and P_a is lower semi-continuous,*

(ii) *if a belongs to the non-atomic part of $(A, \mathcal{A}, \mu)$ then $P(a) \subset \tilde{P}(a)$ where $\tilde{P}(a)$ is an ordered binary relation on $X(a)$ with open lower sections*[15] *in $X(a)$ and if a belongs to an atom of $(A, \mathcal{A}, \mu)$ then the preference relation $P(a)$ is convex.*

Remark 5.2. Let $a \in A$, following Sondermann [26], if $P(a)$ is partially ordered and continuous[16] then there exists an upper semi-continuous function $u_a : X(a) \to \mathbb{R}$ such that $P(a) \subset \{(x, y) \in X(a) \times X(a) \mid u_a(x) < u_a(y)\}$. The function u_a defines an ordered binary relation $\tilde{P}(a)$ on $X(a)$ with open lower sections such that $P(a) \subset \tilde{P}(a)$. It follows that Assumption C'$_p$ is weaker than C$_p$.

Assumption (C'). *Assumption C'$_p$ or Assumption C'$_n$ is satisfied.*

Assumption (M'). *The correspondences X and Y are graph measurable and the correspondence of preferences P is lower semi-graph measurable, that is, for all graph measurable correspondence $V : A \twoheadrightarrow \mathbb{L}$ with open values,*

$$\{(a, x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}).$$

Remark 5.3. Following Proposition A.6, Assumption M' is weaker than Assumption M.

Assumption (SS'). *For almost every agent $a \in A$, there exists $x^0(a) \in X(a)$ and $y^0(a) \in \tilde{Y}(a)$ such that $e(a) + y^0(a) - x^0(a) \in int\mathbb{L}_+$, $X(a)$ is radial at $x^0(a)$ and for all $x \in X(a)$, $P_a(x)$ is radial to $x^0(a)$*[17]*.*

Remark 5.4. If $X(a)$ is star-shaped about $x^0(a)$ and $P_a(x)$ is open in $X(a)$ then $P_a(x)$ is radial to $x^0(a)$.

Theorem 5.1. *If an economy $\mathcal{E}$ satisfies Assumptions C', M', P, S, B, LNS and FD, then a Walrasian quasi-equilibrium (x^*, y^*, p^*) exits, with $p^* > 0$. If moreover $\mathcal{E}$ satisfies SS' then (x^*, y^*, p^*) is a Walrasian equilibrium of $\mathcal{E}$.*

Assumption (Ss'). *For almost every agent $a \in A$, one of the two following properties holds.*

(i) *There exists $x^0(a) \in X(a)$ and $y^0(a) \in \tilde{Y}(a)$ such that $e(a)+y^0(a)-x^0(a) \in \mathbb{L}_+$, $X(a)$ is star-shaped about $x^0(a)$ and for all $x \in X(a)$, $P_a(x)$ is radial to $x^0(a)$.*

(ii) $\{e(a)\} + Y(a) - X(a) \subset -\mathbb{L}_+$.

Corollary 5.1. *If an economy satisfies Assumptions C', M', P, S, B, MON and E, then a Walrasian quasi-equilibrium (x^*, y^*, p^*) exists, with $p^* \gg 0$. If moreover $\mathcal{E}$ satisfies Ss' then (x^*, y^*, p^*) is a Walrasian equilibrium.*

5.2. **Satiation equilibria.** Hereafter, we introduce an auxiliary concept of quasi-equilibria for an economy.

Definition 5.1. A *satiation quasi-equilibrium* of an economy $\mathcal{E}$ is an element (x^*, y^*, p^*) of $\mathrm{S}^1(X) \times \mathrm{S}^1(Y) \times \mathbb{L}^*$ such that $p^* \neq 0$ and satisfying the following properties.

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[14]We recall that P_a is convex if $X(a)$ is convex and for all $x \in X(a)$, $x \notin \mathrm{co}\, P_a(x)$.

[15]That is for all $y \in X(a)$, $\{x \in X(a) \mid (x, y) \in \tilde{P}(a)\}$ is open in $X(a)$.

[16]That is for all $x \in X(a)$, $P_a(x)$ and $P_a^{-1}(x)$ are open in $X(a)$.

[17]A subset P of $\mathbb{L}$ is radial to $x^0 \in X$ if for all $y \in P$ the segment $[y, x^0 + \lambda(y - x^0)]$ still lie in P for some $0 \leqslant \lambda < 1$.

(i) For a.e. $a \in A$, if $(x,y) \in P_a(x^*(a)) \times Y(a)$ then $p^*(x) \geqslant p^*(y) + p^*(e(a))$.

(ii) $\int_A x^*(a)d\mu(a) = \int_A e(a)d\mu(a) + \int_A y^*(a)d\mu(a)$.

Remark 5.5. When the condition (i) is replaced by the following condition

(i') For a.e. $a \in A$, if $(x,y) \in P_a(x^*(a)) \times Y(a)$ then $p^*(x) > p^*(y) + p^*(e(a))$,

then (x^*, y^*, p^*) is called a satiation equilibrium. Indeed, condition (i') means that either agent $a \in A$ is satiated, $P_a(x^*(a)) = \emptyset$ or for all bundle $x \in X(a)$, if x is prefered to $x^*(a)$ then x is not in the bugdet set, $p^*(x) > p^*(e(a)) + \sup p^*(Y(a))$. Note that the consumption bundle $x^*(a)$ is not expected to lie in the budget set, however the consumption plan x^* has to be realizable.

If (x^*, y^*, p^*) is a Walrasian quasi-equilibrium of an economy $\mathcal{E}$, then (x^*, y^*, p^*) is clearly a satiation quasi-equilibrium of $\mathcal{E}$. We provide in the following remark, a suitable *Local Non Satiation* property on $\mathcal{E}$ under which the converse is true.

Remark 5.6. A satiation quasi-equilibrium (x^*, y^*, p^*) of an economy $\mathcal{E}$, is a Walrasian quasi-equilibrium, if we assume that, for almost every agent $a \in A$, for all bundle $x \in X(a)$, $x \in \overline{\text{co}}\, P_a(x)$.

Following this remark, to prove Theorem 3.1, it is sufficient to prove the following lemma.

Lemma 5.1. *If $\mathcal{E}$ is an economy satisfying Assumptions C', M', P, S, B and FD then a satiation quasi-equilibrium (x^*, y^*, p^*) exists, with $p^* > 0$.*

5.3. **Existence of satiation equilibria for integrably bounded economies.** As an auxiliary result, we propose to first prove existence of a satiation equilibrium for integrably bounded economies, that is economies satisfying the following assumption.

Assumption (IB). *The consumption sets correspondence X and the production sets correspondence Y are integrably bounded*[18].

This first step allows us to isolate the crucial aspect of the new approach, which is the approximation of economies with a measure space of agents (measurable correspondences) by a sequence of economies with a finite set of agents (resp. simple correspondences). Moreover, the framework of integrably bounded economies allows us to deal with both non-ordered but convex preferences and partially ordered but non-convex preferences. This auxiliary result will be applied in the next subsection to prove Lemma 5.1.

Lemma 5.2. *If $\mathcal{E}$ is an economy satisfying Assumptions C', M', P, S and IB, then a satiation quasi-equilibrium exists.*

Proof. Following Proposition A.7, we can suppose without any loss of generality that for almost every $a \in A$, $Y(a) = \tilde{Y}(a)$ and $e(a) = 0$. Following Proposition A.2, the correspondences X, Y are measurable and following Proposition A.1, there exist a sequence $(f_k)_{k\in\mathbb{N}}$ of measurable selections of X and a sequence $(g_k)_{k\in\mathbb{N}}$ of measurable selections of Y such that for all $a \in A$, $X(a) = \text{cl}\{f_k(a) \mid k \in \mathbb{N}\}$ and $Y(a) = \text{cl}\{g_k(a) \mid k \in \mathbb{N}\}$. We let for all $(k,q) \in \mathbb{N}^2$, $R_{k,q}(a) := \{x \in X(a) \mid P_a(x) \cap B(f_k(a), r_q) = \emptyset\}$, where $r_q = 1/(q+1)$ and $B(f_k(a), r_q)$ is the open ball centered in $f_k(a)$ and of radius r_q. For all $(k,q) \in \mathbb{N}^2$, $R_{k,q}$ is graph measurable with closed values, following Proposition A.1 it is then measurable.

Note that for almost every agent $a \in A$, for all $x \in \mathbb{L}$,

$$[d(x, X(a)) = 0 \Leftrightarrow x \in X(a)] \quad \text{and} \quad [d(x, Y(a)) = 0 \Leftrightarrow x \in Y(a)],$$

[18]That is, there exists an integrable function $h : A \to \mathbb{R}_+$ such that for almost every $a \in A$, for all $(x,y) \in X(a) \times Y(a)$, $\max\{\|x\|, \|y\|\} \leqslant h(a)$.

and if $x \in X(a)$,

$$d(x, R_{k,q}(a)) > 0 \Longleftrightarrow P_a(x) \cap B(f_k(a), r_q) \neq \emptyset.$$

Following Assumption IB, there exists an integrable function $h : A \to \mathbb{R}_+$ such that for almost every $a \in A$, for all $(x, y) \in X(a) \times Y(a)$, $\max\{\|x\|, \|y\|\} \leqslant h(a)$. Applying Theorem 4.2 and Corollary 4.1, there exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of measurable partitions $\sigma^n = (A_i^n)_{i\in S^n}$ of $(A, \mathcal{A})$, and a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets $A^n = \{a_i^n \mid i \in S^n\}$ subordinated to the measurable partition σ^n, satisfying the following properties.

Fact 5.1. For all $a \in A$,

(i) for all $n \in \mathbb{N}$, $h^n(a) \leqslant 1 + h(a)$ and for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} (f_k^n(a), g_k^n(a)) = (f_k(a), g_k(a)) \ ;$$

(ii) for all sequence $(x^n)_{n\in\mathbb{N}}$ of $\mathbb{L}$ converging to $x \in \mathbb{L}$,

$$\lim_{n\to\infty} d(x^n, X^n(a)) = d(x, X(a)) \ , \quad \lim_{n\to\infty} d(x^n, Y^n(a)) = d(x, Y(a))$$

and

$$\forall (k, q) \in \mathbb{N}^2 \quad \lim_{n\to\infty} d(x^n, R_{k,q}^n(a)) = d(x, R_{k,q}(a)).$$

We construct now a sequence of economies with a finite set of consumers. We distinguish two cases. In the first case (Claim 5.1) preferences are possibly non-ordered but convex, in the second case (Claim 5.2) preferences are ordered but possibly non-convex.

Claim 5.1. If $\mathcal{E}$ satisfies C_n, then a satiation quasi-equilibrium exists.

Proof. For all $n \in \mathbb{N}$, we let $I^n := \{i \in S^n \mid \mu(A_i^n) \neq 0\}$ be the finite set of consumers of the following *finite* economy $\mathcal{E}^n = (\langle \mathbb{L}^*, \mathbb{L}\rangle, (X_i^n, Y_i^n, P_i^n)_{i\in I^n})$. The consumption set of the consumer $i \in I^n$ is given[19] by $X_i^n := \mu(A_i^n)X(a_i^n)$ and the production set is given by $Y_i^n := \mu(A_i^n) Y(a_i^n)$. Preferences are defined by the relation $P_i^n := \mu(A_i^n)P(a_i^n)$. For all $n \in \mathbb{N}$, the economy $\mathcal{E}^n$ satisfies the assumptions of Theorem 2.1.1 in Oiko Nomia [21]. It follows that, for all $n \in \mathbb{N}$, there exists

$$((x_i^n)_{i\in I^n}, (y_i^n)_{i\in I^n}, p^n) \in \prod_{i\in I^n} X_i^n \times \prod_{i\in I^n} Y_i^n \times \mathbb{L}^* \ ,$$

satisfying $\|p^n\| = 1$, $\sum_{i\in I^n} x_i^n = \sum_{i\in I^n} y_i^n$ and for all $i \in I^n$, if $(x, y) \in P_i^n(x_i^n) \times Y_i^n$ then $p^n(x - y) \geqslant 0$. Let, for all $n \in \mathbb{N}$,

$$x^n := \sum_{i\in I^n} \frac{x_i^n}{\mu(A_i^n)} \chi_{A_i^n} \quad \text{and} \quad y^n := \sum_{i\in I^n} \frac{y_i^n}{\mu(A_i^n)} \chi_{A_i^n}.$$

For each $n \in \mathbb{N}$, we have defined integrable selections $x^n \in S^1(X^n)$ and $y^n \in S^1(Y^n)$ satisfying[20]

$$\int_A x^n(a)d\mu(a) = \int_A y^n(a)d\mu(a). \tag{1}$$

$$\forall a \in \bigcup_{i\in I^n} A_i^n \quad (x, y) \in P_a^n(x^n(a)) \times Y^n(a) \Rightarrow p^n(x) \geqslant p^n(y) \,. \tag{2}$$

Following (i) of Fact 5.1, the sequences $(x^n)_{n\in\mathbb{N}}$ and $(y^n)_{n\in\mathbb{N}}$ are integrably bounded. Applying Theorem A.1 there exist integrable functions $x^*, y^* : A \to \mathbb{L}$, such that $\int_A x^* d\mu = \lim_{n\to\infty} \int_A x^n d\mu$ and $\int_A y^* d\mu = \lim_{n\to\infty} \int_A y^n d\mu$. Moreover, for a.e. $a \in A$, $x^*(a) \in \mathrm{Ls}\,\{x^n(a)\}$ and $y^*(a) \in \mathrm{Ls}\,\{y^n(a)\}$. Since, for all $n \in \mathbb{N}$, $\|p^n\| = 1$, there exists a subsequence of $(p^n)_{n\in\mathbb{N}}$ converging to p^*, with $\|p^*\| = 1$.

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[19] The consumer a_i^n *represents* the coalition A_i^n.

[20] Following the notations of Section A, $P^n := \psi(\sigma^n, A^n)(P)$, that is, for all $a \in A$, $P^n(a) = P(a_i^n)$, where $i \in I^n$ is such that $a \in A_i^n$.

We propose to prove that (x^*, p^*) is a satiation quasi-equilibrium of $\mathcal{E}$. We let $A_0 := \bigcup_{n\in\mathbb{N}} \bigcup_{i\in S^n\setminus I^n} A_i^n$, then we easily check that $\mu(A_0) = 0$. Let now A' be a subset of $A \setminus A_0$ with $\mu(A \setminus A') = 0$ and such that all *almost every where* assumptions and properties are satisfied for all $a \in A'$.

To prove condition (ii) of Definition 5.1, it is sufficient to prove that $(x^*, y^*) \in \mathrm{S}^1(X) \times \mathrm{S}^1(Y)$. First let us prove that, for all $a \in A'$, $x^*(a) \in X(a)$. Let $a \in A'$, by construction, we have that for every $n \in \mathbb{N}$, $x^n(a) \in X^n(a)$, and thus, for every $n \in \mathbb{N}$, $d(x^n(a), X^n(a)) = 0$. Since $x^*(a) \in \mathrm{Ls}\,\{x^n(a)\}$, we apply Fact 5.1 to get that $d(x^*(a), X(a)) = 0$. We prove similarly that $y^* \in \mathrm{S}^1(Y)$. In fact we proved that for almost every $a \in A$, $\mathrm{Ls}\,(X^n(a)) \subset X(a)$ and $\mathrm{Ls}\,(Y^n(a)) \subset Y(a)$.

We will now prove that (x^*, p^*) satisfies condition (i) of Definition 5.1. Let $a \in A'$ and $(x, y) \in P_a(x^*(a)) \times Y(a)$. Since $Y(a) = \mathrm{cl}\,\{g_k(a) \mid k \in \mathbb{N}\}$, there exist a subsequence $(g_{\psi(k)}(a))_{k\in\mathbb{N}}$ converging to y. To prove that $p^*(x) \geqslant p^*(y)$, it is sufficient to prove that for all k and q large enough, there exists[21] $z \in \overline{B}(x, 2r_q)$ such that $p^*(z) \geqslant p^*\left(g_{\psi(k)}(a)\right)$. Let $j \in \psi(\mathbb{N})$ and $q \in \mathbb{N}$. Since $X(a) = \mathrm{cl}\,\{f_k(a) \mid k \in \mathbb{N}\}$ there exists $k \in \mathbb{N}$ such that $f_k(a) \in B(x, r_q)$. In particular $x \in B(f_k(a), r_q) \cap P_a(x^*(a))$ and $d(x^*(a), R_{k,q}(a)) > 0$. Applying Fact 5.1, for all n large enough, $d(x^n(a), R^n_{k,q}(a)) > 0$. It follows that there exists $z^n \in P^n_a(x^n(a)) \cap B(f^n_k(a), r_q)$. Thus, applying (2), for all n large enough, $p^*(z^n) \geqslant p^n\left(g^n_j(a)\right)$. Now the sequence $(f^n_k(a))_{n\in\mathbb{N}}$ converges to $f_k(a)$, thus $(z^n)_{n\in\mathbb{N}}$ is bounded. Passing to a subsequence if necessary, $(z^n)_{n\in\mathbb{N}}$ converges to $z \in \mathbb{L}$ which satisfies $p^*(z) \geqslant p^*(g_j(a))$ and $d(z, f_k(a)) \leqslant r_q$, that is, $z \in \overline{B}(x, 2r_q)$. □

We consider now the case of ordered but possibly non-convex preferences.

Claim 5.2. If $\mathcal{E}$ satisfies C_p, then a satiation quasi-equilibrium exists.

Proof. The purely atomic part of A is noted A^{pa} and the non-atomic part of A is noted A^{na}. Under Assumption C'$_p$, for almost every $a \in A^{na}$, there exists an ordered binary relation $\tilde{P}(a)$ on $X(a)$ such that $P(a) \subset \tilde{P}(a)$. We let, for every $a \in A^{pa}$, $\tilde{P}(a) := P(a)$. We define the correspondence $\tilde{R}$ from A into $\mathbb{L} \times \mathbb{L}$ by, for almost every $a \in A$, $\tilde{R}(a) := \{(z, z') \in X(a) \times X(a) \mid (z', z) \notin \tilde{P}(a)\}$.

In order to use the same *limit* argument as Claim 5.1, we define preferences satisfying the *convex property*. This construction is borrowed from Hildenbrand [16]. We let, for each $a \in A$, $\hat{X}(a) := \mathrm{co}\, X(a)$ and we define $\hat{P} : A \to \mathbb{L} \times \mathbb{L}$ by, for almost every $a \in A$,

$$\hat{P}(a) := \{(x, x') \in \hat{X}(a) \times \hat{X}(a) \mid x' \in X(a) \quad \text{and} \quad x \notin \mathrm{co}\, \tilde{R}_a(x')\}.$$

Note that for all $a \in A^{pa}$, $\hat{X}(a) = X(a)$ and $\hat{P}(a) = P(a)$. For almost every $a \in A^{na}$, the preferences $\tilde{P}(a)$ have open lower sections, it follows that for almost every $a \in A^{na}$, for each $y \in \hat{X}(a)$, $\hat{P}_a^{-1}(y)$ is open in $\hat{X}(a)$. Moreover, the binary relation $\tilde{R}(a)$ is a complete pre-order on $X(a)$. We check then, that for almost every $a \in A$, $\hat{P}(a)$ satisfies the following convex property,

$$\forall x \in \hat{X}(a) \quad x \notin \mathrm{co}\, \hat{P}_a(x).$$

We are now ready to construct the sequence of economies with a finite set of consumers. For all $n \in \mathbb{N}$, we note $\mathcal{E}^n$ the following *finite* economy $\mathcal{E}^n = (\langle \mathbb{L}^*, \mathbb{L}\rangle, (X^n_i, Y^n_i, P^n_i)_{i\in I^n})$ where $I^n := \{i \in S^n \mid \mu(A^n_i) \neq 0\}$ is the finite set of consumers. The consumption set of the consumer $i \in I^n$ is given by $X^n_i := \mu(A^n_i)\hat{X}(a^n_i)$ and the production set is given by $Y^n_i := \mu(A^n_i)\,[Y(a^n_i) + (1/n)\overline{B}]$, where $\overline{B}$ is the closed unit ball in $\mathbb{L}$. Preferences are defined by the binary relation $P^n_i := \mu(A^n_i)\hat{P}(a^n_i)$. For all $n \in \mathbb{N}$, the economy $\mathcal{E}^n$ satisfies the assumptions of Theorem 2.1.1 in Oiko Nomia [21]. It follows that, for all

[21] For each $y \in \mathbb{L}$ and $r > 0$, we let $\overline{B}(y, r) = \{z \in \mathbb{L} \mid d(z, y) \leqslant r\}$.

$n \in \mathbb{N}$, there exists

$$((x_i^n)_{i\in I^n}, (y_i^n)_{i\in I^n}, p^n) \in \prod_{i\in I^n} X_i^n \times \prod_{i\in I^n} Y_i^n \times \mathbb{L}^* ,$$

satisfying $\|p^n\| = 1$, $\sum_{i\in I^n} x_i^n = \sum_{i\in I^n} y_i^n$ and for all $i \in I^n$, if $(x, y) \in P_i^n(x_i^n) \times Y_i^n$ then $p^n(x - y) \geqslant 0$. For all $n \in \mathbb{N}$, for all $i \in I^n$, there exists $\xi_i^n \in \overline{B}$ such that $y_i^n - (\mu(A_i^n)/n)\xi_i^n \in \mu(A_i^n)Y(a_i^n)$. For all $n \in \mathbb{N}$, we let $\xi^n := \sum_{i\in I^n} \mu(A_i^n)\xi_i^n \in \mu(A)\overline{B}$ and

$$x^n := \sum_{i\in I^n} \frac{x_i^n}{\mu(A_i^n)}\chi_{A_i^n} \quad \text{and} \quad y^n := \sum_{i\in I^n} \left(\frac{y_i^n}{\mu(A_i^n)} - \frac{1}{n}\xi_i^n\right)\chi_{A_i^n}.$$

For each $n \in \mathbb{N}$, we have defined integrable selections $x^n \in S^1(X^n)$ and $y^n \in S^1(Y^n)$ satisfying

$$\int_A x^n(a)d\mu(a) = \int_A y^n(a)d\mu(a) + (1/n)\xi^n \tag{3}$$

and for all $a \in \bigcup_{i\in I^n} A_i^n$,

$$(x, z) \in P_a^n(x^n(a)) \times (Y^n(a) + (1/n)\overline{B}) \Rightarrow p^n(x) \geqslant p^n(z). \tag{4}$$

Since, for all $n \in \mathbb{N}$, $\|p^n\| = 1$, there exists a subsequence of $(p^n)_{n\in\mathbb{N}}$ converging to p^*, with $\|p^*\| = 1$. For all $a \in A$, we let $B(a) = \{x \in X(a) \mid p^*(x) \leqslant \sup p^*(Y(a))\}$ and $\beta(a) = \{x \in X(a) \mid p^*(x) < \sup p^*(Y(a))\}$. We define the correspondences D, G and H by, for all $a \in A$, $D(a) := \{x \in B(a) \mid P_a(x) \cap B(a) = \emptyset\}$, $G(a) := \{x \in X(a) \mid P_a(x) \cap B(a) = \emptyset\}$ and $H(a) := \{x \in X(a) \mid P_a(x) \cap \beta(a) = \emptyset\}$. When replacing X by $\hat{X}$ and P by $\hat{P}$, we define $\hat{G}$. Moreover, for each $n \in \mathbb{N}$, when replacing X by X^n, P by P^n, Y by Y^n and p^* by p^n, we define $B^n(a)$, $\beta^n(a)$, $D^n(a)$, $G^n(a)$. Similarly when replacing P^n by $\tilde{P}^n$, we define $\tilde{D}^n$ and $\tilde{G}^n$. We define $\hat{G}^n$ when X^n is replaced by $\hat{X}^n$ and P^n by $\hat{P}^n$. We assert that for all $n \in \mathbb{N}$,

$$\forall a \in A^{na} \quad \hat{G}^n(a) \subset \operatorname{co}[\tilde{G}^n(a)] \subset \operatorname{co}[G^n(a)] \tag{5}$$

and for all $a \in A^{pa}$, $\hat{G}^n(a) = \tilde{G}^n(a) = G^n(a)$. Indeed, if $a \in A^{pa}$ then $\hat{P}^n(a) = P^n(a)$ and the result follows. Now let $a \in A^{na}$ and $x \in \hat{G}^n(a)$. The set $X^n(a)$ is compact, the strict-preference relation $\tilde{P}^n(a)$ is irreflexive, transitive with open lower sections. Hence, following a classical maximal argument, the set $\tilde{D}^n(a)$ is non-empty. Let $\tilde{x} \in \tilde{D}^n(a)$, then $\tilde{x} \in B^n(a)$, and since $x \in \hat{G}^n(a)$, we have that $(x, \tilde{x}) \notin \hat{P}^n(a)$, that is, $x \in \operatorname{co}\tilde{R}_a^n(\tilde{x})$. Since $\tilde{R}^n(a)$ is transitive and complete and $\tilde{x} \in \tilde{D}^n(a)$, it is straightforward to verify that $\tilde{R}_a^n(\tilde{x}) \subset \tilde{G}^n(a) \subset G^n(a)$, and thus $x \in \operatorname{co}[G^n(a)]$.

Since (x^n, p^n) satisfies (4), it follows[22] that for almost every $a \in A$, $x^n(a) \in \hat{G}^n(a) \subset \operatorname{co} G^n(a)$. Following (i) of Fact 5.1, the sequences $(x^n)_{n\in\mathbb{N}}$ and $(y^n)_{n\in\mathbb{N}}$ are integrably bounded. Applying Theorem A.1, there exist integrable functions $x^*, y^* : A \to \mathbb{L}$, such that $\int_A x^* d\mu = \lim_{n\to\infty} \int_A x^n d\mu$ and $\int_A y^* d\mu = \lim_{n\to\infty} \int_A y^n d\mu$. Moreover for a.e. $a \in A$, $x^*(a) \in \operatorname{Ls}\{x^n(a)\}$ and $y^*(a) \in \operatorname{Ls}\{y^n(a)\}$. Following almost verbatim the arguments of Claim 5.1, we prove that (x^*, y^*) lies in $\in \mathrm{S}^1(X) \times \mathrm{S}^1(Y)$. Moreover, with (3) we get that $\int_A x^* d\mu = \int_A y^* d\mu$. Once again, following verbatim the arguments of Claim 5.2, we prove that for almost every $a \in A$, $\operatorname{Ls}(H^n(a)) \subset H(a)$. Applying Carathéodory Convexity Theorem, for almost every $a \in A$,

$$\operatorname{Ls}(\operatorname{co}(H^n(a))) \subset \operatorname{co}\operatorname{Ls}(H^n(a)) \subset \operatorname{co} H(a).$$

It follows[23] that for a.e. $a \in A$, if $a \in A^{na}$ then $x^*(a) \in \operatorname{co} H(a)$, and if $a \in A^{pa}$ then $x^*(a) \in H(a)$. The correspondence β is graph measurable with open values (in $X(a)$), it

[22] This is the reason why we introduce the unit ball $\overline{B}$ in the definition of Y_i^n.

[23] Note that for a.e. agent $a \in A$, $x^n(a) \in \operatorname{co} G^n(a) \subset \operatorname{co} H^n(a)$.

follows from Assumption M' that the correspondence H is graph measurable. We apply now the Lyapunov Theorem,

$$\int_A y^* \in \int_{A^{na}} \mathrm{co}\,[H(a)]d\mu(a) + \int_{A^{pa}} H(a)d\mu(a) = \int_A H(a)d\mu(a).$$

That is, there exists $\bar{x} \in \mathrm{S}^1(X)$ such that for almost every agent $a \in A$, $\bar{x}(a) \in H(a)$ and $\int_A \bar{x} \in Y_\Sigma$. It follows that $(\bar{x}, p^*)$ is a satiation quasi-equilibrium of the economy $\mathcal{E}$. □

The end of the proof of Lemma 5.2 is a direct consequence of Claims 5.1 and 5.2. □

5.4. **Proof of Lemma 5.1.** Let $\mathcal{E}$ be an economy satisfying Assumptions C, M, P, S, B and FD. In order to apply Lemma 5.2, we are led to truncate economies such that consumption and production sets correspondences are integrably bounded.

Claim 5.3. There exists $\bar{x} \in \mathrm{S}^1(X)$ and $\bar{y} \in \mathrm{S}^1(Y)$ such that for a.e. $a \in A$, $\bar{x}(a) = e(a) + \bar{y}(a)$.

Proof. We let $F : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined for all $a \in A$ by $F(a) := X(a) \cap (\{e(a)\} + Y(a))$. The correspondence F is graph measurable with non-empty and closed values. Thus, applying Proposition A.1, there exist $\bar{x} \in \mathrm{S}(X)$ a measurable selection of X and $\bar{y} \in \mathrm{S}(Y)$ a measurable selection of Y such that $\bar{x}(a) = \bar{y}(a)$. We propose to prove that both functions $\bar{x}$ and $\bar{y}$ are integrable. Since Y_Σ is non-empty, there exists $\underline{y} \in \mathrm{S}^1(Y)$. For each $n \in \mathbb{N}$, we let $A^n := \{a \in A \mid \|\bar{y}(a)\| \leqslant n\}$ and we let the function $y^n : A \to \mathbb{L}$ defined by $y^n(a) := \bar{y}(a)$ if $a \in A^n$, and $y^n(a) = \underline{y}(a)$ elsewhere. The function y^n is an integrable selection of Y, that is, $y^n \in \mathrm{S}^1(Y)$. For each $n \in \mathbb{N}$, we let $u^n := \int_A y^n(a)d\mu(a)$ and we check that

$$u^n \in Y_\Sigma \cap \left(\left\{ \int_A \inf(\underline{x}(a), \underline{y}(a))d\mu(a) \right\} + \mathbb{L}_+ \right).$$

Following Assumption B, $A(Y_\Sigma) \cap \mathbb{L}_+ = \{0\}$ and it follows that the sequence $(u^n)_{n\in\mathbb{N}}$ is bounded. We can suppose (extracting a subsequence if necessary) that $(u^n)_{n\in\mathbb{N}}$ is convergent to $u^* \in Y_\Sigma$. Applying Theorem A.2, there exists an integrable function $\hat{y} : A \to \mathbb{L}$, such that $\int_A \hat{y} d\mu \leqslant u^*$ and for a.e. $a \in A$, $\hat{y}(a) \in \mathrm{Ls}\,\{y^n(a)\}$. Since for a.e. $a \in A$, the sequence $(y^n(a))_{n\in\mathbb{N}}$ converges to $\bar{y}(a)$, it follows that $\hat{y} = \bar{y}$. □

We are now ready to construct a sequence of bounded economies. For each $n \in \mathbb{N}$, we let $\mathcal{E}^n$ be the economy $\mathcal{E}^n := ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X^n, Y^n, P^n, e))$, where for all $a \in A$, $X^n(a) := X(a) \cap K_{\bar{x}}(a, n)$, $Y^n(a) := Y(a) \cap K_{\bar{y}}(a, n)$ and $P^n(a) := P(a) \cap (X^n(a) \times X^n(a))$ with $K_z(a, n) := \{x \in \mathbb{L} \mid \|x\| \leqslant \max(\|z(a)\|, n)\}$, for each integrable function $z : A \to \mathbb{L}$.

For each $n \in \mathbb{N}$, $\mathcal{E}^n$ satisfies Assumptions C', M', P, S and IB of Lemma 5.2. It follows that for each $n \in \mathbb{N}$, there exists $(x^n, p^n) \in \mathrm{S}^1(X) \times \mathbb{L}^*$ with $\|p^n\| = 1$ satisfying $v^n := \int_A x^n \in X_\Sigma \cap (\{\omega\} + Y_\Sigma)$ and such that there exists $A^n \subset A$ with $\mu(A \setminus A^n) = 0$, with for all $a \in A^n$,

$$(x, y) \in P_a^n(x^n(a)) \times Y^n(a) \Longrightarrow p^n(x) \geqslant p^n(e(a)) + p^n(y). \tag{6}$$

We can thus suppose (extracting a subsequence if necessary) that $(p^n)_{n\in\mathbb{N}}$ converges to $p^* \in \mathbb{L}^*$ with $\|p^*\| = 1$. Applying Assumption B, we can (extracting a subsequence if necessary) as well assume that the sequence $(v^n)_{n\in\mathbb{N}}$ converges to $v^* \in \{\omega\} + Y_\Sigma$. Applying Theorem A.2, there exists an integrable function $x^* : A \to \mathbb{L}$, such that $\int_A x^* d\mu \leqslant v^*$ and $x^*(a) \in \mathrm{Ls}\,\{x^n(a)\}$. Since for a.e. $a \in A$, $X(a)$ is closed, we have that $x^* \in \mathrm{S}^1(X)$, and thus $\int_A x^* - \omega \in Y_\Sigma - \mathbb{L}_+$. Now two cases may occur, production sets are free-disposal (Assumption FD (a)) or preferences are weakly monotone (Assumption FD (b)). We deal with the first situation since the proof of the other one is similar and classic. Assume thus that the total production set is free-disposal, that is, $-\mathbb{L}_+ \subset A(Y_\Sigma)$. It follows that there exists $y^* \in \mathrm{S}^1(Y)$ such that

$$\int_A x^* = \omega + \int_A y^*.$$

We propose to prove that (x^*, y^*, p^*) is a satiation quasi-equilibrium of $\mathcal{E}$. Condition (ii) of Definition 5.1 is already proved. We will now prove condition (i), that is, for almost every $a \in A$,

$$(x, y) \in P_a(x^*(a)) \times Y(a) \Longrightarrow p^*(x) \geqslant p^*(e(a)) + p^*(y).$$

Let $a \in A \setminus (\cup_{n\in\mathbb{N}} A^n)$ be such that $P_a(x^*(a)) \neq \emptyset$ and let $(x, y) \in P_a(x^*(a)) \times Y(a)$. For all n large enough, $x^*(a) \in X^n(a)$ and $(x, y) \in P^n_a(x^*(a)) \times Y^n(a)$. We may assume (extracting a subsequence if necessary) that $(x^n(a))_{n\in\mathbb{N}}$ converges to $x^*(a)$. Since P_a is lower semi-continuous, applying (6) we get that $p^*(x) \geqslant p^*(e(a)) + p^*(y)$.

Appendix A. Measurability and integration of correspondences

We consider $(A, \mathcal{A}, \mu)$ a measure space and (D, d) a complete separable metric space.

A.1. **Measurability of correspondences.** A correspondence (or a multifunction) $F : A \twoheadrightarrow D$ is *measurable* if for all open set $G \subset D$, $F^-(G) = \{a \in A \mid F(a) \cap G \neq \emptyset\} \in \mathcal{A}$. The correspondence F is said to be *graph measurable* if $\{(a, x) \in A \times D \mid x \in F(a)\} \in \mathcal{A} \otimes \mathcal{B}(D)$. A function $f : A \to D$ is a *measurable selection* of F if f is measurable and if, for almost every $a \in A$, $f(a) \in F(a)$. The set of measurable selections of F is noted $S(F)$.

Following Castaing and Valadier [8] and Himmelberg [17], we recall the two following classical characterizations of measurable correspondences.

Proposition A.1. *Consider $F : A \twoheadrightarrow D$ a correspondence with non-empty closed values. The following properties are equivalent.*

- **(i)** *The correspondence F is measurable.*
- **(ii)** *There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable selections of F such that for all $a \in A$, $F(a) = cl\{f_n(a) \mid n \in \mathbb{N}\}$.*
- **(iii)** *For each $x \in D$, the function $\delta_F(., x) : a \mapsto d(x, F(a))$ is measurable.*

Proposition A.2. *Consider $F : A \twoheadrightarrow D$ a correspondence.*

- **(i)** *If F has non-empty closed values then the measurability of F implies the graph measurability of F.*
- **(ii)** *If $(A, \mathcal{A}, \mu)$ is complete then the graph measurability of F implies the measurability of F.*
- **(iii)** *If F has non-empty closed values and $(A, \mathcal{A}, \mu)$ is complete then measurability and graph measurability of F are equivalent.*

Following Aumann [4], graph measurable correspondences (that are not supposed to have closed values) have measurable selections.

Proposition A.3. *Consider F a graph measurable correspondence from A into D with non-empty values. If $(A, \mathcal{A}, \mu)$ is complete then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of measurable selections of F, such that for all $a \in A$, $(z_n(a))_{n\in\mathbb{N}}$ is dense in $F(a)$.*

A.2. **Measurability of preferences.** Let P be a correspondence from A into $D \times D$. For each function $x : A \to D$ the *upper section relatively to* x is noted $P_x : A \twoheadrightarrow D$ and is defined by $a \mapsto \{y \in D \mid (x(a), y) \in P(a)\}$. For each function $y : A \to D$ the *lower section relatively to* y is noted $P^y : A \twoheadrightarrow D$ and is defined by $a \mapsto \{x \in D \mid (x, y(a)) \in P(a)\}$.

Let $X : A \twoheadrightarrow D$ be a correspondence. A *correspondence of preferences in* X is a correspondence P from A into $D \times D$ satisfying for all $a \in A$, $P(a) \subset X(a) \times X(a)$. For each $a \in A$, we note P_a the correspondence[24] from $X(a)$ into $X(a)$ defined by $x \mapsto \{y \in X(a) \mid (x, y) \in P(a)\}$. For each $y \in X(a)$ the lower inverse image of y by P_a

[24] Remark that the graph of P_a and $P(a)$ coincide.

is noted $P_a^{-1}(y) = \{x \in X(a) \mid y \in P_a(x)\}$. The correspondence of preferences P is graph measurable if

$$\{(a,x,y) \in A \times D \times D \mid (x,y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D) \otimes \mathcal{B}(D).$$

The correspondence of preferences P in X is *Aumann measurable* if

$$\forall (x,y) \in S(X) \times S(X) \quad \{a \in A \mid (x(a), y(a)) \in P(a)\} \in \mathcal{A}.$$

The correspondence of preferences P in X is *lower graph measurable* if for all measurable selection y of X, the correspondence P^y is graph measurable, that is

$$\forall y \in S(X) \quad G_{P^y} = \{(a,x) \in A \times D \mid (x, y(a)) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D).$$

The correspondence of preferences P in X is *upper graph measurable* if for all measurable selection x of X, the correspondence P_x is graph measurable, that is

$$\forall x \in S(X) \quad G_{P_x} = \{(a,y) \in A \times D \mid (x(a), y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D).$$

We propose to compare these three concepts of measurability of preferences.

Proposition A.4. *Let P be a correspondence of preferences in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph. Then the graph measurability of P implies the lower and upper graph measurability of P, and lower or upper graph measurability of P implies the Aumann measurability of P.*

Proof. This is a direct consequence of Projection Theorem in Castaing and Valadier [8]. □

Under additional assumptions, the converse is true.

Proposition A.5. *Let P be a correspondence of preferences in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph. Moreover, we suppose that for a.e. $a \in A$, $X(a)$ is a closed connected subset of D, $P(a)$ is an irreflexive and transitive binary relation on $X(a)$ and for each $x \in X(a)$, $P_a(x)$ and $P_a^{-1}(x)$ are open in $X(a)$. If one of the two following properties holds,*

1. *for a.e. $a \in A$, $X(a) = (\mathbb{R}_+)^\ell$ where*[25] *$D = \mathbb{R}^\ell$ and $P(a)$ is strictly monotone* [26]*,*
2. *for a.e. $a \in A$, $P(a)$ is negatively transitive,*

then the Aumann measurability of P implies the upper and lower graph measurability of P, and the lower and upper graph measurability of P implies the graph measurability of P.

Remark A.1. In Aumann [3] and Schmeidler [24], property 1 is satisfied. In Hildenbrand [15], for all $a \in A$, $P(a)$ is ordered and then property 2 is satisfied.

Proof. Suppose that P is Aumann measurable. We distinguish two cases. Under Property 1, $(\mathbb{Q}_+)^\ell$ is dense in $X(a)$ for all $a \in A$, hence if $(x,y) \in P(a)$ then there exists $r \in (\mathbb{Q}_+)^\ell$ such that $(x,r) \in P(a)$ and $r < y$. It follows that, if $x \in S(X)$ is a measurable selection of X, then $G_{P_x} = \bigcup_{r \in \mathbb{Q}_+^\ell} G_1(r) \cap G_2(r) \in \mathcal{A} \times \mathcal{B}(\mathbb{R}^\ell)$, where $G_1(r) := \{(a \in A \mid (x(a), r) \in P(a)\} \times (\mathbb{R}_+)^\ell$ and $G_2(r) := A \times \{y \in D \mid r < y\}$. Similarly we can prove that $G_{P^x} \in \mathcal{A} \times \mathcal{B}(\mathbb{R}^\ell)$.

Under Property 2, to prove that P is both upper and lower graph measurable, we can follow almost verbatim the proof of Lemma in Appendix in Podczeck [22]. The graph of X is measurable, then Proposition A.2 implies that X has a Castaing representation, that is there exists a sequence $(h_i)_{i \in \mathbb{N}}$ of measurable selections of X, such that for all $a \in A$, $X(a) = \operatorname{cl}\{h_i(a) \mid i \in \mathbb{N}\}$. Now suppose that a measurable selection $x \in S(X)$ has been given. Consider any $a \in A$ and let $y \in X(a)$. If $(x(a), y) \in P(a)$, then following Debreu [11], there exists $i \in \mathbb{N}$ such that $(x(a), h_i(a)) \in P(a)$ and $(h_i(a), y) \in P(a)$. By the continuity of $P(a)$, for each $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $d(y, h_j(a)) \leqslant 1/n$ and $(h_i(a), h_j(a)) \in P(a)$. Conversely, if for some $i \in \mathbb{N}$, $(x(a), h_i(a)) \in P(a)$ and for each $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $d(y, h_j(a)) \leqslant 1/n$ and $(h_i(a), h_j(a)) \in P(a)$, then $y \in \operatorname{cl} P_a(h_i(a)) \subset P_a(x(a))$. It follows that

$$G_{P_x} = G_X \cap \bigcup_{i \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} [(A(i,j) \times D) \cap G_j] \ ,$$

[25] For some integer $\ell \in \mathbb{N}$.

[26] That is for all $x \in X(a)$, for all $m \in (\mathbb{R}_+)^\ell$, $x + m \in P_a(x) \cup \{x\}$.

where $A(i,j) = \{a \in A \mid (x(a), h_i(a)) \in P(a)\} \cap \{a \in A \mid (h_i(a), h_j(a)) \in P(a)\}$ and $G_j := \{(a,y) \in A \times D \mid d(a, h_j(a)) \leqslant 1/n\}$. Since P is Aumann measurable, for each $(i,j) \in \mathbb{N}^2$, $A(i,j) \in \mathcal{A}$. Finally following [8] or [17], for each $(j,n) \in \mathbb{N}^2$, $\{(a,y) \in A \times D \mid d(a, h_j(a)) \leqslant 1/n\} \in \mathcal{A} \times \mathcal{B}(D)$, and P is upper graph measurable. Similarly we prove that P is lower graph measurable.

Suppose now that P is both upper and lower graph measurable. Let $(a,x,y) \in G_P$, that is $(x,y) \in P(a)$. We distinguish two cases. Under property 2 there exists $i \in \mathbb{N}$ such that $(x, h_i(a)) \in P(a)$ and $(h_i(a), y) \in P(a)$. Since $P(a)$ is transitive, the converse is true, and G_P coincide with

$$\bigcup_{i \in \mathbb{N}} \left\{ (a,x,y) \in A \times D \times D \mid (a,x) \in G_{P^{h_i}} \quad \text{and} \quad (a,y) \in G_{P_{h_i}} \right\}.$$

It follows that P is graph measurable.

Under property 1 there exists $r \in (\mathbb{Q}_+)^\ell$ such that $(x,r) \in P(a)$ and $r < y$. Since preferences are monotone the converse is true and

$$G_P = \bigcup_{r \in (\mathbb{Q}_+)^\ell} G_{P^r} \cap \{(a,x,y) \in A \times \mathbb{R}^\ell \times \mathbb{R}^\ell \mid r < y\}.$$

It follows that P is graph measurable. □

We recall that the correspondence P_a is lower semi-continous if for all open set $V \subset D$, $\{x \in X(a) \mid P_a(x) \cap V \neq \emptyset\}$ is open in $X(a)$.

We introduce a notion of measurability of preferences, close to the notion of lower semi-continuity.

Definition A.1. The correspondence of preferences P in X is lower semi-graph measurable if for all graph measurable correspondence $V : A \twoheadrightarrow D$ with open values, the following set is measurable

$$\{(a,x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} \in \mathcal{A} \times \mathcal{B}(D).$$

We propose to compare this measurability notion with the other notions introduced before.

Proposition A.6. *Let P be a correspondence of preferences in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph.*

- **(i)** *The graph measurability of P implies the lower semi-graph measurability of P.*
- **(ii)** *If for a.e. $a \in A$, for all $x \in X(a)$, $P_a(x)$ is open in $X(a)$, then the lower graph measurability of P implies the lower semi-graph measurability of P.*
- **(iii)** *If for a.e. $a \in A$, for all $x \in X(a)$, $P_a(x)$ is closed in $X(a)$, then the lower semi-graph measurability of P implies the lower graph measurability of P.*

Proof. The part (i) is a direct consequence of Projection Theorem in Castaing and Valadier [8]. Indeed, the set $\{(a,x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\}$ coincide with

$$\pi \left[G_P \cap \{(a,x,y) \in A \times D \times D \mid y \in V(a)\} \right],$$

where $\pi : A \times D \times D \to A \times D$ is the projection $(a,x,y) \mapsto (a,x)$.

Suppose now that P is lower graph measurable and that for a.e. $a \in A$, for all $x \in X(a)$, $P_a(x)$ is open in $X(a)$. Let $(a,x) \in G_X$ such that $P_a(x) \cap V(a) \neq \emptyset$. Following Proposition A.3, there exists a sequence $(z_n)_{n \in \mathbb{N}}$ of measurable selections of X, such that for all $a \in A$, $(z_n(a))_{n \in \mathbb{N}}$ is dense in $X(a)$. The set $P_a(x) \cap V(a)$ is open in $X(a)$, it follows that there exists $n \in \mathbb{N}$ such that $z_n(a) \in P_a(x) \cap V(a)$. The converse is true and then the set $\{(a,x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\}$ coincide with

$$\bigcup_{n \in \mathbb{N}} \left[G_{P^{z_n}} \cap (\{a \in A \mid z_n(a) \in V(a)\} \times D) \right].$$

That is P is lower semi-graph measurable.

Suppose now that the correspondence P is lower semi-graph measurable and that for a.e. $a \in A$, for all $x \in X(a)$, $P_a(x)$ is closed in $X(a)$. Let $y \in S(X)$ be a measurable selection of X. Let $(a,x) \in G_{P^y}$, that is, $x \in X(a)$ and $y \in P_a(x)$. Let $n \in \mathbb{N}$, and consider $V_n(a) := \{z \in D \mid d(z, y(a)) < 1/(n+1)\}$. Then for all $n \in \mathbb{N}$, $P_a(x) \cap V_n(a) \neq \emptyset$. Conversely, is for all $n \in \mathbb{N}$,

$P_a(x) \cap V_n(a) \neq \emptyset$, then $y(a) \in \mathrm{cl}\, P_a(x)$. Since $P_a(x)$ is closed in $X(a)$, then $(a,x) \in G_{P^y}$. Thus

$$G_{P^y} = \bigcap_{n\in\mathbb{N}} \{(a,x) \in G_X \mid P_a(x) \cap V_n(a) \neq \emptyset\}.$$

And the correspondence P si lower graph measurable. □

A.3. **Integration of correspondences.** We suppose in this section that $(A, \mathcal{A}, \mu)$ is a finite complete measure space. If $F : A \twoheadrightarrow \mathbb{L}$ is a correspondence from A to $\mathbb{L}$, the set of integrable selections of F is noted $S^1(F)$. We note F_Σ the following (possibly empty) set $F_\Sigma := \int_A F(a)d\mu(a) := \left\{v \in \mathbb{L} \;\middle|\; \exists x \in S^1(F) \quad v = \int_A x(a)d\mu(a)\right\}$.

Proposition A.7. *Consider $F : A \twoheadrightarrow \mathbb{L}$ a graph measurable correspondence. If F_Σ is non-empty, we let $G : A \twoheadrightarrow \mathbb{L}$ the correspondence defined by*

$$\forall a \in A \quad G(a) := cl\left[\overline{co}\, F(a) + A(F_\Sigma)\right].$$

If F_Σ is non-empty, closed and convex then $G_\Sigma = F_\Sigma$ and for all $p \in \mathbb{L}^*$, if there exists an integrable selection g^* of G such that for a.e. $a \in A$, $p\,(g^*(a)) = \sup p\,(G(a))$, then there exists an integrable selection f^* of F satisfying for a.e. $a \in A$, $p\,(f^*(a)) = \sup p\,(F(a))$ and $\int_A f^* = \int_A g^*$.

Proof. Since $(A, \mathcal{A}, \mu)$ is complete, following Proposition A.2, the correspondence F is measurable. Following Rockafellar and Wets [23], the correspondence G is measurable with closed-values. Once again applying Proposition A.2, G is graph measurable and $F_\Sigma \subset G_\Sigma$. Moreover if $p \in \mathbb{L}$ then for all $a \in A$, $\sup p\,(G(a)) = \sup p\,(F(a)) + \sup p\,(A(F_\Sigma))$. Note that, since $A(F_\Sigma)$ is a cone containing zero, $\sup p\,(A(F_\Sigma)) \in \{0, +\infty\}$.

Suppose now that that F_Σ is non-empty, closed and convex, and suppose that there exists $v \in G_\Sigma$ such that $v \notin F_\Sigma$. Since F_Σ is closed convex, by a separation argument there exists $p \in \mathbb{L} \setminus \{0\}$ such that $p\,(v) > \sup p\,(F_\Sigma)$. It follows that $\sup p\,(A(F_\Sigma)) = 0$ and following Theorem C in Hildenbrand [15],

$$\sup p\,(F_\Sigma) = \int_A \sup p\,(F(a))\, d\mu(a) = \int_A \sup p\,(G(a))\, d\mu(a) = \sup p\,(G_\Sigma)\,.$$

Thus $p\,(v) > \sup p\,(G_\Sigma)$ and this contradicts the fact that $v \in G_\Sigma$. The rest of the proof of Proposition A.7 is a direct consequence of this result. □

We are now ready to present two versions of Fatou's Lemma in several dimensions. The first one is due to Artstein [1].

Theorem A.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions from A to $\mathbb{L}$, integrably bounded and such that $\lim_{n\to\infty} \int_A f_n$ exists. Then there exists an integrable function f from A to $\mathbb{L}$ such that*

$$\int_A f = \lim_{n\to\infty} \int_A f_n \quad \text{and} \quad \text{for a.e. } a \in A \quad f(a) \in Ls\{f_n(a)\}.$$

The second one is due to Cornet and Topuzu [10]. This version of Fatou's Lemma generalizes a version of Schmeidler [25] to more general positive cones.

Theorem A.2. *Let $C \subset \mathbb{L}$ be a pointed closed convex cone. We note $\geqslant$ the partial order induced[27] by C. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions from A to $\mathbb{L}$ integrably bounded from below [28] and such that $\lim_{n\to\infty} \int_A f_n$ exists. Then there exists an integrable function f from A to $\mathbb{L}$ such that*

$$\int_A f \leqslant \lim_{n\to\infty} \int_A f_n \quad \text{and} \quad \text{for a.e. } a \in A \quad f(a) \in Ls\{f_n(a)\}.$$

For related results we refer to Balder [5] and Balder and Hess [7].

[27]For all $(x,y) \in \mathbb{L}^2$, $x \geqslant y$ whenever $x - y \in C$.

[28]That is, there exists an integrable function g such that for each $n \in \mathbb{N}$, for almost every $a \in A$, $f_n(a) \geqslant g(a)$.

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