

Portfolio Selection with Random Transaction Costs

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April 9, 2002

ABSTRACT

I develop a model of portfolio selection in continuous time where transaction costs are random. In the model, the agent faces a trade off between getting good terms of trade and holding a well balanced portfolio. First, I formulate the relevant control problem and prove that the value function is the unique viscosity solution of the associated Hamilton-Jacobi-Bellman equation. Next, I present a numerical procedure to solve the equation and a proof that the numerical solution converges to the solution of the problem. The actual implementation of the procedure fully characterizes the optimal consumer behavior.

*The University of Chicago. This paper is part of my Phd dissertation. Special thanks to my advisor Jose Scheinkman. I am indebted to conversations with Lars Hansen, George Constantinides, Jesus Saa-Requejo, Jesus Santos and Thaleia Zariphopoulou. Financial support by C.N.P.Q. is greatly acknowledged. All errors are mine.

I. Introduction

The literature of portfolio selection in a continuous time setting starts with Merton (1973). He shows that in the absence of transaction costs, the best strategy for the investor is to keep his holdings in the stock and in the money market account at a constant proportion. To implement this strategy the investor should trade continuously. The first objection to this setup, is that this strategy breaks down in the presence of an arbitrarily small transaction cost. The wild nature of the Brownian motion that drives the stock price movements implies that the strategy has infinite cost. The first model with fixed transaction costs was developed by Constantinides (1986). He shows that the investor copes with transaction costs by refraining to transact too often. The optimal strategy is to transact only when the proportion of stock relative to bonds in the portfolio gets too high or too low. Constantinides also reports that the loss in consumption relative to the Merton paradigm is small. With a 2% transaction cost consumption decreases by 0.4%; with 5% consumption decreases by 1%. Formal proofs of the results in the Constantinides paper can be found in Davis and Norman (1990). For a rather different proof of the same results see Fleming and Soner (1993) pg. 342.

Were not for the development of the notion of viscosity solutions ((Crandall, Ishii, and Lions 1992) and (Fleming and Soner 1993)) for nonlinear partial differential equations, anything slightly more complex than the Constantinides model would be impossible to analyze. After the breakthrough, more complex models appeared. Fleming and Zariphopoulou (1991) analyzed a model with different borrowing and lending rates, Zariphopoulou (1994) studies a model with transaction costs and borrowing constraints. For a model of option pricing with transaction costs, see Davis, Panas, and Zariphopoulou (1993). A common characteristic of all the models above is the nature of the optimal policies. The optimal policy implies that there is a region where it is optimal not to transact. As soon as the stock holdings try to escape out of this region, the agent is to purchase or sell stock, so as to keep the holdings just within the prescribed region. This paper is a first attempt to look at these policies in the presence of random transaction costs.

Transaction costs usually have a fixed component like brokerage fees, as well as a random component in the form of the bid-ask spread. Furthermore, the bid-ask spread can be viewed as a measure of market liquidity. Suppose large shocks hit the market making the agent's portfolio unbalanced. Suppose also that the market lose liquidity so that the bid-ask spread is temporarily high. Should the agent pay high transaction fees and rebalance the portfolio at once, or exercise his/her option and wait for better terms of trade? This paper is able to give a definite answer to this question: Do wait.

The chain of events described above clearly links high volatility with high bid-ask spread. This feature, besides being intuitive, is present in many branches of the market microstructure literature. The details will be shown in section II. There, I start with a discrete time model that have this behaviour. Unfortunately, some features vanish when the model is passed to the continuous time limit. The readers can judge by themselves.

Sections III– IV formulates the control problem and show that it has a unique solution. The numerical methodology is developed in section V. First, in subsection A, I obtain the relevant Markov chain and its

boundary behaviour. Next, in subsection B, I prove the convergence of the scheme. The results of the numerical model are presented in section VI.

II. Modeling Transaction Costs

In the model, the agent faces a decision of purchasing and selling stock. He faces proportional transaction costs which are random. When he purchases y dollars worth of stock, his money market account is debited $y(1 + K_t)$ dollars. When he sells y dollars worth of stock, his money market account is credited $y(1 + K_t)^{-1}$ dollars. I will be looking for a process K_t which is nonnegative. In this section I develop the dynamics for the stock price S_t and transaction cost K_t . I approach this problem by first developing a discrete time model, where intuition is easier to grasp. The discrete time model is only a device to capture some stylized facts. In the second subsection I proceed to take its continuous time limit. In later sections, I will only work with the continuous time process.

A. Discrete Time Model

I start with a discrete time, standard log-normal process for the stock price. For $m = 0, 1, 2, \dots$ define

$$S_{m+1} = S_m + \mu S_m + \sigma S_m Z_m \tag{1}$$

$$S_0 \text{ given, } Z_m \text{ i.i.d. } N(0, 1)$$

Next, I postulate the behavior of the random transaction costs. The idea is to capture some stylized facts hinted by the market microstructure literature. In the inventory branch of the literature I cite Stoll (1978), Ho and Stoll (1983) and O'Hara and Oldfield (1986). These all imply that the market maker responds to a more volatile market by widening the bid-ask spread. The same qualitative result appear in some information based models. The story behind the well known papers by Glosten and Milgrom (1985) and Easley and O'Hara (1992) goes like this: suppose a group of traders learn privately that the market price of a certain stock is below its real value. Acting in self interest they will send to the market maker an abnormal flow of buy orders. He, in turn, will respond by increasing the bid and ask prices, as well as by widening the spread. The overall effect is a positive association between the size of the spread and the absolute value of the stock price change. The effect also tend to be temporary, in the sense that, once the market maker learns the information conveyed by the flow of orders, the spread is reduced. Empirical tests of the results above have been carried out by Bollerslev and Melvin (1994), Hasbrouk (1991) and Hausman, Lo, and MacKinlay (1992). Hasbrouk finds evidence linking large trades with wide spreads, what is consistent with the Easley and O'Hara model. Hausman, Lo and MacKinlay find positive links between stock price volatility and bid-ask spreads in the U.S.. The same relationship was uncovered by Bollerslev and Melvin in foreign exchange data. The idea, hence, is that transaction costs are higher when the market is volatile. A sequence of abnormally large shocks hitting the market are supposed to make transaction costs grow temporarily. A calm period, on the other hand, should make transaction costs smaller. The situation

is reminiscent of an ARCH like modeling. I assume that current transaction costs K_{m+1} depends only on the past shocks (Z_m, Z_{m-1}, \dots) and that (K_{m+1}, Z_{m+1}) is markovian.

$$K_{m+1} = f(Z_m, Z_{m-1}, \dots) = g(Z_m, K_m)$$

What are the features I want from the function g ? Well, if $|Z_m|$ is large, I want K_{m+1} to increase, at least temporarily. Whereas, if $|Z_m|$ is small, I want K_{m+1} to decrease. On top of that, I also want the flexibility to let transaction costs respond asymmetrically with respect to positive and negative shocks. Like the EGARCH model, I assume that a large negative shock increases transaction costs more than a shock that is equally large but positive. The rationale in the EGARCH model is to capture the so called “leverage effect”. Here, it just means that it is particularly expensive to transact during crashes.¹ Summing up:

$$\left\{ \begin{array}{ll} \text{If } Z_m \text{ is large and positive} & \Rightarrow K_{m+1} \uparrow \\ \text{If } Z_m \text{ is small} & \Rightarrow K_{m+1} \downarrow \\ \text{If } Z_m \text{ is large and negative} & \Rightarrow K_{m+1} \uparrow\uparrow \end{array} \right.$$

Before I proceed, I should warn the reader that the situation is only analogous to ARCH modeling. In my model the volatility of the stock prices is always constant. I completely abstract from stochastic volatility considerations, trying to isolate the effect of random transaction costs. First, I specialize the function $g(Z_m, K_m)$. With a view toward the continuous time limit I pick the “square root” process.

$$\begin{aligned} K_{m+1} &= K_m + \beta(\alpha - K_m) + \lambda \sqrt{K_m} \times \\ &\quad \times [\rho Z_m + \gamma (|Z_m| - (2/\pi)^{1/2})] \end{aligned} \quad (2)$$

K_0 given

the constants satisfy,

$$\begin{aligned} \alpha, \beta, \lambda &> 0 \\ -1 &< \rho < 1 \\ 2\beta\alpha &> \lambda^2 \\ \gamma &= \left(\frac{1 - \rho^2}{1 - 2/\pi} \right)^{1/2} \end{aligned}$$

The response of $K_{m+1}/\sqrt{K_{m+1}}$ to shocks Z_m are plotted in figure 1 for $\rho < 0$. I favor the case $\rho < 0$. The choice of the process K_m is ad-hoc. It is dictated, however, by the desired response to shocks described above as well as by the properties of the continuous time limit of the system 1–2.

Figure 1 about here.

¹No empirical evidence to support this particular feature will be presented. This gives me flexibility and can be undone by setting correlation parameter ρ equal to zero.

B. Continuous Time Model

I now turn to the continuous time limit of the system 1–2. All the missing details can be found in Nelson (1990), which I follow closely. I will be considering 3 kinds of processes (where X stands for S, K and Z).

- Sequences of discrete time processes $\{X_{mh}^h\}$ that depend on h and the discrete time index mh , $m = 0, 1, 2, \dots$. I set $X_0^h = X_0 \forall h$.
- Sequences of continuous time processes $\{X_t^h\}$ defined as

$$P^h[X_t^h = X_{mh}^h, mh \leq t < (m+1)h] = 1$$

That is $\{X_t^h\}$ is formed as step functions from the processes $\{X_{mh}^h\}$ with jumps at $h, 2h, \dots$

- Limit diffusion processes $\{X_t\}$ to which $\{X_t^h\}$ will converge weakly as $h \downarrow 0$.

Now rewrite the system (1–2) as

$$S_{(m+1)h}^h = S_{mh}^h + \mu S_{mh}^h h + \sigma S_{mh}^h Z_{mh}^h \quad (3)$$

$$K_{(m+1)h}^h = K_{mh}^h + \beta(\alpha - K_{mh}^h)h + \lambda \sqrt{K_{mh}^h} \times \\ \times [\rho Z_{mh}^h + \gamma (|Z_{mh}^h| - (2h/\pi)^{1/2})] \quad (4)$$

$$(S_0^h, K_0^h) = (S_0, K_0) \text{ given} \\ Z_{mh}^h \text{ i.i.d. } N(0, h)$$

The following proposition gives the continuous time limit of the process above. The proof is a simple check that the conditions of theorem 3.1 in Nelson (1990) are satisfied.

Proposition 1 $(S_t^h, K_t^h) \implies (S_t, K_t)$ (weakly) as $h \downarrow 0$. Where (S_t, K_t) satisfy:

$$dS_t = \mu S_t dt + \sigma S_t dW_{1,t} \quad (5)$$

$$dK_t = \beta(\alpha - K_t) dt + \lambda \sqrt{K_t} dW_{2,t} \quad (6)$$

$$S_0, K_0 \text{ given}$$

and $[W_{1,t}, W_{2,t}]$ is a 2-dimensional standard Brownian motion satisfying

$$\begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix} \begin{bmatrix} dW_{1,t} & dW_{2,t} \end{bmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} dt. \quad (7)$$

I claim that the choice of a square root process for the transaction costs is reasonable. With this choice, I get a process K_t that is stationary, is positive with probability 1 and it mean reverts. When properly initialized, the mean of K_t is equal to the mean reversion point α .

The last property leads to a very natural way to access the effects of random transaction costs as opposed to fixed transaction costs. Namely, I will be comparing the effects of fixed costs $[(1 + \alpha), (1 + \alpha)^{-1}]$ as opposed to random costs $[(1 + K_t), (1 + K_t)^{-1}]$ initialized at $K_0 = \alpha$.

This setup has a drawback and is to be regarded as a first attempt to modeling random transaction costs. The reason being that all the nice intuition of the discrete time model 1–2 have vanished out of thin air. The continuous time model 5–7 has only a flavor of the association between transaction costs and volatility. In the limit all that remained was a negative association between prices and transaction costs (if $\rho < 0$). The easiest way to see this is the case, is by considering a standard Euler approximation for the system 5–7. Approximating $(W_{1,t}, W_{2,t})$ by correlated normals (Z_1, Z_2) one sees that $E[Z_2 \mid |Z_1|] = 0$. The effect, thus, is absent in all stages of the approximation and therefore absent in the continuous time limit.

III. Investment-Consumption Model

The first sub-section poses the control problem for the consumer and establishes some properties of the value function. In the second sub-section I write down the Hamilton-Jacobi-Bellman equation for the value function.

A. The Control Formulation

In the model, the agent faces a decision of purchasing and selling stock that incur in proportional transaction costs that are random.

The market has 2 securities. The first is a risk free money market account, which pays a constant interest rate r

$$B(t) = e^{rt} \tag{8}$$

The second security is a stock $S(t)$. It's dynamics is given together with the dynamics of the transaction costs $K(t)$ by the system 5–7. $[W_{1,t}, W_{2,t}]$ is a two-dimensional Brownian Motion on a complete probability space (Ω, F_t, F, P) . F_t is the completion of $\sigma(W_{1,t}, W_{2,t}; s \leq t)$.

The amounts held in the money market and stock accounts are denoted $X(t)$ and $Y(t)$ respectively. The consumer chooses processes $(C(t), L(t), M(t))$.

- $C(t)$ is the dollar consumption process.
- $L(t)$ is the cumulative dollar transfer into the stock account.
- $M(t)$ is the cumulative dollar transfer into the money market account.

Transfers between the stock account $Y(t)$ and the money market account $X(t)$ incur transaction costs equal to $(1 + K(t))$. More specifically, a instantaneous transfer into the stock account of size $dL(t)$ reduces the bond account by $(1 + K(t))dL(t)$. A instantaneous transfer out of the stock account of size $dM(t)$ increases the bond account by $(1 + K(t))^{-1}dM(t)$. My bookkeeping always penalizes the money market account.

The consumption process $C(t)$ drains directly the bond account. Transfers out of the money market account to consumption incur no transaction costs.

Everything considered, the processes $X(t), Y(t)$ and $K(t)$ are given by:

$$dX(t) = (rX(s) - C(s)) ds - (1 + K(s)) dL(s) + (1 + K(s))^{-1} dM(s) \quad (9)$$

$$dY(t) = \mu Y(s) ds + \sigma Y(s) dW_{1,s} + L(t) - M(t) \quad (10)$$

$$dK(t) = \beta(\alpha - K(s)) ds + \lambda \sqrt{K(s)} dW_{2,s} \quad (11)$$

with initial conditions: $X(0) = x, Y(0) = y$ and $K(0) = k$.

The agent has Von Neumann-Morgestern preferences

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} U(C(t)) dt \right]$$

over the consumption process $\{C(t); t \geq 0\}$, where δ is the subjective discount factor and the utility function $U(C(t))$ has the following standard properties:

- $U : \mathbb{R}^+ \mapsto \mathbb{R}, U \in C^2$ with $U(0) = 0, U' > 0, U'' < 0$.
- $0 \leq U(c) \leq M(1+c)^\gamma$ for some $M > 0, \gamma \in (0, 1)$
- U satisfy the Inada conditions $U'(0) = \infty, U'(\infty) = 0$

The investor solves the following problem.

$$V(x, y, k) = \sup_{A(x, y, k)} \mathbb{E}^{(x, y, k)} \left[\int_0^\infty e^{-\delta t} U(C(t)) dt \right] \quad (12)$$

where the expectation is taken given the initial condition (x, y, k) and $A_{(x, y, k)}$ is the set of triples $(L(t), M(t), C(t))$ satisfying

A-1 $(L(t), M(t))$ are right continuous, adapted, non-decreasing processes. I set $L(0) = M(0) = 0$.

A-2 $C(t)$ is adapted, continuous, non-negative and $\int_0^\infty C(t) dt < \infty$.

A-3 $X(t), Y(t) \geq 0$.

A-3 is admittedly restrictive. It does not allow leveraged or short positions in the stock. One justification is that the parameters in the numerical computations will be such that the consumer never let $X(t) \leq 0$ or $Y(t) \leq 0$. The

real reason, however, is that the natural alternative assumption leads to a pathology. It would be natural to assume instead

$$0 \leq Z(t) \equiv \begin{cases} X(t) + Y(t)(1 + K(t))^{-1} & \text{if } Y(t) \geq 0 \\ X(t) + Y(t)(1 + K(t)) & \text{if } Y(t) < 0 \end{cases}$$

which says that, should the investor be called upon liquidating his position in the stock, at the current transaction cost, he remains solvent.

But consider two sequences of points $a_n = (x_n, y_n, k_n) = (n, -1/n, n^2 - 1)$ and $b_n = (n, 0, n^2 - 1)$. Along b_n the consumer gets infinitely rich. It is feasible never to transact and consume arbitrarily large amounts directly from the money market account. Along a_n he is always broke. The only feasible strategy is to liquidate his stock position immediately and consume zero forever. Acting otherwise risks making $Z(t) < 0$ which is not allowed.

The pathology is that you may have an arbitrarily small short position in the stock ($-1/n$) but transaction costs are so high ($n^2 - 1$) that you are in fact insolvent. This implies that $|a_n - b_n| \rightarrow 0$ but $|V(a_n) - V(b_n)| \rightarrow U(\infty) - U(0) > 0$. This shows that under the hypothesis $Z \geq 0$ the value function is not uniformly continuous, a property that is essential for my uniqueness result. I close this section showing some elementary properties of the value function (All proofs are in the appendix).

Proposition 2 (a) $V(x, y, k)$ is concave in (x, y) .

(b) $V(x, y, k)$ is strictly increasing in (x, y) , non-increasing in k .

(c) $V(x, y, k)$ is uniformly continuous.

B. The Hamilton-Jacobi-Bellman Equation

This sub-section develops an heuristic argument that suggests that the value function satisfy equation 16 below. The argument is heuristic because it assumes a priori that $V(x, y, k) \in C^2$. It turns out that the value function will satisfy 16 only in a weak (viscosity) sense.

Consider the following strategy: sell ϵ worth of stock instantly and continue optimally thereafter. This implies a jump from (x, y, k) to $(x + \epsilon(1 + k)^{-1}, y - \epsilon, k)$. Hence

$$V(x, y, k) \geq V(x + \epsilon(1 + k)^{-1}, y - \epsilon, k)$$

dividing by ϵ and letting $\epsilon \downarrow 0$, I get

$$\frac{\partial V}{\partial y} - (1 + k)^{-1} \frac{\partial V}{\partial x} \geq 0 \tag{13}$$

Analogously, buying ϵ worth of stock instantly and proceeding optimally thereafter yields

$$V(x, y, k) \geq V(x - \epsilon(1 + k), y + \epsilon, k)$$

dividing by ϵ and letting $\epsilon \downarrow 0$, this implies

$$(1+k) \frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} \geq 0 \quad (14)$$

Before I derive the last equation, let me introduce some notation. Let

$$\begin{aligned} g(x, y, k) &= [rx \ \mu y \ \beta(\alpha - k)]^\top \\ dV &= \left[\frac{\partial V}{\partial x} \ \frac{\partial V}{\partial y} \ \frac{\partial V}{\partial k} \right]^\top \\ \Sigma(y, k) &= \begin{bmatrix} \sigma^2 y^2 & \lambda \rho \sigma \sqrt{k} y \\ \lambda \rho \sigma \sqrt{k} y & \lambda^2 k \end{bmatrix} \\ D^2 V &= \begin{bmatrix} \frac{\partial^2 V}{\partial y^2} & \frac{\partial^2 V}{\partial y \partial k} \\ \frac{\partial^2 V}{\partial y \partial k} & \frac{\partial^2 V}{\partial k^2} \end{bmatrix} \end{aligned}$$

$g(x, y, k)$ is the drift and $\Sigma(y, k)$ is the nonzero part of the covariance matrix of the system 9–11 when no controls are applied. Notice that $D^2 V$ is not the Hessian matrix of V , since it does not include partials with respect to x .

Last, consider the strategy S : In $[0, t)$ do not transact and consume at a constant rate c ; thereafter proceed optimally. This yields

$$V(x, y, k) \geq \int_0^t e^{-\delta s} U(c) ds + E^S [e^{-\delta t} V^S(X(t), Y(t), K(t))]$$

where $E^S [e^{-\delta t} V^S(X(t), Y(t), K(t))]$ means that one let the system 9–10 evolve freely, just draining the money market account at rate c . By Ito's lemma,

$$\begin{aligned} &E^S [e^{-\delta t} V^S(X(t), Y(t), K(t))] - V(x, y, k) \\ &= \int_0^t e^{-\delta s} (-\delta V + g^\top dV + \frac{1}{2} \text{Tr} \Sigma . D^2 V - c \frac{\partial V}{\partial x}) ds \end{aligned}$$

Hence

$$\int_0^t e^{-\delta s} (-\delta V + g^\top dV + \frac{1}{2} \text{Tr} \Sigma . D^2 V + U(c) - c \frac{\partial V}{\partial x}) ds \leq 0$$

dividing by t and letting $t \downarrow 0$, I get $\forall c \geq 0$,

$$-\delta V + g^\top dV + \frac{1}{2} \text{Tr} \Sigma . D^2 V + U(c) - c \frac{\partial V}{\partial x} \leq 0$$

which implies

$$-\delta V + g^\top dV + \frac{1}{2} \text{Tr} \Sigma . D^2 V + \max_{c \geq 0} \{U(c) - c \frac{\partial V}{\partial x}\} \leq 0 \quad (15)$$

Finally I define the operator \mathcal{L} by

$$\mathcal{L}V(x,y,k) \equiv -\delta V + g^\top dV + \frac{1}{2}\text{Tr}\Sigma.D^2V + \max_{c \geq 0} \{U(c) - c \frac{\partial V}{\partial x}\},$$

combine equations 13–15 above to get

$$\min\{-\mathcal{L}V, (1+k)\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}, \frac{\partial V}{\partial y} - (1+k)^{-1}\frac{\partial V}{\partial x}\} = 0 \quad (16)$$

The form of the equation 16 strongly suggests, and the numerical results show, that the (x,y,k) space will split in three regions.

[NT] where $\mathcal{L}V = 0$. Here it is optimal not to transact.

[BS] where $(1+k)V_x - V_y = 0$. Here it is optimal to buy stock immediately, forcing (x,y,k) to return to the boundary of the **NT** region.

[SS] where $V_y - (1+k)^{-1}V_x = 0$. Here it is optimal to sell stock immediately, forcing (x,y,k) to return to the boundary of the **NT** region.

The boundaries between the regions are not known a priori and have to be determined together with $V(x,y,k)$.

The form of the variational inequality also suggests that the optimal strategy involves singular controls. At time 0, if the agent finds himself either at the **BS** or the **SS** regions, he jumps immediately to the boundary of the **NT** region along the lines $x(1+k) = y$ and $x(1+k)^{-1} = y$, respectively. Afterwards he exercises the controls $L(t), M(t)$ just enough to keep (x,y,k) inside the **NT** region. The controls therefore behave like the local time of the Brownian motions $(W_{1,t}, W_{2,t})$. It can be shown ((Davis, Panas, and Zariphopoulou 1993)) that the optimal controls never jumps, except, perhaps at $t = 0$.

IV. The Viscosity Property

In this section I introduce the notion of viscosity solutions. I argue that the value function defined by 12 is the unique viscosity solution of the variational inequality 16. For a first class survey in the theory, the reader is referred to the “User’s guide” by Crandall, Ishii, and Lions (1992). Another useful source of results is Ishii and Lions (1990). The standard reference concerning control problems is the book by Fleming and Soner (1993).

The theory applies to PDE’s of the form $F(x,u,Du,D^2u) = 0$ where $F : R^N \times R \times R^N \times R^{N \times N} \rightarrow R$, where $R^{N \times N}$ denotes the set of symmetric $N \times N$ matrices and u is a real valued function defined in a open subset $\Omega \subseteq R^N$. The theory allows F to be fully non-linear and the solution u to be merely continuous. However F is required to satisfy the monotonicity condition

$$F(x,r,p,X) \leq F(x,s,p,Y) \quad \text{whenever } r \leq s \text{ and } Y \leq X$$

where $r, s \in R$; $x, p \in R^N$; $X, Y \in R^{N \times N}$ and $R^{N \times N}$ is equipped with its usual order. The condition on (r, s) is named properness, while the condition on (X, Y) is referred as degenerate elliptic.

In control problems where state constraints are present the relevant notion is that of constrained viscosity solution. It was introduced by Capuzzo-Dolcetta and Lions (1990). See also Katsoulakis (1994).

Definition 1 A continuous function $u : \Omega \rightarrow R$ is a constrained viscosity solution of

$$F(x, u(x), Du(x), D^2u(x)) = 0$$

if

1. u is a viscosity subsolution of $F = 0$ on $\bar{\Omega}$; that is, if for all $\psi \in C^2(\Omega)$ and all points x_0 such that $u(x_0) - \psi(x_0)$ attains a local maximum one has

$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0$$

2. u is a viscosity supersolution of $F = 0$ on Ω ; that is, if for all $\psi \in C^2(\Omega)$ and all points x_0 such that $u(x_0) - \psi(x_0)$ attains a local minimum one has

$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0$$

The reader should notice the judicious use of Ω and $\bar{\Omega}$ in the definition.

With this definition I can now establish the following proposition that follows the lines of (Davis, Panas, and Zariphopoulou 1993, Tourin and Zariphopoulou 1994, Zariphopoulou 1994) closely. The argument is nowadays standard, so its proof omitted.

Proposition 3 The value function $V(x, y, k)$ defined by equation 12 is a constrained viscosity solution of

$$\min\{-\mathcal{L}V, (1+k)\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}, \frac{\partial V}{\partial y} - (1+k)^{-1}\frac{\partial V}{\partial x}\} = 0$$

The last result in this section proves that the value function as defined by equation 12 is the unique constrained viscosity solution of the variational inequality 16. As is common in this theory, the result is stated in the form of a comparison result. The proof identical to theorem 3.3 in Zariphopoulou (1994).

Proposition 4 Let u be a continuous viscosity subsolution of 16 on R_+^3 and v be a bounded below, uniformly continuous viscosity supersolution of 16 on R_{++}^3 . Then $u \leq v$ in R_+^3 .

A. The HARA Case

In the case where the utility function is HARA, a further characterization of the boundaries of the three transaction regions is possible. Furthermore, in this case one can have a closed form solution to $\max_{c \geq 0} \{U(c) - c \frac{\partial V}{\partial x}\}$. This makes the numerical implementation run ten times faster than in the general case.

To simplify the notation, I assume the value function to be smooth. The differentials of $V(x, y, k)$ should be understood in the viscosity sense. For $0 < \gamma < 1$, let

$$U(c) = c^\gamma / \gamma$$

The linearity of the system 9–10 with respect to $(C(t), L(t), M(t))$ implies that if $\theta > 0$ and $(C, L, M) \in A_{(x, y, k)}$ then $(\theta C, \theta L, \theta M) \in A_{(\theta x, \theta y, k)}$

$$\begin{aligned} V(\theta x, \theta y, k) &= \sup_{A_{(\theta x, \theta y, k)}} \mathbb{E}^{(\theta x, \theta y, k)} \left[\int_0^\infty e^{-\delta t} c_t^\gamma / \gamma dt \right] \\ &= \sup_{A_{(x, y, k)}} \mathbb{E}^{(x, y, k)} \left[\int_0^\infty e^{-\delta t} (\theta c_t)^\gamma / \gamma dt \right] \\ &= \theta^\gamma v(x, y, k). \end{aligned}$$

Hence, $V(x, y, k)$ inherits the homothetic property of $U(c)$. The homothetic property implies that

$$V_x(\theta x, \theta y, k) = \theta^{\gamma-1} V_x(x, y, k)$$

and

$$V_y(\theta x, \theta y, k) = \theta^{\gamma-1} V_y(x, y, k)$$

thus, if

$$(1+k) \frac{\partial V}{\partial x}(x, y, k) - \frac{\partial V}{\partial y}(x, y, k) = 0$$

or

$$\frac{\partial V}{\partial y}(x, y, k) - (1+k)^{-1} \frac{\partial V}{\partial x}(x, y, k) = 0$$

the same holds for all points $(\theta x, \theta y, k)$. This strongly suggests that for fixed k , the boundaries between the no transaction and the transaction regions are straight lines through the point $(0, 0, k)$.

V. Numerical Model

This section describes a methodology to discretize the control problem. The essence of the procedure is to replace the continuous time process (X_t, Y_t, K_t) by a sequence of Markov chains. The original continuous domain is to

be replaced by a sequence of discrete and bounded grids. The spacing of the grids has to converge to zero and the bound has to converge to infinity. In this sense the grid will approximate the original domain. The most important feature of the approximating scheme is what is called local consistency. One has to find transition probabilities for the chain such that, as the step size converges to zero, the drift and the volatility of the Markov chain converge to those of the original process, for all control policies.

The general methodology for the discretization scheme can be found in Kushner and Dupuis (1992). For a problem similar to ours in mathematical character the reader can consult Hindy, Huang, and Zhu (1993). For the convergence of the methodology there are too branches in the literature. Barles and Souganidis (1991) use a viscosity formulation whereas Kushner and Dupuis follow a more probabilistic formulation. I use the Markov chain structure of the latter and the viscosity ideas of the former. The proof of convergence of the procedure is deferred to the next subsection. The reader is warned that the Barles and Souganidis ideas require a comparison theorem that I do not have. They require that comparison holds for upper and lower semicontinuous functions, while proposition 4 proves it only for uniformly continuous functions.

A. The Markov Chain

It turns out that the covariance structure of the original process lacks a key property required by the numerical approximation scheme. The property is called diagonal dominance. Basically it is required that the principal components of the matrix is stable. In a 2×2 matrix, the property required is that the diagonal terms of the matrix are bigger in absolute value than the off diagonal terms, for all values of y and k . That is $\forall y, k > 0$

$$\begin{aligned}\sigma^2 y^2 &> |\rho| \lambda \sigma y \sqrt{k} \\ \lambda^2 k &> |\rho| \lambda \sigma y \sqrt{k}\end{aligned}$$

Even if I try rotating the coordinate system, so as the matrix becomes closer to a diagonal matrix, this properties still refuses to hold. There are no linear transformation of the variables y and k that satisfy the property in the whole domain. I am thus forced to make a change of variables the variables X_t, Y_t and K_t . Start by making the transformations:

$$\begin{aligned}x_t &= \log X_t \\ y_t &= \log Y_t \\ k_t &= a \sqrt{K_t}\end{aligned}$$

where a is a positive constant to be chosen later. After the transformation, the original system 9– 11 has the following form

$$dx_t = (r - c_t)dt - \left(1 + \frac{k_t^2}{a^2}\right)e^{-x_t} dL(t) + \left(1 + \frac{k_t^2}{a^2}\right)^{-1} e^{-x_t} dM(t)$$

$$\begin{aligned}
dy_t &= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_{1,t} + e^{-y_t} dL(t) - e^{-y_t} dM(t) \\
dk_t &= \left\{ \frac{a^2}{2k_t} \left(\beta\alpha - \frac{\lambda^2}{4}\right) - \frac{\beta k_t}{2} \right\} dt + \frac{a\lambda}{2} dW_{2,t}
\end{aligned}$$

A check on the boundary classification for the process k_t show that zero is an entrance boundary as long as $\lambda^2 < 2\beta\alpha$. This is exactly the restriction imposed on the parameters of the square root process so as to make it stationary. Notice that the transaction processes $dM(t)$ and $dL(t)$ are in dollar units. The exponentials e^{-y_t} and e^{-x_t} make the transformation for percentage units. Finally, for the same reason, a choice of c_t at a given moment, will give rise to a utility of $U(e^{x_t} c_t)$.

For the numerical implementation, next I need to replace the unbounded processes $(x_t, y_t, k_t) \in [-\infty, \infty] \times [-\infty, \infty] \times [0, \infty]$ by a reflected version $(\tilde{x}_t, \tilde{y}_t, \tilde{k}_t) \in [\underline{M}^x, \overline{M}^x] \times [\underline{M}^y, \overline{M}^y] \times [0, \overline{M}^k]$. For that purpose, introduce reflection processes $\underline{Z}^x(t), \overline{Z}^x(t), \underline{Z}^y(t), \overline{Z}^y(t)$ and $\overline{Z}^k(t)$ active at the boundaries of the region. Dropping the tilde, the process suitable for numerical implementation becomes :

$$\begin{aligned}
dx_t &= (r - c_t)dt - \left(1 + \frac{k_t^2}{a^2}\right)e^{-x_t} dL(t) + \left(1 + \frac{k_t^2}{a^2}\right)^{-1} e^{-x_t} dM(t) + d\underline{Z}^x(t) - d\overline{Z}^x(t) \\
dy_t &= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_{1,t} + e^{-y_t} dL(t) - e^{-y_t} dM(t) + d\underline{Z}^y(t) - d\overline{Z}^y(t) \\
dk_t &= \left\{ \frac{a^2}{2k_t} \left(\beta\alpha - \frac{\lambda^2}{4}\right) - \frac{\beta k_t}{2} \right\} dt + \frac{a\lambda}{2} dW_{2,t} - d\overline{Z}^k_t
\end{aligned}$$

Again, the processes $\underline{Z}^x(t), \overline{Z}^x(t), \underline{Z}^y(t), \overline{Z}^y(t)$ and $\overline{Z}^k(t)$ are nondecreasing and increase only when the state variables hit the boundaries of the region $\{[\underline{M}^x, \overline{M}^x] \times [\underline{M}^y, \overline{M}^y] \times [0, \overline{M}^k]\}$.

In this new setting, the control problem is restated as

$$V(x, y, k) = \sup_{A_{x,y,k}} \mathbf{E}^{x,y,k} \left[\int_0^\infty e^{-\delta t} U(x_t, c_t) dt \right] \quad (17)$$

given the dynamics and the set of admissible controls. Next let

$$\begin{aligned}
g(x, y, k; c) &= \begin{bmatrix} r - c \\ \mu - \frac{\sigma^2}{2} \\ \frac{a^2}{2k} \left(\beta\alpha - \frac{\lambda^2}{4}\right) - \frac{\beta k}{2} \end{bmatrix} \\
\Sigma(y, k) &= \begin{bmatrix} \sigma^2 & \frac{a}{2} \lambda \rho \sigma \\ \frac{a}{2} \lambda \rho \sigma & \frac{\lambda^2 a^2}{4} \end{bmatrix}
\end{aligned}$$

and

$$\mathcal{L}_c V(x, y, k) \equiv g^\top(c) dV + \frac{1}{2} \text{Tr} \Sigma . D^2 V,$$

It is useful to define the operators

$$\begin{aligned}\mathcal{B}V(x,y,k) &\equiv (1 + \frac{k^2}{a^2})^{-1} e^{-y} \frac{\partial V}{\partial y} - e^{-x} \frac{\partial V}{\partial x} \\ \mathcal{S}V(x,y,k) &\equiv (1 + \frac{k^2}{a^2})^{-1} e^{-x} \frac{\partial V}{\partial x} - e^{-y} \frac{\partial V}{\partial y}\end{aligned}$$

then the H-J-B equation becomes:

$$\max\{-\delta V + \max_{c \geq 0} \{\mathcal{L}_c V + U(x,c)\}, \mathcal{B}V, \mathcal{S}V\} = 0 \quad (18)$$

Before I proceed, let me comment on the viscosity property of the new H-J-B equation. By inspecting the proofs of the comparison result, the reader will see that the argument goes on unchanged for the new processes. This is because the value functions of the problems have the relation $V(x,y,k) = W(e^x, e^y, \frac{k^2}{a^2})$. More precisely, a similar relation hold for the solution of the Merton problems with the two classes of variables. I could proceed with the arguments of the session IV, using the solution of the corresponding Merton problem as the source for the comparison result.

Also, due to the continuity of the consumption process, I can always add a nonbinding upper bound of the form $0 \leq c \leq \bar{c}$ on the H-J-B equation above on any given compact domain. Finally let me add that signs have been reversed between the formulation 16 and 18, so now subsolutions satisfy $\max\{\dots\} \geq 0$ and supersolutions satisfy $\max\{\dots\} \leq 0$.

I will work with sequences of grids indexed by the step size h . Abusing the notation on the k variable slightly:

$$\begin{aligned}G^h &= \{(i,j,k) : x = i \times h, y = j \times h, k = k \times h \\ &\quad ; i = -\underline{N}^x, \dots, -1, 0, 1, \dots, \bar{N}^x \\ &\quad ; j = -\underline{N}^y, \dots, -1, 0, 1, \dots, \bar{N}^y \\ &\quad ; k = 0, 1, \dots, \bar{N}^k\}\end{aligned}$$

where $\underline{N}^x = \underline{M}^x/h$, $\bar{N}^x = \bar{M}^x/h$, $\underline{N}^y = \underline{M}^y/h$, $\bar{N}^y = \bar{M}^y/h$, $\bar{N}^k = \bar{M}^k/h$. Each grid point (i,j,k) clearly correspond to a state (x,y,k) where $x = i \times h$, $y = j \times h$, $k = k \times h$.

The set of allowed strategies:

$$A^h = \{(L,M,c); \Delta L = 0, h, \Delta M = 0, h, 0 \leq c \leq \bar{c}\}$$

or

$$A^h = \{(L,M,c); \Delta L = 0, h, \Delta M = 0, h, c = l \times h \text{ for } l = 0, \dots, N(l)\}$$

The choice will be dictated by the utility function. If one can solve the maximization problem in closed form, the first choice is to be preferred. The second form, simplifies the numerical procedure, looking for the solution

of the maximization problem in a grid. In both cases, the upper bound in consumption should be chosen so as it is never binding. Again, such a \bar{c} always exist for a given compact domain.

The continuous time process (x_t, y_t, k_t) is to be approximated by Markov chains $\{(x_n, y_n, k_n); n = 1, 2, \dots\}$ where the index n denotes time. Next I describe the set of allowed transitions and their probabilities. These will differ depending on the transaction regions. I start with the case where it is optimal not to transact.

NO TRANSACTION REGION ($\Delta L = \Delta M = 0$)

In this case, transitions will be allowed for the 6 closest neighbors, two diagonal states depending on the sign of the correlation of the Brownian motions and also a transition to the current state. The directions of transitions are represented by 9 vectors in R^3 . In the case $\rho < 0$, I define:

$$\begin{aligned}
v_0 &= (0, 0, 0) & : & (i, j, k) \mapsto (i, j, k) \\
v_1 &= (1, 0, 0) & : & (i, j, k) \mapsto (i+1, j, k) \\
v_2 &= (-1, 0, 0) & : & (i, j, k) \mapsto (i-1, j, k) \\
v_3 &= (0, 1, 0) & : & (i, j, k) \mapsto (i, j+1, k) \\
v_4 &= (0, -1, 0) & : & (i, j, k) \mapsto (i, j-1, k) \\
v_5 &= (0, 0, 1) & : & (i, j, k) \mapsto (i, j, k+1) \\
v_6 &= (0, 0, -1) & : & (i, j, k) \mapsto (i, j, k-1) \\
v_7 &= (0, 1, -1) & : & (i, j, k) \mapsto (i, j+1, k-1) \\
v_8 &= (0, -1, 1) & : & (i, j, k) \mapsto (i, j-1, k+1)
\end{aligned}$$

If one is interested in the case $\rho \geq 0$, the only modification required in all that follows is to substitute the vectors v_7 and v_8 by

$$\begin{aligned}
\tilde{v}_7 &= (0, 1, 1) & : & (i, j, k) \mapsto (i, j+1, k+1) \\
\tilde{v}_8 &= (0, -1, -1) & : & (i, j, k) \mapsto (i, j-1, k-1)
\end{aligned}$$

Next define quantities $q_i^{0,h}$ and $q_i^{1,h}$ for $i = 1, \dots, 8$. These are the building blocks of the transition probabilities.

$$\begin{aligned}
q_1^{0,h}((i, j, k); c) &= (r - c)^+ \\
q_2^{0,h}((i, j, k); c) &= (r - c)^- \\
q_3^{0,h}(i, j, k) &= \left(\mu - \frac{\sigma^2}{2}\right)^+ \\
q_4^{0,h}(i, j, k) &= \left(\mu - \frac{\sigma^2}{2}\right)^- \\
q_5^{0,h}(i, j, k) &= \left\{ \frac{a^2}{2kh} \left(\beta\alpha - \frac{\lambda^2}{4} \right) - \frac{\beta kh}{2} \right\}^+ \\
q_6^{0,h}(i, j, k) &= \left\{ \frac{a^2}{2kh} \left(\beta\alpha - \frac{\lambda^2}{4} \right) - \frac{\beta kh}{2} \right\}^- \\
q_7^{0,h}(i, j, k) &= 0 \\
q_8^{0,h}(i, j, k) &= 0
\end{aligned}$$

with the usual the notation $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$

$$\begin{aligned}
q_1^{1,h}(i, j, k) &= 0 \\
q_2^{1,h}(i, j, k) &= 0 \\
q_3^{1,h}(i, j, k) &= \frac{\sigma^2}{2} - \frac{a}{4}|\rho|\sigma\lambda \\
q_4^{1,h}(i, j, k) &= \frac{\sigma^2}{2} - \frac{a}{4}|\rho|\sigma\lambda \\
q_5^{1,h}(i, j, k) &= \frac{\lambda^2 a^2}{8} - \frac{a}{4}|\rho|\sigma\lambda \\
q_6^{1,h}(i, j, k) &= \frac{\lambda^2 a^2}{8} - \frac{a}{4}|\rho|\sigma\lambda \\
q_7^{1,h}(i, j, k) &= \frac{a}{4}|\rho|\sigma\lambda \\
q_8^{1,h}(i, j, k) &= \frac{a}{4}|\rho|\sigma\lambda
\end{aligned}$$

These quantities are non negative for a suitable choice of the parameter a . For instance $a = \frac{2\sigma}{\lambda}$.² The last piece needed to construct the transition probabilities is the normalization factor:

$$\begin{aligned}
Q^h(i, j, k) &= \max_{0 \leq c \leq \bar{c}} \sum_{m=1}^8 [q_m^{1,h} + hq_m^{0,h}] \\
&= h|r - \bar{c}| + h|\mu - \frac{\sigma^2}{2}| \\
&\quad + h \left| \frac{a^2}{2kh} (\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta kh}{2} \right| \\
&\quad + \sigma^2 + \frac{a^2 \lambda^2}{4} - \frac{a}{2}|\rho|\sigma\lambda
\end{aligned}$$

Notice that with this definition, the quantity $Q^h(i, j, k)$ does not depend on the control policy. The transition probabilities are:

$$P_m^h(i, j, k; c) = \frac{q_m^{1,h} + hq_m^{0,h}}{Q^h(i, j, k)}$$

for $m = 1, \dots, 8$, and

$$P_0^h(i, j, k; c) = 1 - \sum_{m=1}^8 P_m^h(i, j, k)$$

The interpolation time of the chain is defined as:

$$\Delta t^h(i, j, k) = \frac{h^2}{Q^h(i, j, k)}$$

which also does not depend on the controls.

²This is the exact place where the procedure for the original process would break down. It would be impossible to guarantee that the corresponding quantities are positive in all the relevant region.

All this generate a chain that is locally consistent with the reflected process. To show it, define $\Delta x_n^h \equiv x_{n+1}^h - x_n^h$, $\Delta y_n^h \equiv y_{n+1}^h - y_n^h$, $\Delta k_n^h \equiv k_{n+1}^h - k_n^h$. Let E_n^h denote the expectation conditional on the n -th-time and state (x_n^h, y_n^h, k_n^h) . Then, it is easy to check that

$$\begin{aligned}
E_n^h[\Delta x_n^h] &= (r - c)\Delta t^h(i, j, k) \\
E_n^h[\Delta y_n^h] &= (\mu - \frac{\sigma^2}{2})\Delta t^h(i, j, k) \\
E_n^h[\Delta k_n^h] &= \left\{ \frac{a^2}{2k_n^h}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta k_n^h}{2} \right\} \Delta t^h(i, j, k) \\
E_n^h[(\Delta x_n^h)^2] &= o(\Delta t^h(i, j, k)) \\
E_n^h[(\Delta y_n^h)^2] &= \sigma^2 \Delta t^h(i, j, k) + (\mu - \frac{\sigma^2}{2})h\Delta t^h(i, j, k) \\
&= \sigma^2 \Delta t^h(i, j, k) + o(\Delta t^h(i, j, k)) \\
E_n^h[(\Delta k_n^h)^2] &= \frac{\lambda^2 a^2}{4} \Delta t^h(i, j, k) + \left\{ \frac{a^2}{2k_n^h}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta k_n^h}{2} \right\} h\Delta t^h(i, j, k) \\
&= \frac{\lambda^2 a^2}{4} \Delta t^h(i, j, k) + o(\Delta t^h(i, j, k)) \\
E_n^h[\Delta k_n^h \Delta y_n^h] &= -\frac{a}{2} |\rho| \sigma \lambda \Delta t^h(i, j, k)
\end{aligned}$$

This estimation, combined with the fact that

$$\begin{aligned}
(E_n^h[\Delta X_n^h])^2 &= o(\Delta t^h(i, j, k)) \\
(E_n^h[\Delta Y_n^h])^2 &= o(\Delta t^h(i, j, k)) \\
(E_n^h[\Delta K_n^h])^2 &= o(\Delta t^h(i, j, k))
\end{aligned}$$

make the first and second moments of the chain close to those of the continuous process (x_t, y_t, k_t) . This property is called local consistency. Next the transaction cases.

SELL STOCK REGION ($\Delta L = 0$, $\Delta M = h$)

In this case, the chain should jump along the direction $((1 + \frac{(kh)^2}{a^2})^{-1} e^{-x}, -e^{-y}, 0)$. However, in general, the new state does not belong to the grid. To overcome this difficulty, I introduce a randomization scheme. The allowed transition directions are:

$$\begin{aligned}
u_1 = (1, 0, 0) & : (i, j, k) \mapsto (i+1, j, k) \\
u_2 = (0, -1, 0) & : (i, j, k) \mapsto (i, j-1, k)
\end{aligned}$$

associated to this directions are the following probabilities:

$$P_{S_1}^h(i, j, k) = \frac{(1 + \frac{(kh)^2}{a^2})^{-1} e^{-ih}}{e^{-ih}(1 + \frac{(kh)^2}{a^2})^{-1} + e^{-jh}}$$

$$Ps_2^h(i, j, k) = \frac{e^{-jh}}{e^{-ih}(1 + \frac{(kh)^2}{a^2})^{-1} + e^{-jh}}$$

BUY STOCK REGION ($\Delta L = h, \Delta M = 0$)

In this case, the chain should jump along the direction $(-e^{-x}, (1 + \frac{(kh)^2}{a^2})^{-1}e^{-y}, 0)$. Again, I need a randomization scheme. The allowed transition directions are:

$$\begin{aligned} w_1 &= (-1, 0, 0) : (i, j, k) \mapsto (i-1, j, k) \\ w_2 &= (0, +1, 0) : (i, j, k) \mapsto (i, j+1, k) \end{aligned}$$

associated to this directions are the following probabilities:

$$\begin{aligned} Pb_1^h(i, j, k) &= \frac{e^{-ih}}{e^{-ih} + (1 + \frac{(kh)^2}{a^2})^{-1}e^{-jh}} \\ Pb_2^h(i, j, k) &= \frac{(1 + \frac{(kh)^2}{a^2})^{-1}e^{-jh}}{e^{-ih} + (1 + \frac{(kh)^2}{a^2})^{-1}e^{-jh}} \end{aligned}$$

The next step is to discretize the consumers maximization problem:

$$V^h(i, j, k) \equiv \max_{A^h} E_{i,j,k}^h \sum_{n=0}^{\infty} e^{-\delta t_n^h} U(e^{x_n} c_n^h) \Delta t_n^h$$

where $t_n^h = \sum_{0 \leq m \leq n} \Delta t_m^h$. Notice that this is the discrete time analogous of 12.

I am now able to describe the discrete time dynamic programming equation. Given a current state $s = (i, j, k)$ it has the following iterative form:

$$\begin{aligned} V^h(s) &= \max \{ Pb_1^h(s)V(s+w_1) + Pb_2^h(s)V(s+w_2), \\ &Ps_1^h(s)V(s+u_1) + Ps_2^h(s)V(s+u_2), \\ &\max_{0 \leq c \leq \bar{c}} \{ e^{-\delta \Delta t^h} \sum_{m=0}^8 P_m^h V(s+v_m) + U(e^{hi}c)\Delta t^h \} \} \end{aligned} \quad (19)$$

The behavior of the chain in the boundaries will be described shortly. First, let me denote:

$$\begin{aligned} D_i^- V(i, j, k) &= \frac{V(i, j, k) - V(i-1, j, k)}{h} \\ D_i^+ V(i, j, k) &= \frac{V(i+1, j, k) - V(i, j, k)}{h} \\ D_j^- V(i, j, k) &= \frac{V(i, j, k) - V(i, j-1, k)}{h} \\ D_j^+ V(i, j, k) &= \frac{V(i, j+1, k) - V(i, j, k)}{h} \end{aligned}$$

$$\begin{aligned}
D_k^- V(i, j, k) &= \frac{V(i, j, k) - V(i, j, k-1)}{h} \\
D_k^+ V(i, j, k) &= \frac{V(i, j, k+1) - V(i, j, k)}{h} \\
d_{jj}^2 V(i, j, k) &= \frac{V(i, j+1, k) - 2V(i, j, k) + V(i, j-1, k)}{h^2} \\
d_{kk}^2 V(i, j, k) &= \frac{V(i, j, k+1) - 2V(i, j, k) + V(i, j, k-1)}{h^2} \\
d_{jk}^2 V(i, j, k) &= -\frac{2V(i, j, k) + V(i, j+1, k-1) + V(i, j-1, k+1)}{2h^2} \\
&\quad + \frac{V(i, j+1, k) + V(i, j-1, k) + V(i, j, k+1) + V(i, j, k-1)}{2h^2} \\
D_{jk}^2 V(i, j, k) &= \begin{bmatrix} d_{jj}^2 V(i, j, k) & d_{jk}^2 V(i, j, k) \\ d_{jk}^2 V(i, j, k) & d_{kk}^2 V(i, j, k) \end{bmatrix} \\
\mathcal{L}_c^h V(i, j, k) &\equiv (r-c)^+ D_i^+ V(i, j, k) + (r-c)^- D_i^- V(i, j, k) \\
&\quad + (\mu - \frac{\sigma^2}{2})^+ D_j^+ V(i, j, k) + (\mu - \frac{\sigma^2}{2})^- D_j^- V(i, j, k) \\
&\quad + \left\{ \frac{a^2}{2kh} \left(\beta\alpha - \frac{\lambda^2}{4} \right) - \frac{\beta kh}{2} \right\}^+ D_k^+ V(i, j, k) \\
&\quad + \left\{ \frac{a^2}{2kh} \left(\beta\alpha - \frac{\lambda^2}{4} \right) - \frac{\beta kh}{2} \right\}^- D_k^- V(i, j, k) \\
&\quad + \frac{1}{2} \text{Tr} D_{jk}^2 V(i, j, k) \Sigma
\end{aligned}$$

Using this notation I can express the Bellman equation as:

$$\begin{aligned}
0 &\geq -\frac{1 - e^{-\delta\Delta t^h}}{\Delta t^h} V^h(s) + \max_{0 \leq c \leq \bar{c}} \{ \mathcal{L}_c^h V^h(s) + U(s, c) \} \\
0 &\geq P s_1^h(s) D_i^+ V(s) - P s_2^h(s) D_j^- V(s) \\
0 &\geq P b_2^h(s) D_i^+ V(s) - P b_1^h(s) D_j^- V(s)
\end{aligned}$$

where one of this inequalities hold as an equality for each state s . Clearly the quantity $\frac{1 - e^{-\delta\Delta t^h}}{\Delta t^h}$ approximates the discount factor δ as h approaches zero. This is the discrete time analog of the original variational inequality. The numerical scheme is complete with the specifications of the boundary conditions.

Given the reflected processes,

$$(x_t, y_t, k_t) \in [\underline{M}^x, \overline{M}^x] \times [\underline{M}^y, \overline{M}^y] \times [0, \overline{M}^k]$$

I divide the specifications in two classes. The first one deals with the natural boundaries of the original domain, that is the region where one of the original processes $X(t), Y(y), K(t)$ reaches zero. The behavior of the processes (x_t, y_t, k_t) at the lower boundaries $\underline{M}^x, \underline{M}^y$ and 0, will mimic the behavior of the original processes at zero.

The second one deals with the upper boundaries $\overline{M}^x, \overline{M}^y, \overline{M}^k$. Again the behavior of the processes (x_t, y_t, k_t) at this points will mimic the behavior of $X(t), Y(y), K(t)$ at (∞, ∞, ∞) .

The convergence result to be presented holds for any specification of the boundary behavior. While true in theory, one should look for a non distorting boundary specification, for it will affect the actual numerical implementation. In the actual implementation, one start with an arbitrary value function. Then the computation follows the scheme in 19 for all points internal to the grid. Next the boundary values of the new value function is computed as described below.

NATURAL BOUNDARIES

1. $(i, j, k) = (-\underline{N}^x, -\underline{N}^y, 0)$. I set

$$V^h(-\underline{N}^x, -\underline{N}^y, 0) = \frac{U(0)}{\delta}$$

2. $(i, j, k) = (\cdot, \cdot, 0)$. The fact that zero is an entrance boundary for the k_t process dictates

$$V^h(\cdot, \cdot, 0) = V^h(\cdot, \cdot, 1)$$

3. $(i, j, k) = (-\underline{N}^x, \cdot, \cdot)$. I assume it is optimal to sell stock. So, for $j > -\underline{N}^y$.

$$\begin{aligned} V^h(-\underline{N}^x, j, k) &= V^h(-\underline{N}^x + 1, j, k)Ps_1^h(-\underline{N}^x + 1, j, k) \\ &+ V^h(-\underline{N}^x, j - 1, k)Ps_2^h(-\underline{N}^x, j - 1, k) \end{aligned}$$

4. $(i, j, k) = (\cdot, -\underline{N}^y, \cdot)$. I assume it is optimal to buy stock. So, for $i > -\underline{N}^x$.

$$\begin{aligned} V^h(i, -\underline{N}^y, k) &= V^h(i - 1, -\underline{N}^y, k)Ps_1^h(i - 1, -\underline{N}^y, k) \\ &+ V^h(i, -\underline{N}^y + 1, k)Ps_2^h(i, -\underline{N}^y + 1, k) \end{aligned}$$

UPPER BOUNDARIES

1. $(i, j, k) = (\cdot, \cdot, \overline{N}^k)$. I impose

$$V(\cdot, \cdot, \overline{N}^k) = V(\cdot, \cdot, \overline{N}^k - 1).$$

For the other two upper boundaries, first notice that if I write $W(X, Y, K)$ for the value function of the original processes, then it should be the case that at the infinity

$$\frac{\partial W}{\partial X} = \frac{\partial W}{\partial Y} = 0$$

because the marginal utility of wealth should decrease to zero. Using the change of variables relating V and W , one concludes that

$$V(\log X, \log Y, k) = W(X, Y, K).$$

2. $(i, j, k) = (\bar{N}^x, \cdot, \cdot)$. I set

$$V(\bar{N}^x, \cdot, k) = V(\bar{N}^x - 1, \cdot, k)e^h$$

3. $(i, j, k) = (\cdot, \bar{N}^y, \cdot)$. I set

$$V(\cdot, \bar{N}^y, k) = V(\cdot, \bar{N}^y - 1, k)e^h$$

4. $(i, j, k) = (\bar{N}^x, \bar{N}^y, \cdot)$. I set

$$V(\bar{N}^x, \bar{N}^y, k) = V(\bar{N}^x - 1, \bar{N}^y - 1, k)e^{2h}$$

B. Convergence of the Scheme

Let $B(G^h)$ denote the space of real valued functions on the lattice G^h . For what follows, let me introduce the mappings

$$\begin{aligned} \mathcal{N}^{h,c} &: B(G^h) \mapsto B(G^h) \text{ for } 0 \leq c \leq \bar{c} \\ \mathcal{N}^h &: B(G^h) \mapsto B(G^h) \\ \mathcal{B}^h &: B(G^h) \mapsto B(G^h) \\ \mathcal{S}^h &: B(G^h) \mapsto B(G^h) \end{aligned}$$

defined as

$$\begin{aligned} \mathcal{N}^{h,c}V(s) &\equiv e^{-\delta\Delta t^h(s)} \sum_{m=0}^8 P_m^h(s, c)V(s + v_m) + U(s, c)\Delta t^h(s) \\ \mathcal{N}^hV(s) &\equiv \max_{0 \leq c \leq \bar{c}} \mathcal{N}^{h,c}V(s) \\ \mathcal{B}^hV(s) &\equiv Pb_1^h(s)V(s + w_1) + Pb_2^h(s)V(s + w_2) \\ \mathcal{S}^hV(s) &\equiv Ps_1^h(s)V(s + u_1) + Ps_2^h(s)V(s + u_2) \end{aligned}$$

define also $\Delta^h = \min_{s \in G^h} \Delta t^h(s)$. Clearly $\Delta^h > 0$. Define also the usual operator norm, using the L^∞ norm on the space of bounded functions $B(G^h)$.

$$\|\mathcal{T}\| = \sup_{V \in B} \frac{\|\mathcal{T}V\|_\infty}{\|V\|_\infty}$$

In terms of this operator the discrete Bellman equation can be written as:

$$V^h(s) = \max\{\mathcal{N}^hV^h(s), \mathcal{B}^hV^h(s), \mathcal{S}^hV^h(s)\}$$

Naturally, a fixed point of this operator equation will be the solution of the Bellman equation. In order to prove the convergence of the numerical scheme I need to establish the existence of a fixed point for this equation and to establish some properties of the operators $\mathcal{N}^{h,c}$, \mathcal{N}^h , \mathcal{B}^h and \mathcal{S}^h .

I say that $V \leq V'$ if $V(x,y,k) \leq V'(x,y,k)$ for all $(x,y,k) \in G^h$. An operator \mathcal{T} is called monotone if $V \leq V'$ implies $\mathcal{T}V \leq \mathcal{T}V'$.

Proposition 5 $\mathcal{N}^{h,c}$, \mathcal{N}^h , \mathcal{B}^h and \mathcal{S}^h are monotone operators.

Proposition 6 \mathcal{B}^h and \mathcal{S}^h are contractions with norms $\|\mathcal{B}^h\| \leq 1$ and $\|\mathcal{S}^h\| \leq 1$.

Proposition 7 \mathcal{N}^h is a strict contraction.

The strict contraction property of \mathcal{N}^h , the contraction property of \mathcal{B}^h and \mathcal{S}^h , combined with the fact that there are states where it is optimal not to transact, imply that there is a unique fixed point in the operator equation.

The next property I am concerned is the stability of the scheme. To establish it, I have to show that, given a compact domain and a sequence of grids G^h in this domain, there exists a $h_0 > 0$, such that for all $0 < h < h_0$ the solution of the operator equation has a bound independent of h . For that, it suffices that the operators \mathcal{N}^h , \mathcal{B}^h and \mathcal{S}^h have such a bound. It remains to show the property for the operator \mathcal{N}^h .

Proposition 8 For all $0 < h < h_0$, the operator \mathcal{N}^h has a bound independent of h .

The last property of the scheme I have to deal with is consistency. Usually, a scheme $T^hV = f$ is said to be consistent with a partial differential equation $T^hV = f$ if for any smooth function ϕ the property $\lim_{h \rightarrow 0} T^h\phi - T\phi = 0$ holds pointwise for each grid point. In the presence of gradient constraints, the property needed for the convergence result is a slightly modified version of the usual property.

Proposition 9 The numerical scheme is consistent in the following sense: Given a smooth function $\phi^h(i, j, k) \equiv \phi(hi, hj, hk)$ in a grid point, there exists positive constants $a(s)$ and $b(s)$, such that

$$\begin{aligned} \mathcal{N}^h\phi^h(s) - \phi^h(s) &= \Delta^h(s) \{-\delta\phi^h(s) + \max_{0 \leq c \leq \bar{c}} \{\mathcal{L}_c \phi^h(s) + U(x, c)\} + O(h)\phi^h(s)\} \\ \mathcal{B}^h\phi^h(s) - \phi^h(s) &= [\mathcal{B}\phi^h(s) + o(h)]ha(s) \\ \mathcal{S}^h\phi^h(s) - \phi^h(s) &= [\mathcal{S}\phi^h(s) + o(h)]hb(s) \end{aligned}$$

I am now in a position to state the main result of this section, which says that the solution of the discrete problem converge to the solution of the continuous problem. The reader is warned that in order for the result to be tight, one would need a comparison theorem for upper and lower semicontinuous functions or to prove that V_* and V^* are uniformly continuous with sublinear growth.

Proposition 10 The sequence $V^h(i, j, k)$ of solutions of the operator equation converge to the value function $V(x, y, k)$ of the H-J-B equation uniformly in any compact subset of $Q = (-\infty, \infty) \times (-\infty, \infty) \times (0, \infty)$ as $h \downarrow 0$, $\underline{M}^x \downarrow -\infty$, $\underline{M}^y \downarrow -\infty$, $\overline{M}^x \uparrow \infty$ and $\overline{M}^y \uparrow \infty$ such as $hi \rightarrow x$, $hj \rightarrow y$ and $hk \rightarrow k$.³

³With a little abuse of notation on the k variable.

VI. Numerical Results

This chapter describes the results of the algorithm presented in the last section. I am going to emphasize the distinction between the optimal policies in the Random Transaction Costs case (RTC for short) and those obtained in the Fixed Transaction Cost case (FTC). Notice that the FTC case or the Constantinides (1986) case can be viewed as the system 9–11 with $\lambda = 0$ and $k = \alpha$. Thus the same algorithm apply to both cases. This is important for the exercise I present, since any adjustments needed in the algorithm can be applied to both cases. For instance, I mentioned in the last section that the convergence of the algorithm takes place irrespective of the outward boundary condition. However the choice there will affect the actual result. So by imposing the same condition for both cases I hope that both results are affected in a similar way.

A. The Setup

I implemented the algorithm in Fortran 90 in a 300 MHz Pentium II workstation. It takes about 24 hours or 6000 steps for the algorithm to converge. I proceeded as follows:

1. Set the initial guess. I always set $V_{old} = V_{Merton}$, the solution to the Merton problem.
2. Using the algorithm and V_{old} produce V_{new} .
3. If $\max\left\{\frac{V_{new}-V_{old}}{V_{old}}\right\} < 10^{-6}$, stop. Else, set $V_{old} = V_{new}$ and go to 2.

I used the utility function $U(c) = \frac{c^\gamma}{\gamma}$. This choice reduces the computational time in about 10 times as compared to a general utility function. The reason is that the maximization problem $\max_{c \geq 0} \{U(c) - cV_x\}$ has a closed form expression in the HARA case. Please recall that in this case, subsection A shows the no transaction [NT] region a wedge, for every value of the transaction cost.

For the main exercise of the thesis I set the time unit to be 1 year and used the parameters in table A.

Table I about here

I used a step of size $h = 0.025$. The grid spanned a stock and bond holdings from US\$ 0.17 to US\$ 90.02 in 251 steps. The transaction costs span was from 0.03% to 18% in 28 steps. The nonlinear transformation I used in the previous section implies that the grid points are more dense close to the origin.

The transaction costs grid includes the value of 2.95% in the position 12. This is roughly the mean reverting level and is close to the middle of the grid. The parameters I have chosen for the transaction costs process implies a reasonable stationary distribution. Please refer to figure 2.

B. Main Findings

Next I present the main findings of the paper. That is, the best way to cope with random transaction costs is to look for bargains. Figures 3–7 clearly demonstrate the situation. Figure 3 is just a reminder of the general setup. It depicts a contour plot of the value function and the 3 distinct transaction regions for a given level of transaction cost.

Figure 4 shows the various transaction lines for the Constantinides case. Notice that even for a very high transaction cost (18%) transactions do take place. Contrast this with the RTC situation shown in picture 5 below. The values of transaction costs depicted are $k_1 = 0.03\%$, $k_2 = 1.56\%$ and $k_3 = 1.98\%$, all below the mean reverting value of 3%. For the last value, no stock is ever purchased, the Buy Stock line degenerates to the x axis. Stocks are sold in this case only under extreme unbalanced positions, roughly with the Stock holdings over 10 times the bond holdings.

The whole picture is shown in the figures 6–7. These are plots of the angles of the lines obtained using least squares. They clearly show the optimal way to deal with Random Transaction Costs. Do not engage in transactions even if the costs are fair. It is best to wait for bargains.

Let me also add that the waiting period to come back to the market is non trivial. Figure 6 shows that nearly all transactions take place when the transaction cost is below a value somewhere between $k_2 = 1.56\%$ and $k_3 = 1.98\%$. Figure 2 implies that this happens only about 10% of the time. Furthermore, figure 8 shows that the waiting period until transaction costs fall back to these levels are significant. Facing a 5% transaction cost, the consumer should wait on average, 3 to 4.5 weeks for it to go to the range 1.56% – 1.98%.

Before I proceed with more quantitative results, let me present a better description of the transaction lines produced by the algorithm. Recall that I proved that in the HARA case, the transaction regions are separated by straight lines through the origin. First notice that figure 5 show a slightly strange behaviour near the outward boundaries. This is probably caused by boundary condition imposed, together with the relative coarseness of the grid there. Figures 9 and 10 above show the lines closer and closer to the origin. Figure 9 depicts nearly straight lines, avoiding the irregularities of figure 5. Figure 10 show that the behaviour is still very reasonable near the origin. The transaction regimes are still clearly separated.

C. Consumption Loss

The next study shows the overall loss incurred by the consumer in different settings. The studies consider an initial holding of US\$4.48 in the bond account and US\$5.75 in the stock account. Where applicable RTC start at 2.95%. I show the value function and the consumption loss in a certainty equivalent formulation. By that I mean that if I consider the problem

$$V(x_0) = \max_{c \geq 0} \int_0^{\infty} e^{-\delta t} U(c_t) dt$$

subject to

$$dx_t = (r - c_t)x_t dt \quad x_0 \text{ given}$$

then optimal consumption is a constant and since $V(x_0)$ is increasing, it can be written as

$$c^* = c(x_0^{-1}(V))$$

The consumption loss is computed using the value functions obtained in the algorithm, benchmarked against

$$c^0 = c(x_0^{-1}(V_{Merton}))$$

Figure 11 depicts the situation for two cases of RTC. Namely, the case of negative correlation ($\rho = -0.8$) and the zero correlation case. The difference agrees with the intuition that consumers are worse off in the negative correlation case. However, negative correlation appears to be a minor distraction.

The unavoidable conclusion is that the consumer is much better off in the RTC case. Notice the nearly zero slope of the RTC case.⁴ Consumers do not seem to be bothered by a temporarily high transaction cost for they know to wait. In figure 11, I also show an example of naive behaviour. This comes from a Monte Carlo study I describe next.

D. A Monte Carlo Study

In the processing of the algorithm, I obtain the optimal consumption policy and the transaction lines. The system then becomes suitable to simulations. Notice that the algorithm ties a given consumption process in the grid to transition probabilities of the Markov chain. Evolving the chain, starting in the no transaction region, I compute the utility achieved in each step and also check if one of the transaction lines have been crossed. If this is the case, the algorithm applies a randomization scheme to bring the process back to the no transaction region. Meanwhile, I record that as an instance of a transaction. I did 1 million simulations, stopping the system at a time t such that:

$$\int_0^t e^{-\delta s} ds = \frac{0.999}{\delta}$$

What I call naive behaviour is to ignore that transaction costs are random. I solved the FTC case for the various transaction costs in the grid. Thus, obtaining the consumption process and transaction lines for several Constantinides problems. Next I introduced these into the dynamics of RTC. That is, the naive consumer checks every time his/her holdings and the current transaction cost. Believing the cost will remain fixed at the current level forever, he/she acts optimally. However, transaction costs are random. The table II below show the results. The Monte Carlo started with US\$4.48 in the bond account, US\$5.75 in the stock account and RTC at 2.95%. The \pm intervals have a coverage of 99%.

⁴They are equal to zero within the precision set for the algorithm.

One can see that the naive behaviour can hurt the consumer rather seriously. The two values for the optimal policy corresponds to the value found in the Monte Carlo study and the value function computed with the algorithm. There is a slight upward bias in the Monte Carlo setting. This implies that if one uses these values as a control variate for the naive policy, the consumption loss will be even stronger. Figure 11 show that a consumer facing a RTC averaging 3%, but despising the RTC effect, ends up being as well off a consumer facing a FTC of around 7%.

Table II about here

The next exhibit is the table A. It sheds light on the source of the well being of the consumer in the RTC case. Glancing at figures 4–5 one could imagine that the consumer transacts rather less in the RTC case. This conclusion is not warranted. The Monte Carlo study shows that the optimal policy for the RTC case transacts more often than the naive policy. The source of well being comes from placing the buy/sell orders at bargain spreads. The table below show the yearly average of both strategies.

Table III about here

E. Comparative Statics

I finish this section by presenting some comparative statics. I showed that the ability to choose the best moment to transact is very valuable to the consumer. However, the consumer may have to keep an unbalanced position while the RTC is temporarily high. The bigger the volatility of the RTC process the fastest the consumer will see bargain spreads again. Figures 12–13 confirm this intuition. The consumer is better off with high volatility in the spreads, and again, the higher the volatility, the wider is the no transaction region, since the incentive to wait for bargains is larger.

The last comparative statics study deals with correlation. I found that the transaction lines vary slightly with correlation in a reasonable way. Please look at figure 14. With negative correlation, the Sell Stock line is a little above than in the zero correlation case. Close to the line, the consumer reasons: should stock prices go up I will have to sell. But if ρ is negative, RTC will be smaller than in the $\rho = 0$ case, so I can wait a little longer. The reverse reasoning applies in the Buy Stock line, making the consumer less willing to wait.

VII. Conclusion and Extensions

I have shown that the best way to cope with bad terms of trade is to wait. This finding has a theoretical interest on its own, but also raises questions in two other areas, namely, option pricing and market microstructure.

I have shown that in the balance between keeping a desirable portfolio and waiting for better terms of trade, the consumer prefers to wait. However, in a option hedging situation, like the one described in Davis, Panas,

and Zariphopoulou (1993), the urge to rebalance the portfolio much stronger, so one cannot tell which effect will dominate.

In the information based branch of market microstructure literature, one usually see models where the noise traders are much less responsive to bid-ask spread fluctuations than predicted in this paper. Increasing the elasticity of the response will impair the ability of the informed traders to disguise themselves into noise traders. The result is likely to be an improvement in the information flow to the markets.

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Appendix A. Proofs

Proof of Proposition 2:

(a) By the linearity of the system 9–11 with respect to $(C(t), L(t), M(t))$, it follows that if

$$(C^1, L^1, M^1) \in A_{(x_1, y_1, k)}$$

and

$$(C^2, L^2, M^2) \in A_{(x_2, y_2, k)}$$

then

$$\lambda(C^1, L^1, M^1) + (1 - \lambda)(C^2, L^2, M^2) \in A_{(\lambda(x_1, y_1, k) + (1 - \lambda)(x_2, y_2, k))}.$$

this, together with the concavity of $U(c)$ yields the desired result.

(b) If $x > x'$ and $(C(t), L(t), M(t)) \in A_{(x', y, k)}$ then,

$$(C(t) + \frac{\delta}{\delta - r}(x - x'), L(t), M(t)) \in A_{(x, y, k)}.$$

this is because it is feasible to behave as if you were poorer. The extra cash is then consumed at a constant rate. $\frac{\delta}{\delta - r}x$ is the optimal consumption for the trivial problem where the only asset is a money market account, whose value starts at x . Hence $V(x, y, k) > V(x', y, k)$.

If $y > y'$ it is feasible to sell the extra stock, what strictly increases your money market holdings and apply the argument above.

For the monotonicity in k , suppose $k' > k$. A standard comparison principle for SDE's yields that if $K(t)$ is a solution of 11 with $K(0) = k$ and $K'(t)$ is a solution with $K'(0) = k'$ then $K'(t) \geq K(t)$ almost surely. Hence, $k' > k$ implies $A_{(x, y, k')} \subseteq A_{(x, y, k)}$. Now, let $J(x, y, k; k')$ be the value of acting as if transaction costs started at k' while in fact they started at k . It follows that

$$V(x, y, k) \geq J(x, y, k; k') \geq V(x, y, k')$$

Notice that it is feasible never to engage in any transactions and just consume optimally from the bond. Thus

$$\frac{1}{\delta} U\left(\frac{\delta}{\delta - r}x\right) \leq V(x, y, k) \leq W_M(x, y)$$

where $W_M(x, y)$ is the solution to the Merton Problem. This shows that $V(x, y, k)$ cannot be concave in k .

(c) I first establish continuity in (x, y) . Consider sequences $(x_n, y_n) \rightarrow (x, y)$.

If $x, y > 0$, $V(x_n, y_n, k) \rightarrow V(x, y, k)$ by concavity.

If $x = 0$, $y > 0$ then $V(0, y_n, k) \rightarrow V(0, y, k)$ also by concavity. Hence, if $x_n \downarrow 0$ then $V(0, y + \frac{x_n}{1+k}, k) \downarrow V(0, y, k)$. But due to the possibility of lump transactions

$$V(0, y + \frac{x_n}{1+k}, k) \geq V(x_n, y, k) \geq V(0, y, k)$$

The case $y = 0, x > 0$ is handled similarly.

If $x = y = 0$, I appeal to the solution to the Merton problem to get

$$0 \leq V(0, 0, k) \leq W_M(x, y) \rightarrow 0 \text{ as } (x, y) \rightarrow (0, 0).$$

For the continuity of $V(x, y, k)$ in k , it is convenient to treat first the case k large. Let $Z(t) = \frac{2}{\lambda} \sqrt{K(t)}$. By Ito's lemma

$$dZ(t) = \left(\frac{\beta}{\lambda} (\alpha - K(t)) - \frac{\lambda}{2} \right) K^{-1/2}(t) dt + dW_{2,t}$$

now find k^* such that the drift is smaller than -1 for $k \geq k^*$. Notice that the drift is decreasing in k . Let $d\tilde{Z}(t) = -dt + dW_{2,t}$, fix $k \geq k^*$ and for $0 < a_n \downarrow 0$, define the stopping times

$$\tau_n(k) = \inf\{t \geq 0; Z(t) = \frac{2\sqrt{k}}{\lambda}, Z(0) = \frac{2\sqrt{k+a_n}}{\lambda}\}$$

$$\tilde{\tau}_n = \inf\{t \geq 0; \tilde{Z}(t) = \frac{2\sqrt{k}}{\lambda}, \tilde{Z}(0) = \frac{2\sqrt{k+a_n}}{\lambda}\}$$

A standard comparison result for SDE's yields that $\tau_n(k) \leq \tilde{\tau}_n$ a.s. and that $\tilde{\tau}_n$ is independent of k . Now notice that the monotonicity of $V(x, y, k)$ in k implies that if $V(x, y, k+a_n) \uparrow V(x, y, k)$ for some sequence $0 < a_n \downarrow 0$, then V is continuous in k .

Now consider the following strategies for the initial points $(x, y, k+a_n)$. Between $[0, \tau_n)$ do nothing, consume zero; proceed optimally thereafter. Let the value of such strategy be $J(x, y, k+a_n)$. Then

$$V(x, y, k+a_n) \geq J(x, y, k+a_n) = E\{e^{-\delta\tau_n} V(X(\tau_n), Y(\tau_n), k)\}$$

$(X(t), Y(t))$ is the solution of equations 9–10 with $C = L = M = 0$. Now, since $\tilde{\tau}_n \xrightarrow{P} 0$ as $a_n \downarrow 0$ it follows that $\tau_n \xrightarrow{P} 0$ and consequently

$$e^{-\delta\tau_n} V(X(\tau_n), Y(\tau_n), k) \xrightarrow{P} V(x, y, k)$$

thanks to the continuity of V in (x, y) . To finish this part, just claim the dominated convergence theorem, noting that $\tilde{\tau}_n$ is almost surely bounded along some subsequence $\tilde{\tau}_{n_j}$.

The proof implies that the convergence is uniform for $k > k^*$. Thanks to the concavity of $V(\cdot, \cdot, k)$ the convergence is also uniform in (x, y) .

The same proof applies to $k > 0$ arbitrary (with the loss of uniformity). The reason is that, thanks to the stationarity of $K(t)$, it is still true that $\tau_n(k) \xrightarrow{P} 0$ (e.g. Karlin and Taylor (1981)). The case $k = 0$, however, cannot be handled similarly, because $E \tau_n(0) = \infty$ (0 is an entrance boundary).

For continuity in $k = 0$, let $(C(t), L(t), M(t))$ be an optimal strategy for the initial condition $(x, y, 0)$. Lemma 1 shows that $L(t), M(t)$ are continuous except perhaps at $t = 0$. Since at $t = 0$ transactions are costless, all initial jumps with $|L(0^+) - M(0^+)| = |\tilde{L}(0^+) - \tilde{M}(0^+)|$ are equivalent. I normalize things setting either $L(0^+) = 0$ or $M(0^+) = 0$. Consider the case $L(0^+) = \alpha \geq 0$ (so $M(0^+) = 0$).

$$V(x, y, 0) = EV(x - \alpha, y + \alpha, 0) = V(x - \alpha, y + \alpha, 0)$$

for $\varepsilon > 0$, define the stopping time

$$\tau_\varepsilon = \inf\{t \geq 0; K(t) = \varepsilon, K(0) = 0\}$$

then,

$$\begin{aligned} V(x - \alpha, y + \alpha, 0) &= \int_0^{\tau_\varepsilon} e^{-\delta t} U(C(t)) dt + E\{e^{-\delta\tau_\varepsilon} V(X(\tau_\varepsilon), Y(\tau_\varepsilon), \varepsilon)\} \\ &= \int_0^{\tau_\varepsilon} e^{-\delta t} U(C(t)) dt + V(x - \alpha, y + \alpha, \varepsilon) E e^{-\delta\tau_\varepsilon} \end{aligned}$$

$$+ E \{ e^{-\delta \tau_\varepsilon} (V(X(\tau_\varepsilon), Y(\tau_\varepsilon), \varepsilon) - V(x - \alpha, y + \alpha, \varepsilon)) \}$$

$(X(t), Y(t))$ is the solution of equations 9–10 with initial conditions $(x - \alpha, y + \alpha)$. Arguing as above I get

$$V(x - \alpha, y + \alpha, 0) = \lim_{\varepsilon \downarrow 0} V(x - \alpha, y + \alpha, \varepsilon)$$

Now, it suffices to show that

$$\lim_{\varepsilon \downarrow 0} V(x - \alpha, y + \alpha, \varepsilon) = \lim_{\varepsilon \downarrow 0} V(x, y, \varepsilon)$$

Suppose not, assume that for ε sufficient small and $a_\varepsilon > a > 0$

$$\begin{aligned} V(x - \alpha, y + \alpha, \varepsilon) &= V(x, y, \varepsilon) + a_\varepsilon \\ &\geq V(x - \alpha(1 + \varepsilon), y + \alpha, \varepsilon) + a_\varepsilon \end{aligned}$$

or for ε sufficient small and $b_\varepsilon > a > 0$

$$\begin{aligned} V(x, y, \varepsilon) &= V(x - \alpha, y + \alpha, \varepsilon) + b_\varepsilon \\ &\geq V(x, y + \alpha\varepsilon, \varepsilon) \end{aligned}$$

again, the inequalities are due to the feasibility of lump transactions. Both statements contradict the continuity of $V(\cdot, \cdot, \varepsilon)$ for positive ε . The argument with $M(0^+) > 0$ is entirely analogous.

Most of the work has been done. For the uniform continuity of $V(x, y, k)$, just evaluate

$$|V(x, y, k) - V(\tilde{x}, \tilde{y}, \tilde{k})| \leq |V(x, y, k) - V(x, y, \tilde{k})| + |V(x, y, \tilde{k}) - V(\tilde{x}, \tilde{y}, \tilde{k})|$$

I proved above that the first term is uniform continuous. For the second term just write

$$\lim_{k \uparrow \infty} V(x, y, k) = V(x, y, \infty) \geq 0$$

then $V(x, y, \infty)$ is concave and for k sufficiently large

$$|V(x, y, \tilde{k}) - V(\tilde{x}, \tilde{y}, \tilde{k})| \leq |V(x, y, \infty) - V(\tilde{x}, \tilde{y}, \infty)| + 2\varepsilon$$

Uniform continuity follows from the concavity of $V(x, y, \infty)$.

Proof of Proposition 5: The proposition follows immediately from the nonnegativity of $P_m^h(s, c)$, Pb_m^h and Ps_m^h

Proof of Proposition 6: A direct estimation yields the result.

$$\begin{aligned} |\mathcal{B}^h V(s) - \mathcal{B}^h V'(s)| &\leq \sum_{m=1}^2 P b_m^h(s) |V(s + u_m) - V'(s + u_m)| \\ &\leq \max_{s \in G^h} |V(s) - V'(s)| \end{aligned}$$

$$\begin{aligned}
|\mathcal{S}^h V(s) - \mathcal{S}^h V'(s)| &\leq \sum_{m=1}^2 P_s^h(s) |V(s+u_m) - V'(s+u_m)| \\
&\leq \max_{s \in G^h} |V(s) - V'(s)|
\end{aligned}$$

Proof of Proposition 7: Proof: Let a be a constant function on G^h . Then, since probabilities add up to one for all $0 \leq c \leq \bar{c}$

$$\begin{aligned}
\mathcal{N}^h[V(s) + a] &= \max_{0 \leq c \leq \bar{c}} \{e^{-\delta \Delta t^h(s)} \sum_{m=0}^8 P_m^h(s, c) [V(s+v_m) + a] + U(s, c) \Delta t^h(s)\} \\
&\leq \mathcal{N}^h V(s) + e^{-\Delta^h} a.
\end{aligned}$$

This discounting property together with the monotonicity of \mathcal{N}^h form the Blackwell sufficient condition for a strict contraction.

Proof of Proposition 8: Estimate for $\|V\| = 1$,

$$\begin{aligned}
|e^{-\delta \Delta t^h(s)} \mathcal{L}_c^h V(s) + U(s, c) \Delta t^h(s)| &= |e^{-\delta \Delta t^h(s)} \sum_{m=0}^8 P_m^h(s, c) V(s+v_m) \\
&\quad + U(s, c) \Delta t^h(s)| \\
&\leq e^{-\delta \Delta^h} \max_{s \in G^h} |V(s)| + U(s, c) o(h^2) \\
&\leq e^{-\delta \Delta^h} + U(e^{\bar{M}}, \bar{c}) o(h_0^2).
\end{aligned}$$

Proof of Proposition 9: Proof: First estimate

$$\begin{aligned}
D_i^\pm \phi &= \phi_x + o(h) \\
D_j^\pm \phi &= \phi_y + o(h) \\
D_k^\pm \phi &= \phi_k + o(h) \\
D_{jk}^2 \phi &= D^2 \phi + o(h)
\end{aligned}$$

Next by the definition of the probabilities $P_m^h(s, c)$, Pb_m^h and Ps_m^h compute,

$$\begin{aligned}
\mathcal{B}^h \phi^h(s) - \phi^h(s) &= [Pb_2^h(s) D_i^+ \phi^h(s) - Pb_1^h(s) D_j^- \phi^h(s)] h \\
&= [\mathcal{B} \phi^h(s) + o(h)] \frac{h}{e^{-ih} + (1 + \frac{(kh)^2}{a^2})^{-1} e^{-jh}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}^h \phi^h(s) - \phi^h(s) &= [Ps_1^h(s) D_i^+ \phi^h(s) - Ps_2^h(s) D_j^- \phi^h(s)] h \\
&= [\mathcal{S} \phi^h(s) + o(h)] \frac{h}{e^{-ih} (1 + \frac{(kh)^2}{a^2})^{-1} + e^{-jh}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^h \phi^h(s) - \phi^h(s) &= \max_{0 \leq c \leq \bar{c}} \{ -[1 - e^{-\delta \Delta t^h}] \phi^h(s) + \mathcal{L}_c^h \phi^h(s) \Delta t^h + U(s, c) \Delta t^h \} \\
&= \Delta t^h \max_{0 \leq c \leq \bar{c}} \left\{ -\frac{1 - e^{-\delta \Delta t^h}}{\Delta t^h} \phi^h(s) + [\mathcal{L}_c + O(h)] \phi^h(s) + U(s, c) \right\} \\
&= \Delta t^h \{ -\delta \phi^h(s) + \max_{0 \leq c \leq \bar{c}} \{ \mathcal{L}_c \phi^h(s) + U(s, c) \} + O(h) \phi^h(s) \}
\end{aligned}$$

The last equality following from the Theorem of the maximum.

Proof of Proposition 10: Proof: Given a point $s = (x, y, k) \in Q$ define

$$V^*(s) \equiv \limsup V^h(s') \quad (\text{A1})$$

$$V_*(s) \equiv \liminf V^h(s') \quad (\text{A2})$$

where the limits above are taken as $s' \rightarrow s$, $s' \in G^h$ and $h \downarrow 0$.

Since the scheme is stable, V^* is finite in all the domain Q . It is clear that V^* is upper semicontinuous, V_* is lower semicontinuous and $V_* \leq V^*$ on Q . I will show that V^* is a viscosity subsolution and V_* is a viscosity supersolution of the H-J-B equation. The comparison result asserts that $V_* \geq V^*$ on Q . Hence, by uniqueness both are equal to V .

Take a point $s_0 = (x_0, y_0, k_0)$ on Q and let it be a local maximum of $V^* - \phi$, for a smooth function ϕ . Without loss assume that $V^* - \phi = 0$ at s_0 . Since I am taking the limit $h \downarrow 0$, $(\underline{N}^x, \underline{N}^y) \downarrow -\infty$ and $(\bar{N}^x, \bar{N}^y, \bar{N}^k) \uparrow \infty$ in such a way as $h \times (\underline{N}^x, \underline{N}^y) \downarrow -\infty$ and $h \times (\bar{N}^x, \bar{N}^y, \bar{N}^k) \uparrow \infty$, there exists some $h_0 > 0$ such that for $h < h_0$, s_0 is an interior point of the grid G^h . By the definition of limsup, there exists a sequence indexed by h such that, as $h \downarrow 0$

$$s_h \rightarrow s_0 \quad \text{and} \quad V^h(s_h) \rightarrow V^*(s_0)$$

For $h < h_0$, $V^h - \phi$ has a local maximum at s_h . So, for some neighborhood \mathcal{N}_0 of s_0 ,

$$V^h(s_h) - \phi(s_h) \geq V^h(s) - \phi(s)$$

$\forall s \in \mathcal{N}_0$. Modifying ϕ outside \mathcal{N}_0 if necessary, I can assume that the inequality holds globally. Let

$$\xi_h = V^h(s_h) - \phi(s_h) \quad (\text{A3})$$

so that $\xi_h \rightarrow 0$ as $h \downarrow 0$. A and A3 imply

$$V^h(s) \leq \phi(s) + \xi_h \quad (\text{A4})$$

But V^h is the solution of the operator equation

$$V^h(s_h) = \max \{ \mathcal{A}^h V^h(s_h), \mathcal{B}^h V^h(s_h), \mathcal{S}^h V^h(s_h) \}$$

Using the monotonicity of the operators \mathcal{A}^h , \mathcal{B}^h and \mathcal{S}^h together with A4 I conclude that at (s_h)

$$0 \leq \max \{ \mathcal{A}^h[\phi + \xi_h] - [\phi + \xi_h], \mathcal{B}^h[\phi + \xi_h] - [\phi + \xi_h], \mathcal{S}^h[\phi + \xi_h] - [\phi + \xi_h] \}$$

But recall that:

$$\mathcal{A}^h[\phi + \xi_h] - [\phi + \xi_h] = \Delta t^h(s) \{ -\delta[\phi + \xi_h] \}$$

$$\begin{aligned}
& + \max_{0 \leq c \leq \bar{c}} \{ \mathcal{L}_c [\phi + \xi_h] + U(x, c) \} + O(h) [\phi + \xi_h] \\
\mathcal{B}^h [\phi + \xi_h] - [\phi + \xi_h] & = [\mathcal{B}[\phi + \xi_h] + o(h)] ha(s) \\
\mathcal{S}^h [\phi + \xi_h] - [\phi + \xi_h] & = [\mathcal{S}[\phi + \xi_h] + o(h)] hb(s)
\end{aligned}$$

Just divide the first equation by $\Delta t^h(s)$, the second by $ha(s)$ and the third by $hb(s)$ and send $h \downarrow 0$ and conclude that:

$$0 \leq \max \{ -\delta\phi(s_0) + \max_{0 \leq c \leq \bar{c}} \{ \mathcal{L}_c \phi(s_0) + U(x_0, c) \}, \mathcal{B}\phi(s_0), \mathcal{S}\phi(s_0) \}$$

So V^* is a viscosity subsolution. The same argument can be applied to show that V_* is a viscosity supersolution.

Table I
Main Parameters

Risk Aversion	γ	0.4
Time Preference	δ	0.075
Interest Rate	r	0.03
Stock Drift	μ	0.1
Stock Volatility	σ	0.4
Correlation	ρ	-0.8
Mean Reversion Level	α	0.03
Mean Reversion Speed	β	40.0
TC Volatility	λ	0.5

Table II
Consumption Loss

	Value Function	Consumption Loss (%)
Merton (benchmark)	27.25	0
Optimal (value function)	27.13	1.13
Optimal (monte carlo)	27.21 ± 0.02	0.32 ± 0.18
Naive (monte carlo)	26.74 ± 0.02	4.62 ± 0.16

Table III
Yearly Transactions

	Optimal policy	Naive policy
Stock Revenue (US\$)	0.632 ± 0.002	0.599 ± 0.002
Stock Expenditures (US\$)	0.01823 ± 0.00007	0.01215 ± 0.00006
Number of Sells	10.035 ± 0.007	9.823 ± 0.007
Number of Buys	0.551 ± 0.002	0.409 ± 0.002

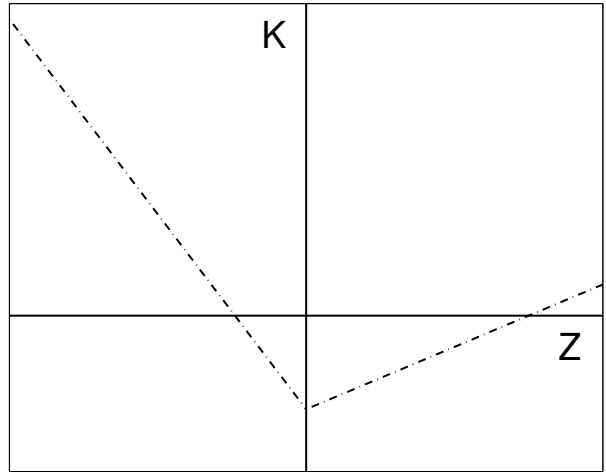


Figure 1. K response to Z shocks

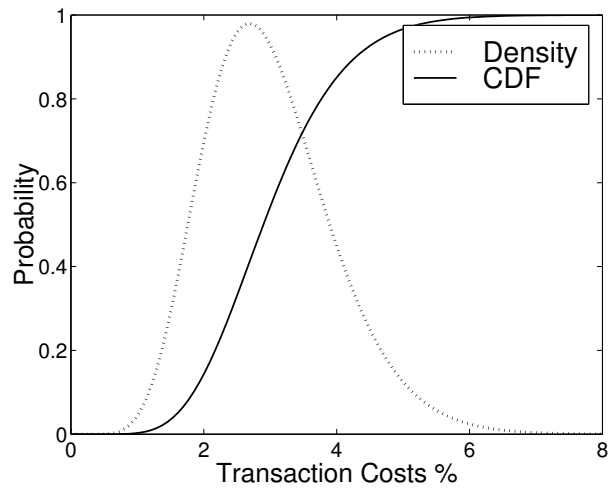


Figure 2. Stationary Distribution

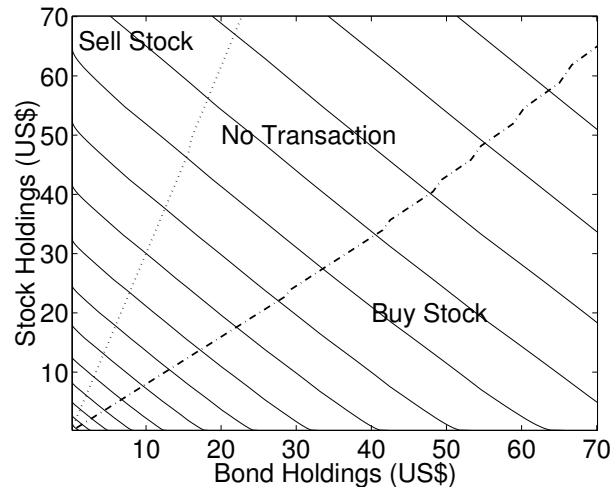


Figure 3. Transaction Regions

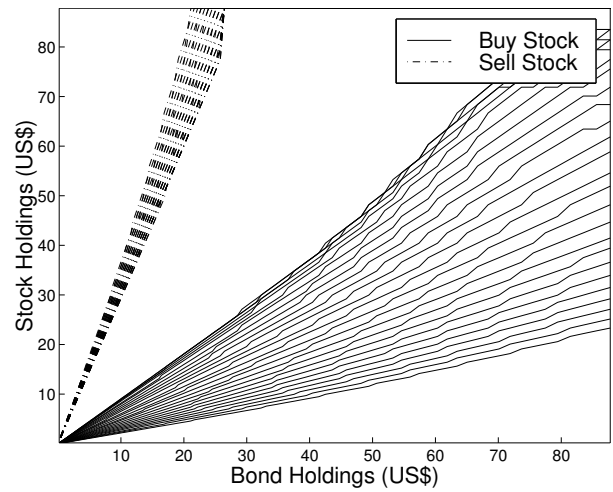


Figure 4. FTC Lines

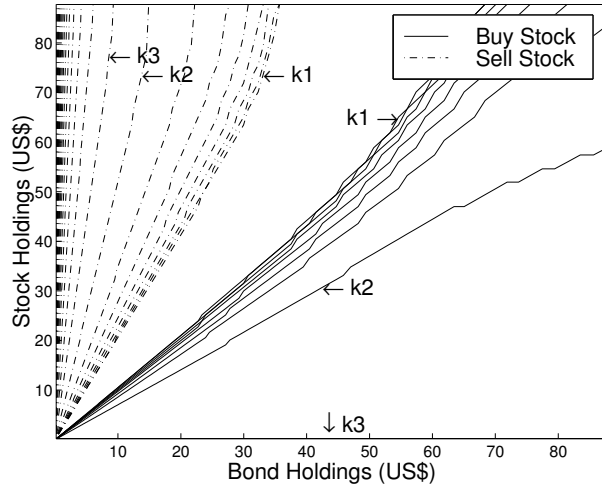


Figure 5. RTC Lines, $k_1 = 0.03\%$, $k_2 = 1.56\%$ and $k_3 = 1.98\%$

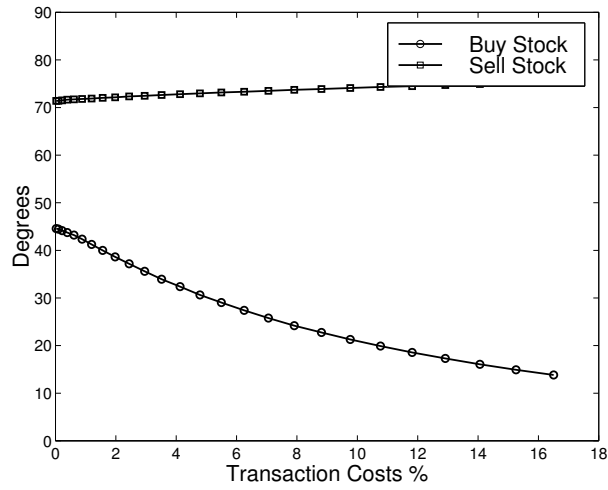


Figure 6. FTC Angles of Lines

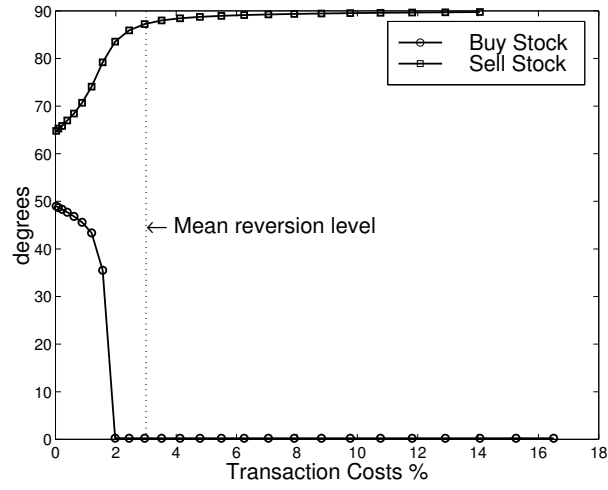


Figure 7. RTC Angles of Lines

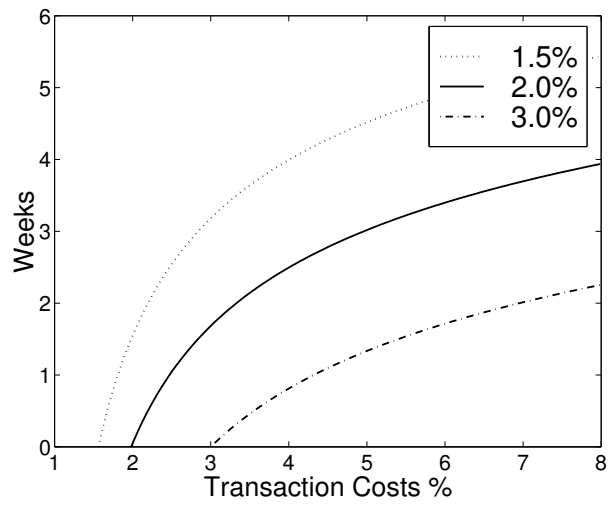


Figure 8. First Passage Times

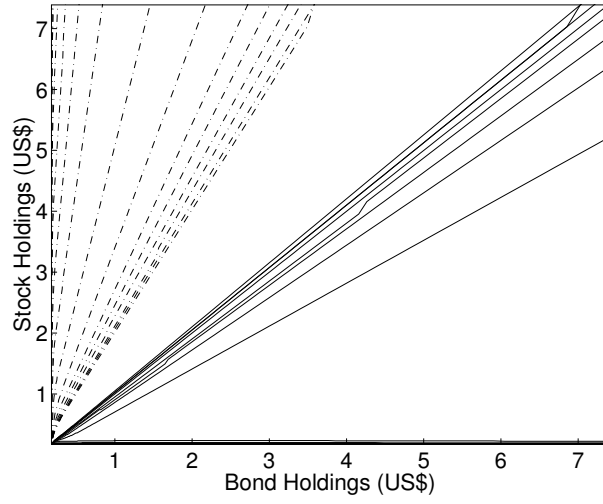


Figure 9. RTC Zoom 1

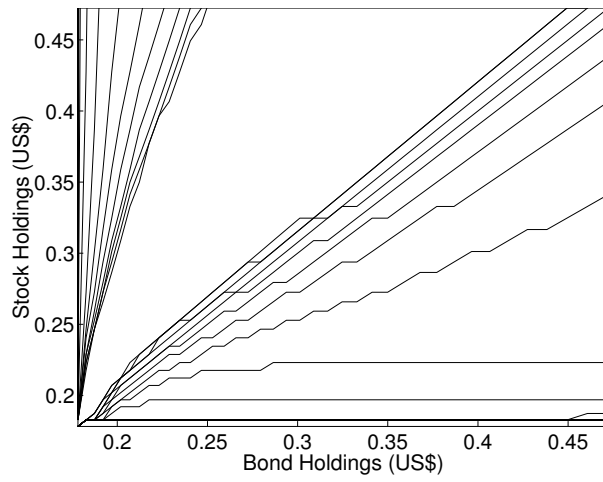


Figure 10. RTC Zoom 2

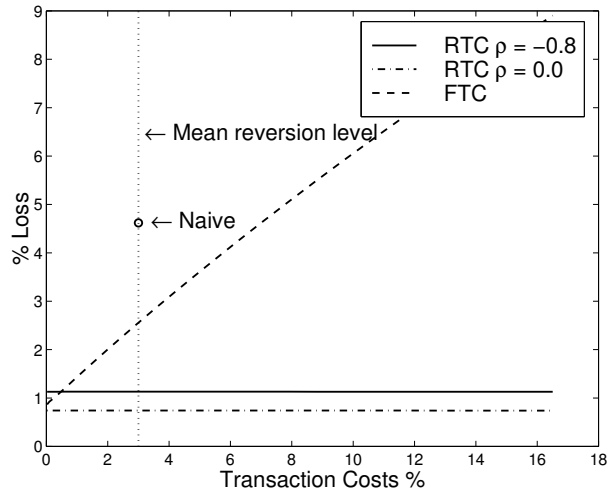


Figure 11. Consumption Loss

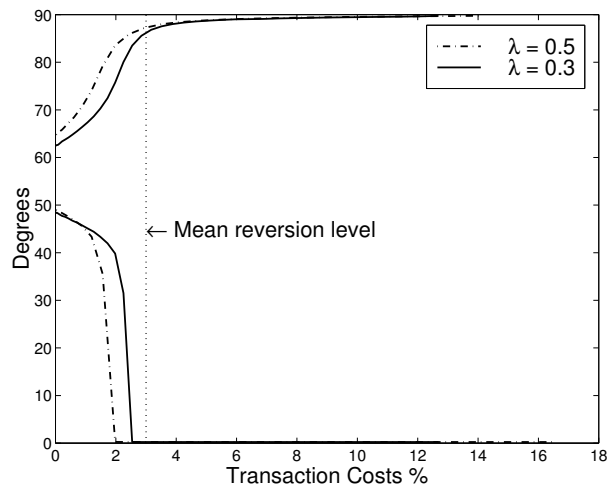


Figure 12. Volatility: Angle of Lines

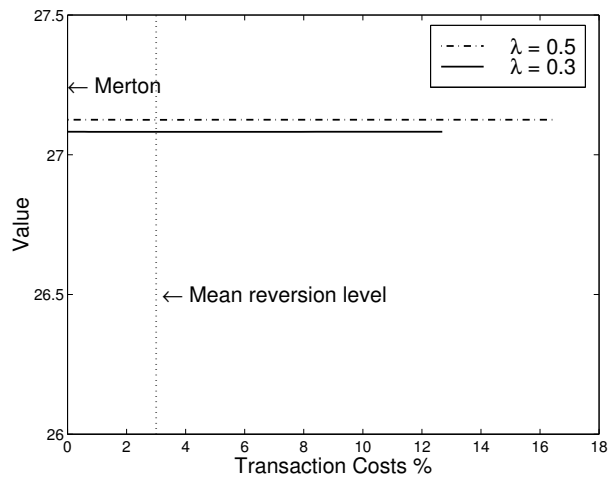


Figure 13. Volatility: Value Function

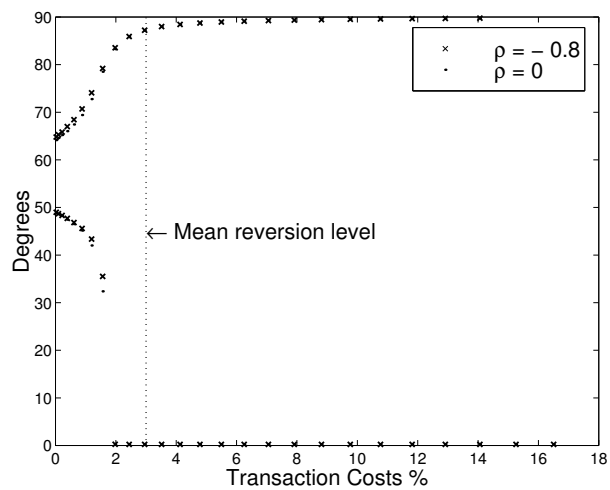


Figure 14. Correlation: Angle of Lines