A CONTRACTIVE METHOD FOR COMPUTING THE STATIONARY SOLUTION OF THE EULER EQUATION

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Abstract.

A contraction method for computing stationary solutions for intertemporal equilibrium models is defined. The method is implemented using a contraction mapping derived from the model's first-order conditions. A deterministic dynamic programming problem is used as a framework in which to illustrate the method. Some numerical examples are provided.

1. Introduction

In intertemporal economic models one of the main difficulties is to find accurate estimatives of the stationary solutions. For instance, in dynamic programming models the traditional method is based on the Bellman approach. This consists in estimating the corresponding value function using a contraction mapping (see Lucas and Stokey (1989)) and then computing the policy function from the approximated value function.

Taylor and Uhlig (1990) described some numerical methods based on the Bellman iterations (i.e., *Value-Function Grid* and *Quadrature Value-Function Grid* methods) and Santos and Vigo-Aguiar (1998) and Maldonado and Svaiter (2001) provided an estimation error for the approximate policy. However, Bellman's method has two main disadvantages. Firstly, it is only useful when the model can be expressed as a representative agent model and, besides that, speed of convergence is slow.

On the other hand, numerical methods based on the solution of the Euler equation have been more efficient in the two aspects above: they can be used even when there is

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no representative agent and they are faster when an adequate approximation scheme is performed (see Judd (1998)). The backward iteration algorithm provides a sequence of functions which converges pointwisely to the stationary solution. Baxter *et.al* (1990) used a discretized version of the original problem and by Santos and Vigo-Aguiar (2000) the distance between the solution of the discretized problem and the solution of the original one is of order one on the grid mesh. On the other hand, Coleman (1990, 1991) used a similar approach, however the discretization was not necessary since linear interpolations were performed. Other approaches of approximations arising from Euler equations are the projection methods (Judd (1992)) and the parameterizing-expectations method (Marcet (1988) and Marcet and Lorenzoni (1998)).

The goal of this paper is to provide an approximation method based on Euler equations to compute accurate estimatives of stationary solutions. It is defined by a contraction mapping in a functional space which has the stationary solution as its fixed point. This technique's major contribution is its capability to approximate stationary solutions in the topology of uniform convergence of continuously differentiable functions. An analogous method was used by Li (1998) for solving the monetary model she was analyzing. Here we are developing a general method which in particular requires a slightly weaker condition than the one presented in her work and the convergence we obtain is in the C^1-topology. Also it is worth noting that this method is not equivalent to the Bellman approach and it is not necessary to make a grid of the state variable set for solving the Euler equation.

To guarantee the convergence of our method we need a condition which is slightly stronger than the determinacy of the steady state condition. For the dynamic programming problem this amounts to the dominant diagonal condition.

The paper is divided as follows. In section 2 we present the deterministic model and the main hypotheses. In section 3 our Euler-equation contraction method is showed. Finally, section 4 implements the algorithm derived from this method and gives some applications. All the proofs are given in the appendix.

2. Basic Framework

The intertemporal equilibrium models that we are going to deal with are described by the following elements: $X \subset \Re^n$ is the state space, $D \subset X \times X \times X$ represents the intertemporal feasibility set and $E : D \to \Re^n$ is the function whose zeros define the temporary equilibria. We will assume that E is a twice continuously differentiable such that E_2 is negative definite on the interior of D.[1]

Let $|.|$ be one of the equivalent norms of $\Re^n$. The associated norm for the real square matrices space of order n ($\mathcal{M}_n$) will be $\| \, . \, \|$.[2] Finally, let $B_r(x) = \{y \in \Re^n \, ; \, |y - x| < r\}$.

An equilibrium path from $x_0 \in X$ is a sequence $(x_t)_{t \geq 0}$ such that

$$E(x_{t-1}, x_t, x_{t+1}) = 0$$

and a stationary solution for (X, D, E) is a function $g: X \to X$ such that:

$$E(x, g(x), g^2(x)) = 0$$

[1] Since E is a function of $(x_1, x_2, x_3) \in D$, E_j is the vector of partial derivatives of E with respect to x_j.

[2] $\|X\| = \sup_{\{x \in \Re^n \, ; \, |x|=1\}} |Xx|, \quad X \in \mathcal{M}_n$

for all $x \in X$.

We say that $\bar{x} \in \Re^n$ is a *steady state* if $g(\bar{x}) = \bar{x}$. We will make the following:

Assumption D. *There exists an interior steady state $\bar{x}$ and $\alpha \in \Re$ such that*[3]
(i) $\|(E_2)^{-1}E_1\| + \|(E_2)^{-1}E_3\| < \alpha < 1$
(ii)$\|(E_2)^{-1}E_3\| < 1/2$

Remark: The condition for the existence of a locally unique stationary equilibrium is that the steady state must be a saddle point of the linearization of $E = 0$. And a necessary and sufficient condition for this is:[4]

$$\|(E_2)^{-1}E_1 + (E_2)^{-1}E_3\| < 1$$

which is weaker than assumption D.

A basic example of this structure is the dynamic programming problem. Following the notation of Stokey and Lucas (1989),

$$E(x_{t-1}, x_t, x_{t+1}) = F_2(x_{t-1}, x_t) + \beta F_1(x_t, x_{t+1})$$

where F is the return function, β is the discount factor and the set D is defined from technological constraints. In this case g represents the policy function. It is easy to verify that for this model the assumption **D (i)** amounts to the dominant diagonal condition at the steady state.

3. Main Result

In this section we will provide an iterative method for computing stationary solutions in a neighborhood of a steady state for the model (X, D, E). This method consists in two stages: (i) defining an implicit map from the temporary equilibrium equation; (ii) showing that this map is a contraction with the stationary solution as the fixed point.

Given $r > 0$, $\gamma > 0$ and $\bar{x} \in \Re^n$ let us denote

$$\mathcal{H} = \{h \in C^2(B_r(\bar{x}))\,;\, h(\bar{x}) = \bar{x},\ \|Dh(x)\| \leq \alpha \text{ and } \|D^2h(x)\| \leq \gamma,\ \forall x \in B_r(\bar{x})\}$$

where $C^l(B_r(\bar{x}))$ is the space of l-th continuously differentiable functions from $B_r(\bar{x})$ into itself and γ is a constant.

Define the norm

$$\|h\|_1 = \sup_{x \in B_r(\bar{x})} \|Dh(x)\|\ , \text{ for all } h \in \mathcal{H}.$$

Let $\bar{\mathcal{H}}$ be the closure of $\mathcal{H}$ with respect to this norm. Therefore, $(\bar{\mathcal{H}}, \|\cdot\|_1)$ is a complete metric space. Observe that, by the definition of the metric (uniform convergence in the first derivative), it is easy to see that $\bar{\mathcal{H}}$ is a subset of $C^1(B_r(\bar{x}))$.

[3] The derivatives are evaluated at $(\bar{x}, \bar{x}, \bar{x})$. Observe that (i) implies (ii) when $n = 1$.

[4] The quadratic equation $x^2 + ax + b = 0$ has a root with modulus greater than one and the other lower than one if and only if $|a| > |1 + b|$.

Lemma 3.1. *Under* **D**, *there exist* $r > 0$, $\gamma > 0$ *and* $\varphi : \mathcal{H} \to \mathcal{H}$ *such that*

$$E(x, \varphi_h(x), h^2(x)) = 0 \tag{*}$$

for all $x \in B_r(\bar{x})$ *and* $h \in \mathcal{H}$.

Theorem 3.2. *Assume* **D**. *Then there exist* $r > 0$ *and* $\gamma > 0$ *such that* $\varphi: \mathcal{H} \to \mathcal{H}$ *defined in Lemma 3.1 is a* η*-contraction, for some* $\eta \in (0, 1)$.

The proof of Theorem 3.2 shows that the map $\varphi: \bar{\mathcal{H}} \to \bar{\mathcal{H}}$ is a $\eta-$contraction and consequently has a fixed point. It is easy to see that such a fixed point is a stationary solution of (X, D, E). Therefore, the map φ provides a recursive method for computing this solution in a neighborhood of the steady state.

The following corollary shows that the recursive method obtained from Theorem 3.2 holds in every neighborhood where assumption **D** is satisfied.

Corollary 3.3. *If the assumption* **D** *is satisfied in a convex neighborhood* $\mathcal{N}$ *of* $\bar{x}$, *then there exist* $\gamma > 0$ *and* $\varphi: \mathcal{H} \to \mathcal{H}$ *satisfying* $(*)$ *(where* $B_r(\bar{x})$ *is replaced by* $\mathcal{N}$*) and there exist* $N \geq 1$ *and* $\eta < 1$ *such that* φ^N *is a* η*-contraction with fixed point* g.

4. The Algorithm and Numerical Examples

In this section we will describe the algorithm derived from Theorem 3.2. But first we will discuss other methods in the literature.

Let us consider the dynamic programming problem. The first method consists in defining the following operator from the Euler equation: given a feasible map $h : X \to X$, let us define $Th : X \to X$ implicitly by

$$F_2(x, Th(x)) + \beta F_1(Th(x), h(Th(x))) = 0.$$

The optimal policy function g is a fixed point of T and the sequence $(T^n(h_0))_n$ converges to g pointwise, where h_0 is constant.[5] Indeed, we claim that this approach is equivalent to Bellman's method. To see this, define the following sequence of functions:

$$v_n(x) = \max_{\{y;\,(x,y)\in A\}} F(x, y) + \beta v_{n-1}(y)$$

for all $n \geq 1$, where A is the feasible set related with the problem and $v_0(x) = F(x, h_0(x))$. From the first order condition and the Envelope Theorem:

$$F_2(x, h_n(x)) + \beta F_1(h_n(x), h_{n-1}(h_n(x))) = 0,$$

[5] A constant function may not be feasible. In this case we have to choose, for instance, a piecewise constant h_0.

where $h_n(x) = \underset{\{y;\,(x,y)\in A\}}{\operatorname{argmax}} F(x,y) + \beta v_{n-1}(y)$. Using the definition of T and this last equation, it is easy to see that $h_n = T^n(h_0)$. Hence, the Bellman method implies that h_n converges to g. In particular, this method is equivalent to Bellman's one.

Baxter *et al.* (1990)) implements that method making a discretization of the state space, whereas Coleman (1990, 1991) used a linear interpolation in each step.

Li (1998) uses a similar method to ours: she defines the same mapping φ of Theorem 3.2 and proves that it is a contraction in the C^0 topology. In Example 2 below we show that our approach has the following advantages: (i) the set of economies where the contraction property holds is larger than hers; (ii) the convergence is in the C^1 topology and, in particular, the stationary solution is C^1.

The Algorithm

We now describe the main steps of the algorithm which implements our contraction method.[6] The main difference of our implementation with respect to the methods above is that it is not necessary to make a discretization of the state space. More precisely, we can give an accurate approximation of the stationary solution for each $x \in X$.

Let $h_0 : X \to X$ be a constant function (for instance, $h_0 \equiv \bar{x}$, the steady state). Fix $x \in X$.

computing h_1: solve

$$E(x, y, h_0^2(x)) = 0.$$

computing $h_2(x)$

first step: find $h_1^2(x)$, i.e., solve

$$E(h_1(x), y, h_0^2(h_1(x))) = 0.$$

second step: solve

$$E(x, y, h_1^2(x)) = 0$$

for finding $h_2(x)$.

In general, we have to proceed as follows:

computing $h_{n+1}(x)$

first step: find h_n^2. In order to do this, we have to compute, using the equilibrium equation, the following sequence

$$h_1 h_n,\ h_1^2 h_n,\ h_2 h_n,\ h_1 h_2 h_n,\ h_1^2 h_2 h_n,\ h_2^2 h_n,\ h_3 h_n, ..., h_{n-1}^2 h_n,\ h_n^2$$

where x was dropped in the notation.

[6]For the examples, we used MATLAB to implement the numerical routines. Upon request, we will provide the MATLAB code by E-mail. The E-mail address is humberto@fgv.br.

second step: solve

$$E(x, y, h_n^2(x)) = 0$$

for finding $h_{n+1}(x)$.

The Examples

1. Deterministic Growth Model

Consider the classical deterministic growth model with utility and production functions:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \text{ and } f(k) = Ak^{\alpha}$$

where $\gamma \geq 0$ (for $\gamma = 1$, $u(c) = \ln c$) and $0 < \alpha \leq 1$.

The Euler equation is given by:

$$E(x, y, z) = -(x^{\alpha} - y)^{-\gamma} + \beta\alpha y^{\alpha-1}(y^{\alpha} - z)^{-\gamma} = 0.$$

For $\alpha = 1$ and $\gamma = 1$ it is possible to determine the operator φ explicitly

$$\varphi_h(k) = \frac{h^2(k) + \beta k}{1+\beta}$$

and the optimal policy function $g(k) = \beta k^{\alpha}$ which is a fixed point of φ. It is important to note that this is not true for the remaining examples and, therefore, this justifies the use of the proposed algorithm.

Using the Mean Value Theorem, we have the following estimative:

$$\|\varphi_{h_1} - \varphi_{h_2}\| \leq \frac{1 + \|Dh_1\|}{1+\beta}\|h_1 - h_2\|,$$

where $\|\cdot\|$ is the sup norm. Taking the domain of φ as $h_1 \in C^1$ such that $\|Dh\| \leq M < \beta$ it results that φ is a contraction. It is important to note that our theorem guarantees a C^1-contraction probably in a smaller domain.

In figures 1 and 2 we show the numerical results of our method in two particular specification of the parameters: $\beta = .95$ and domain $[.05; 1]$. The initial function is $h_0(k) = \min\,\{\bar{k}, k\}$.

In figure 1, $u(c) = \ln(c)$, $\alpha = .34$ and the optimal policy function is the continuous line and the iterations h_1 and h_10 (which is close to g) are the dotted lines (the distance between h_9 and h_{10} is 1.349×10^{-7}). In figure 2, if $\gamma = .2$ and $\alpha = .5$, then the true optimal policy is not known and we only show the iterations h_1 through h_9 in dotted line and h_{10} in continuous line (the distance between h_{10} and g is 1.087×10^{-9}).

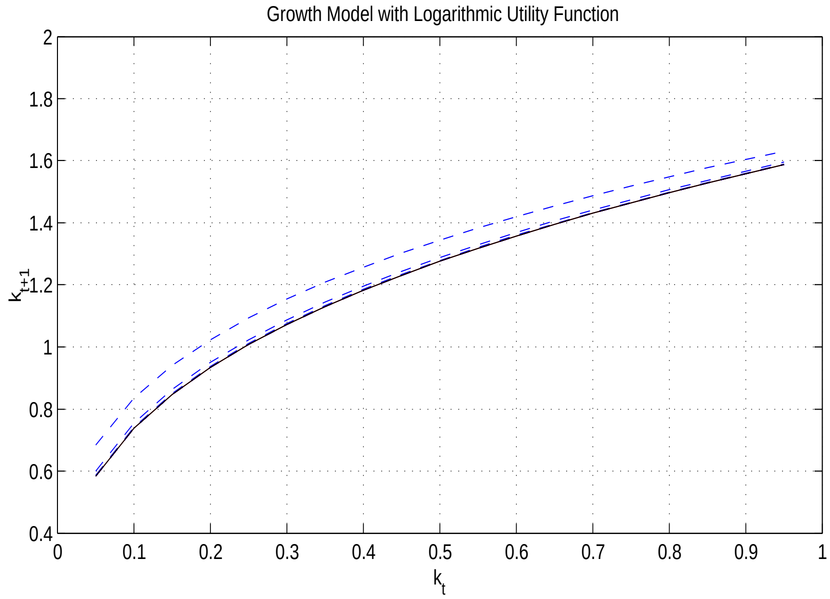

FIGURE 1: Deterministic growth model with $u(c) = \ln c$, $A = 5$, $\alpha = .34$, $\beta = .95$ and iterations$= 10$.

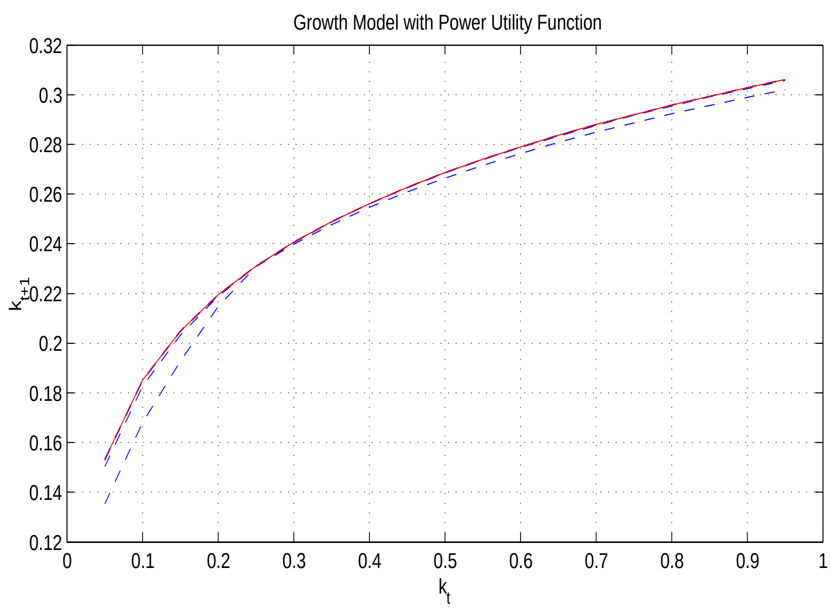

FIGURE 2: Deterministic growth model with $\gamma = .2$, $A = 5$, $\alpha = .5$, $\beta = .95$ and iterations$= 10$.

2. A Monetary Model (Li (1998))

Li (1998) presents a monetary model with preferences given by $U(c) = c^{1-A}/(1-A)$ ($A \geq 0$ and $A \neq 1$) and discount factor $0 < \beta < 1$. The intertemporal equilibrium equation for this model is given by:

$$E(p_{t-1}, p_t, p_{t+1}) = p_t + \phi\left(\frac{p_t}{p_{t+1}}\right)p_{t-1} - 1 = 0$$

where p_t is the (scaled) price of the economy in period t and $\phi(x) = \frac{\beta^{1/A}}{\beta^{1/A}+x^{(A-1)/A}}$.

A stationary solution is defined by a function $g : [0,1] \to [0,1]$ such that

$$1 - x\phi\left(\frac{g(x)}{g \circ g(x)}\right) = g(x),$$

which gives a steady state equilibrium value: $\bar{p} = 1/(1 + \phi(1))$.

Our assumption D imposes the following bounds for the parameter values:

$$\begin{cases} 0 < \phi(1) < 1 - 2|\phi'(1)| & \text{if } \phi'(1) \in (-1, 0) \\ 0 < \phi(1) < 1 & \text{if } \phi'(1) \geq 0. \end{cases}$$

Therefore, the required parameter value set in Li (1998) are strictly included in ours.

Figure 3 shows stationary solution approximations for the foolowing parameter values: $A = .4$ and $\beta = .99$ and the interval of price is $[.05; 1]$. We made ten interactions where the continuous line represents the tenth one (the distance from the ninth is 1.361×10^{-8}).

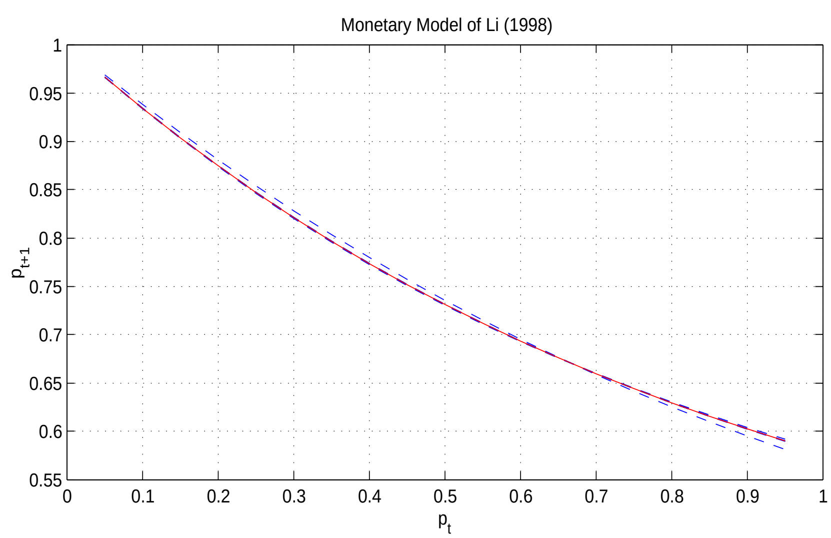

FIGURE 3: Stationary solution of the monetary model of Li (1998) with $A = .4$, $\beta = .99$ and iterations= 10.

3. Growth with Externalities Model (Boldrin and Rustichini (1994))

Boldrin and Rustichini (1994) analyzes a two-sector growth model with labor externality. The utility function is linear and the technological frontier is characterized by:

$$T(x, x', k) = (k^\eta - x')^\alpha (x - \gamma x')^{1-\alpha}$$

where x is the current capital value, x' is the next period capital value, k is the current aggregate capital stock which is considered as an externality in the total number of units of labor and $\alpha, \gamma \in (0, 1)$. In this case the intertemporal equilibrium equation is given by:

$$E(x_t, x_{t+1}, x_{t+2}) = T_2(x_t, x_{t+1}, x_t) + \beta T_1(x_{t+1}, x_{t+2}, x_{t+1}) = 0$$

and the unique interior steady state is

$$\bar{x} = \left(\frac{(\beta - \gamma)(1 - \alpha)}{(\beta - \gamma)(1 - \alpha) + (1 - \gamma)\alpha}\right)^{1/(1-\eta)}.$$

In this model, condition D is:

$$|T_{22} + \beta(T_{11} + T_{13})| > \beta|T_{12}| + |T_{12} + T_{23}|$$

which is stronger than the condition for the existence of a saddle point steady state (see the remark after assumption D).

Figure 4 shows the approximations of the stationary solution around the steady state for parameter values that satisfy assumption D. The parameters are: $\alpha = .5$, $\beta = .95$, $\gamma = .5$, $\eta = .5$ and the interval of capital is $[.05; 1]$. We did ten interactions where the continuous line represents the tenth one (the distance from the ninth is 2.233×10^{-7}).

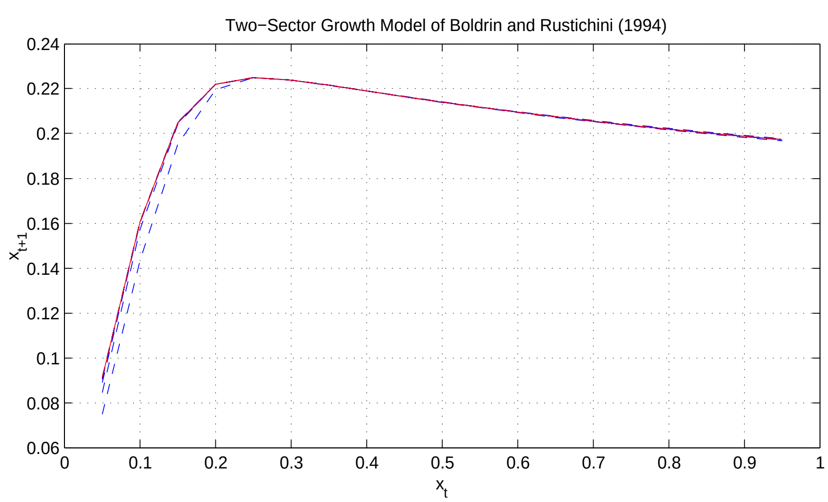

FIGURE 4: Stationary solution of the two-sector growth model of Boldrin and Rustichini (1994) with $\alpha = \eta = \gamma = .5$, $\beta = .95$ and iterations$= 10$.

5. Conclusions

In this paper we provide a recursive method to approximate the stationary solution of the Euler equation for intertemporal deterministic models. Its major difference from classical methods is that it is performed from a contraction mapping in the C^1-topology of a suitable functional space. In particular this implies the continuously differentiability of the stationary solution.

The required hypothesis is an open condition related with the first derivatives of the structural equations evaluated at the steady state. This hypothesis is slightly stronger than the determinacy of the steady state condition.

Another interesting feature of this method is that it needs neither a discretization of the state space nor a piecewise linear approximations of the iterations. Our method is illustrated by numerical examples applied to some models in the literature.

APPENDIX

Proof of Lemma 3.1: Assumption D implies that there exists $r > 0$ such that $B_r(\bar{x}, \bar{x}, \bar{x}) \subset D$ and

$$\sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1}E_1\| + \|E_2^{-1}E_3\| < \alpha \tag{I}$$

and

$$\sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1}E_3\| < 1/2. \tag{II}$$

Let $h \in \mathcal{H}$ and observe that the function $(x_1, x_2) \mapsto E(x_1, x_2, h^2(x_1))$ defined on a neighborhood of $(\bar{x}, \bar{x})$ is twice continuously differentiable,

$$E(\bar{x}, \bar{x}, h^2(\bar{x})) = E(\bar{x}, \bar{x}, \bar{x}) = 0$$

and $E_2(x_1, x_2, h^2(x_1))$ is a negative definite matrix for all (x_1, x_2) in a neighborhood of $(\bar{x}, \bar{x})$. By the Implicit Function Theorem, there exist $\epsilon_1, \epsilon_2 \in (0, r)$ and a continuously differentiable function $\varphi : B_{\epsilon_1}(\bar{x}) \to B_{\epsilon_2}(\bar{x})$ $\;(\varphi = \varphi_h)$ such that

$$E(x_1, x_2, h^2(x_1)) = 0\ ,\ x_i \in B_{\epsilon_i}(\bar{x})\,,\ i = 1, 2 \iff x_2 = \varphi(x_1).$$

Moreover, for each x in $B_{\epsilon_1}(\bar{x})$,

$$D\varphi(x) = -E_2^{-1}(x, \varphi(x), h^2(x))[E_1(x, \varphi(x), h^2(x)) + E_3(x, \varphi(x), h^2(x))Dh(h(x))Dh(x)]$$

$$\Rightarrow \|D\varphi(x)\| \le \sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1}E_1\| + \|E_2^{-1}E_3\| < \alpha\ ,\ \text{for all } x \in B_{\epsilon_1}(\bar{x})$$

because of (I) and the fact that $\|Dh(x)\| \le \alpha < 1$ on $B_r(\bar{x})$.

Observe that

$$\varphi(x) = \bar{x} + \int_0^1 D\varphi(\bar{x} + t(x - \bar{x}))(x - \bar{x})dt\ ,\ \text{for all } x \in B_{\epsilon_1}(\bar{x})$$

$$\Rightarrow |\varphi(x) - \bar{x}| \leq \sup_{t \in [0,1]} \|D\varphi(\bar{x} + t(x - \bar{x}))\| \, |x - \bar{x}| < \epsilon_1$$

and, therefore, we can suppose that $\epsilon = \epsilon_1 = \epsilon_2$.

We claim that we can take $\epsilon = r$. Let $r^* = \sup\ \{\epsilon > 0;\ \varphi$ is defined on $B_\epsilon(\bar{x})\}$. Suppose that $r^* < r$. First, for each $x_1 \in B_{r^*}(\bar{x})$, there exists a unique $x_2 \in B_{r^*}(\bar{x})$ such that $E(x_1, x_2, h^2(x_1)) = 0$. Otherwise, there exist x_2 and $\tilde{x}_2$, $x_2 \neq \tilde{x}_2$, satisfying the last equality. Define $f\colon [0,1] \to \Re$ by

$$f(t) = (x_2 - \tilde{x_2})' E(x_1, x_2 + t(x_2 - \tilde{x_2}), h^2(x_1)).$$

Then $f(0) = f(1) = 0$ and $f'(t) = (\tilde{x_2} - x_2)' E_2 (x_2 - \tilde{x_2}) < 0$. Since E_2 (calculated at $(x_1, x_2 + t(x_2 - \tilde{x_2}), h^2(x_1)))$ is a negative definite matrix, this is a contradiction.

Let x be a point on the border of $B_{r^*}(\bar{x})$. Let $(k_n)_{n \geq 0}$ be a sequence in $B_{r^*}(\bar{x})$ converging to x. Since $(\varphi(x_n))_{n \geq 0}$ is a sequence in $\overline{B_{r^*}}(\bar{x})$ (a compact set), there exists a subsequence converging to a point $y \in B_{r^*}(\bar{x})$. By the continuity of $E(\,\cdot\,, \cdot\,, h^2(\,\cdot\,))$, $E(x, y, h^2(x)) = 0$. However, y is uniquely determinated, implying that the sequence $(\varphi(x_n))_{n \geq 0}$ converges to y. We can apply again the Implicit Function Theorem at $(x, y, h^2(x))$ for the Euler equation. Doing this for all points on the border of $B_{r^*}(\bar{x})$, we can extend φ to a ball centered at $\bar{x}$ which contains (strictly) $B_{r^*}(\bar{x})$. This contradicts the definition of r^*.

Now we choose the constant $\gamma > 0$. Let us calculate the second order derivative of φ:

$$\begin{aligned} D^2\varphi(x) = &- [D(E_2^{-1}E_1) + D(E_2^{-1}E_3) Dh(h(x)) Dh(x)] \\ &- E_2^{-1} E_3 [D^2 h(h(x)) (Dh(x))^2 \, + \, Dh(h(x)) D^2 h(x)] \end{aligned}$$

(calculated at $(x, \varphi(x), h^2(x))$. Taking the supremum on the right side and using (II) we have[7]

$$\|D^2\varphi(x)\| \leq c_1 \, + \, (\alpha^2\gamma + \alpha\gamma) \sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1} E_3\| \, \leq \, c_1 \, + c_2\gamma$$

where $c_1 > 0$ and $0 < c_2 < 1$ (by (II)). Choose γ sufficiently large such that $c_1 + c_2\gamma \leq \gamma$.

Proof of Theorem 3.2: Let $\mathcal{B} \subset \mathcal{M}_n$ be the unit ball. Since E is C^2, there exists $r > 0$ such that the map

$$\Psi : B_r(\bar{x}, \bar{x}, \bar{x}) \times \mathcal{B}^2 \to \mathcal{M}_n$$

defined by

$$\Psi(x_1, x_2, x_3, M_1, M_2) = -E_2^{-1}(x_1, x_2, x_3)[E_1(x_1, x_2, x_3) + E_3(x_1, x_2, x_3) M_1 M_2]$$

is a Lipschitz function, i.e., there exists $L > 0$ such that:

$$\|\Psi(x^1, M^1) - \Psi(x^2, M^2)\| \leq L \sum_{i=1}^{3} |x_i^1 - x_i^2| \, + S_r[\|M_1^1 - M_1^2\| + \|M_2^1 - M_2^2\|] \qquad (III)$$

[7] Observe that one of the components of the derivative of $E_2^{-1} E_1$ and $E_2^{-1} E_3$ involves the derivatives of φ and h which are uniformly bounded by α.

where we are denoting $x^i = (x_1^i, x_2^i, x_3^i) \in B_r(\bar{x})^3$, $M^i = (M_1^i, M_2^i) \in \mathcal{B}^2$, $i = 1, 2$ and $S_r = \sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1} E_3\|$.

The fact that E is C^1 and assumption D (ii) guarantee that we can choose $r > 0$ such that $S_r < 1/2$. Finally, we choose such r satisfying Lemma 3.1 and:

$$\frac{(Lr\alpha + S_r)(2 + \gamma r)}{1 - Lr} = \eta_r < 1$$

(notice that $|\eta_r - 2S_r| \to 0$ when $r \to 0$).

We will prove that the function $\varphi: \mathcal{H} \to \mathcal{H}$ defined by $\varphi(h) = \varphi_h$ is a η_r-contraction map (φ_h defined in Lemma 3.1).

Given $h_1, h_2 \in \mathcal{H}$, by Lemma 3.1, for $i = 1, 2$ and $x \in B_r(\bar{x})$,

$$D\varphi_{h_i}(x) = \Psi(k, \varphi_{h_i}(x), h_i^2(x), Dh_i(h_i(x)), Dh_i(x))$$

Observe that

$$Dh_i^2 = Dh_i(h_i) Dh_i, \;\; i = 1,\, 2$$

and

$$\begin{aligned} Dh_1(h_1)Dh_1 \;-\; Dh_2(h_2)Dh_2 =& (Dh_1(h_1) \;-\; Dh_2(h_1))Dh_1 \\ &+\; (Dh_2(h_1) \;-\; Dh_2(h_2))Dh_1 \;+\; Dh_2(h_2)(Dh_1 \;-\; Dh_2). \end{aligned}$$

Thus,

$$\|h_1^2 - h_2^2\|_1 \le \alpha \|h_1 - h_2\|_1 + \alpha\gamma r \|h_1 - h_2\|_1 + \alpha \|h_1 - h_2\|_1 \qquad \text{(IV)}$$

by the definition of $\|\cdot\|_1$ and the space $\mathcal{H}$.

Therefore, from (III) and (IV)

$$\|\varphi_{h_1} - \varphi_{h_2}\|_1 \le L(r\|\varphi_{h_1} - \varphi_{h_2}\|_1 + r\alpha(2 + \gamma r)\|h_1 - h_2\|_1) + (2 + \gamma r) S_r \|h_1 - h_2\|_1$$

$$\Rightarrow \|\varphi_{h_1} - \varphi_{h_2}\|_1 \le \eta_r \|h_1 - h_2\|_1.$$

So $\varphi: \mathcal{H} \to \mathcal{H}$ is a η_r-contraction. It is easy to see that we can extend continuously φ to $\bar{\mathcal{H}}$. Let φ also denote this extension, then we obtain a η_r-contraction on $\bar{D}_r$. By the Banach Fixed Point Theorem, there exists $g \in \bar{D}_r$ such that $\varphi_g = g$ and

$$\|\varphi^n(h) - g\|_1 \le (\eta_r)^n \|h - g\|_1 \,, \quad \text{for all } h \in D_r.$$

Proof of Corollary 3.3: Fix $r > 0$ and let $\rho > 0$ be such that $\alpha(r + \rho) \le r$ (*i.e.* $\rho \le \frac{1-\alpha}{\alpha} r$) and assumption D holds in $B_{r+\rho}(\bar{x}) \cap C$. From assumption D(i), the choice of ρ guarantees that for each $h \in \bar{D}_{r+\rho}$, $h(x) \in B_r(\bar{x})$ for all $x \in B_{r+\rho}(\bar{x})$. Let $B = B_r(\bar{x})$, $A = B_{r+\rho}(\bar{x}) - B$ and $|.|_i^A$, $|.|_i^B$ the C^i-norm for $i = 0, 1$ on A and B, respectively. For h_1, $h_2 \in D_{r+\rho}(\bar{x})$ we have:

$$|\varphi_{h_1} - \varphi_{h_2}|_1^A \le L\{\rho|\varphi_{h_1} - \varphi_{h_2}|_1^A + |\varphi_{h_1} - \varphi_{h_2}|_0^B + \rho|h_1^2 - h_2^2|_1^A + |h_1^2 - h_2^2|_0^B\}$$

$$+S_{r+\rho}[\gamma|h_1-h_2|_0^A+|h_1-h_2|_1^B]+S_{r+\rho}\gamma|h_1-h_2|_1^A$$

From definitions of $|.|_i^A$ and $|.|_i^B$ we have the following inequalities:

$$|h_1^2-h_2^2|_1^A\leq\alpha|h_1-h_2|_1^B+\alpha\gamma[\rho|h_1-h_2|_1^A+|h_1-h_2|_0^B]+\alpha|h_1-h_2|_1^A;$$

$$|h_1^2-h_2^2|_0^B\leq r(\alpha+1)|h_1-h_2|_1^B\ ;\ |h_1-h_2|_0^A\leq|h_1-h_2|_0^B+\rho|h_1-h_2|_1^A;$$

$$|\varphi_{h_1}-\varphi_{h_2}|_0^B\leq r|\varphi_{h_1}-\varphi_{h_2}|_1^B\leq r\eta_r|h_1-h_2|_1^B.$$

Using these last three inequalities in the first one we get:

$$(1-L\rho)|\varphi_{h_1}-\varphi_{h_2}|_1^A\leq[L\alpha\rho+S_{r+\rho}](1+\gamma\rho)|h_1-h_2|_1^A$$

$$+[L\alpha\rho(1+\gamma)+Lr(\eta_r+\alpha+1)+S_{r+\rho}(r\gamma+1)]|h_1-h_2|_1^B.$$

Defining

$$\begin{cases}\alpha_\rho=[L\alpha\rho+S_{r+\rho}](1+\gamma\rho)(1-L\rho)^{-1}\\ \gamma_\rho=[L\alpha\rho(1+\gamma)+Lr(\eta_r+\alpha+1)+S_{r+\rho}(r\gamma+1)](1-L\rho)^{-1}\end{cases}$$

we have that

$$|\varphi_{h_1}-\varphi_{h_2}|_1^A\leq\alpha_\rho|h_1-h_2|_1^A+\gamma_\rho|h_1-h_2|_1^B.$$

By induction

$$|\varphi_{h_1}-\varphi_{h_2}|_1^A\leq\alpha_\rho^n|h_1-h_2|_1^A+\gamma_\rho\big(\sum_{i=0}^{n-1}\alpha_\rho^{n-1-i}\eta_\rho^i\big)|h_1-h_2|_1^B$$

$$\leq\text{Max}\{\alpha_\rho^n\,,\,\gamma_\rho n(\text{Max}\{\alpha_\rho\,,\,\eta_\rho\})^{n-1}\}|h_1-h_2|_1^{A\cup B}=\gamma_n|h_1-h_2|_1^{A\cup B}.$$

Since $\alpha_\rho\to S_r<1/2$ when $\rho\to0$ and $\eta_r<1$ then, for all $\rho>0$ small enough, we can find $n>1$ such that $\gamma_n<1$, so φ^n is a γ_n-contraction.

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