On the accuracy of the estimated policy function using the Bellman contraction method*

Wilfredo L. Maldonado†
Department of Economics, Universidade Federal Fluminense
Rua Tiradentes 17, Ingá, Niterói
CEP 24210-510, RJ, BRAZIL
Email: willy@impa.br

B. F. Svaiter‡
Instituto de Matemática Pura e Aplicada
Est. Dona Castorina 110, Jardim Botânico, Rio de Janeiro
CEP 22460-320, RJ, BRAZIL
E-mail: benar@impa.br

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Abstract

In this paper we give explicit error bounds for approximations of the optimal policy function in the stochastic dynamic programming problem. The approximated policy function is obtained by using the Bellman contraction method and the error bounds depend on the primitive data of the problem. Neither differentiability of the return function nor interiority of solutions is required. Furthermore, similar error bounds are obtained when Bellman iteration and the computation of the associated policy function are performed inexactly. This shows the robustness of the Bellman method and provides a stopping criterium for computational implementations.

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*Correspondence to Wilfredo L. Maldonado

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1 Introduction

The Bellman method is one of the classical methods used to estimate the value function and the optimal policy function in stochastic dynamic programming problems (Christiano [4], Tauchen [10], Santos and Vigo [8]). This method is based on a contraction mapping and provides a fast algorithm to estimate the value function with high precision. However, until now, only asymptotic convergence results, without numerical error bounds, were obtained for the estimated policy function with this method.

We will prove that the error of the estimated policy function in the n-th step of Bellman method is bounded by

$$\left[\frac{2}{\eta_1}\left(\frac{\|F\|_\infty}{1-\beta}\right)\right]^{1/2}\beta^{n/2}.$$

where F is the return function, β is the discount factor, η_1 is the modulus of strong concavity of F and n is the number of iterations. This estimate holds for a constant function equal to zero as an initial guess for the value function.

There exist other approaches for obtaining good estimates for the optimal policy function. For example, the Euler equation grid method (Baxter *et al.* [1], Coleman [2, 3]), the parameterized expectations method (Marcet and Marshall [6]) and projection methods (Judd [5]). Again, only asymptotic convergence to the optimal policy function was proved for these methods.

Bounds for the distance between the optimal policy function (of the original problem) and the *exact* optimal policy function of a discretized (piecewise linear) version of the problem were obtained by Santos and Vigo [8].

In Santos [7], Euler equation residuals were used to obtain error bounds for an approximated policy function. Either an assumption on the repeated iterations of the approximated policy function (condition NDIV) or a bound on the second derivative of the return function evaluated at the optimal policy function was necessary. Existence of interior solutions and twice differentiablity of the return function were also required.

The error bounds presented in this paper only require boundedness and strong concavity of the return function. These assumptions are quite general and were used by Santos and Vigo [8] and Santos [7]. We require neither differentiability of the return function nor existence of interior solutions.

We also prove the robustness of the Bellman method by considering the use of inexact Bellman operator or inexact solutions in each maximization process. In this case we again obtain error bounds.

Our result has a direct consequence for the use of the Bellman method to estimate the policy function: the number of iterations required to attain a given precision is computed in advance, using only some primitive data of the problem.

This paper is organized into five sections. Section 2 describes the framework we will consider and the hypotheses assumed. Section 3 states the main theorem, which provides an error bound for the estimated policy function when the Bellman contraction method is used. Section 4 shows the robustness of this method under small numerical errors. Conclusions are given in Section 5 and the proofs are given in the appendix.

2 The framework

The stochastic dynamic programming problem is defined using the following elements: the set of values for the endogenous state variables $X \subset \mathbb{R}^l$ (which is a convex Borel set), the set of values for the exogenous shocks $Z \subset \mathbb{R}^k$ (which is a compact set); both are measurable spaces with their σ-algebras denoted by $\mathcal{X}$ and $\mathcal{Z}$ respectively. The evolution of the stochastic shocks is given by the transition function Q defined on $(Z, \mathcal{Z})$ with the Feller property. A (measurable) set $\Omega \subset X \times X \times Z$ describing the feasibility of decisions, *i.e.* if $(x, z) \in X \times Z$ are the current values of the state variable and the shock then $y \in X$ is feasible for the next period if and only if $(x, y, z) \in \Omega$. From this we can define the correspondence $\Gamma : X \times Z \to \mathbb{R}^l$ by $\Gamma(x, z) = \{y \in X;\ (x, y, z) \in \Omega\}$. The one-period return function $F : \Omega \to \mathbb{R}$ is such that $F(x, y, z)$ is the current return if y is chosen for the next period from (x, z). The discount factor is $\beta \in (0, 1)$. With all these elements, the stochastic dynamic programming problem is to find a sequence of contingent plans $(\hat{x}_t)_{t \geq 1}$ (where for all $t \geq 1$, $\hat{x}_t : Z^t \to X$ is a measurable function) such that it solves the following maximization:

$$\begin{aligned} v(x_0, z_0) = \quad & \operatorname{Max} \sum_{t=0}^{\infty} \int_{Z^t} \beta^t F(x_t, x_{t+1}, z_t)\, Q^t(z_0, dz^t) \\ & \text{subject to } (x_t, x_{t+1}, z_t) \in \Omega \text{ for all } t \geq 0 \\ & (x_0, z_0) \in X \times Z \text{ given} \end{aligned}$$

The following hypotheses will be used in this work.

Hypothesis 1. The correspondence Γ is nonempty, compact-valued, continuous and for all $x, x' \in X$, $z \in Z$ and $\alpha \in [0, 1]$ it satisfies:

$$\alpha\Gamma(x, z) + (1 - \alpha)\Gamma(x', z) \subset \Gamma(\alpha x + (1 - \alpha)x', z).$$

Hypothesis 2. The function F is bounded, continuous and there exists $\eta_1 > 0$ such that $F(x, y, z) + (\eta_1/2)|x|^2$ is a concave function in (x, y).

Under these assumptions, the value function v is well-defined and satisfies $\|v\|_\infty \leq \|F\|_\infty/(1-\beta)$. From now on, $\|\cdot\|$ stands for $\|\cdot\|_\infty$.

3 The main result

In this section we will state the main theorem that estimates the approximation error in the optimal policy function computed through the Bellman method. This estimate is obtained using primitive data of the problem. Let T be the Bellman operator on $C(X \times Z)$ (the set of continuous and bounded functions defined in $X \times Z$ with the topology induced by the supremum norm) defined by:

$$TV(x, z) = \operatorname*{Max}_{\{y \in X;\, (x,y,z) \in \Omega\}} F(x, y, z) + \beta \int_Z V(y, z')\, Q(z, dz').$$

It is well known (see Stokey and Lucas [9]) that under hypotheses 1 and 2 this operator is a contraction with modulus β and fixed point v (the value function). The numerical method based on this contraction is defined as follows: let $v_0 \in C(X \times Z)$ be a concave function in x and define the sequence $(v_n)_{n \geq 0}$ by:

$$v_{n+1}(x, z) = Tv_n(x, z), \ \forall\, (x, z) \in X \times Z, \forall\, n \geq 0 \tag{1}$$

Lemma 3.1 *With hypotheses 1 and 2, each v_n defined by (1) is strongly concave[1] in x with modulus η_1 for all $n \geq 1$. In particular the value function v is also strongly concave in x with the same modulus.*

Since for each $n \geq 1$, v_n is strongly concave, we can define:

$$g_n(x, z) = \operatorname*{Argmax}_{\{y \in X;\, (x,y,z) \in \Omega\}} F(x, y, z) + \beta \int_Z v_n(y, z')\, Q(z, dz'), \tag{2}$$

and the optimal policy is a function given by:

$$g(x, z) = \operatorname*{Argmax}_{\{y \in X;\, (x,y,z) \in \Omega\}} F(x, y, z) + \beta \int_Z v(y, z')\, Q(z, dz').$$

[1] A function $f : C \subset \mathbb{R}^n \to \mathbb{R}$ is strongly concave with modulus $\eta \geq 0$ if $f(x) + (\eta/2)|x|^2$ is a concave function.

The main theorem *With hypotheses 1 and 2, the sequence* $(v_n, g_n)_{n\geq 1}$*, defined by (1) and (2) satisfies:*

$$\|g - g_n\| \leq \left(\frac{2}{\eta_1}\|v - v_n\|\right)^{1/2}.$$

In particular,

$$\|g - g_n\| \leq \left[\frac{2}{\eta_1}\left(\frac{\|F\|}{1-\beta} + \|v_0\|\right)\right]^{1/2} \beta^{n/2}.$$

A similar result can be proved substituting Hypothesis 2 by the following:

Hypothesis 3 The function F is bounded, continuous and there exists $\eta_2 > 0$ such that $F(x, y, z) + (\eta_2/2)|y|^2$ is a concave function in (x, y).

Theorem 3.2 *With hypotheses 1 and 3, the sequence* $(v_n, g_n)_{n\geq 0}$ *defined by (1) and (2) satisfies:*

$$\|g - g_n\| \leq \left(\frac{2\beta}{\eta_2}\|v - v_n\|\right)^{1/2}.$$

In particular,

$$\|g - g_n\| \leq \left[\frac{2}{\eta_2}\left(\frac{\|F\|}{1-\beta} + \|v_0\|\right)\right]^{1/2} \beta^{(n+1)/2}.$$

4 Robustness of the approximation method

In this section we will show that the approximation method is robust by jointly considering errors in computing Bellman's operator and errors in computing the policy.

Suppose that Bellman's iteration is performed using a numerical method and that $\widetilde{T}$, an "approximated" operator, is computed.

Hypothesis 4 Let $\widetilde{T} : C(X \times Z) \to C(X \times Z)$. Assume that there exists $\varepsilon \geq 0$ such that for all $f \in C(X \times Z)$

$$\|\widetilde{T}(f) - T(f)\| \leq \varepsilon.$$

Now let $(\tilde{v}_n)_{n\geq 0}$ be a sequence generated by the rule

$$\tilde{v}_{n+1} = \widetilde{T}(\tilde{v}_n).$$

Proposition 4.1 *If the correspondence Γ is nonempty, compact-valued, and continuous, the function F is bounded and continuous, and $\widetilde{T}$ satisfies Hypothesis 4, then the sequence $(\tilde{v}_n)_{n\geq 0}$ satisfies*

$$\|\tilde{v}_n - v\| \leq \frac{\varepsilon}{1-\beta} + \beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right).$$

Remark The aplication $\widetilde{T}$ does not have to satify the usual assumptions of monotonicity and discounting (see Santos and Vigo [8]).

Proposition 4.1 also says that if

$$\beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \ll \frac{\varepsilon}{1-\beta}$$

then more than n iterations may not appreciably improve the accuracy of the estimated value function.

Suppose that an *inexact* maximization is used to compute the policy associated to $\tilde{v}_n$. That is, take a tolerance $\tau \geq 0$ and define $\widetilde{G}_n(x,z)$ as those $\tilde{y} \in \Gamma(x,z)$ such that $\forall y \in \Gamma(x,z)$

$$F(x,\tilde{y},z) + \beta \int_Z \tilde{v}_n(\tilde{y},z')Q(z,dz') \geq F(x,y,z) + \beta \int_Z \tilde{v}_n(y,z')Q(z,dz') - \tau. \tag{3}$$

Observe that $\widetilde{G}_n$ can be a correspondence. In general, practical computation does not provide *the whole* set $\widetilde{G}_n(x,z)$. Instead, the inexact maximization will provide some $\tilde{y}_n \in \widetilde{G}_n(x,z)$. Even so, we have the following estimation.

Theorem 4.2 *Let $\tilde{g}_n : X \times Z \to X$ be a selection of $\widetilde{G}_n$, that is, $\tilde{g}_n(x,z) \in \widetilde{G}_n(x,z)$ for all (x,z). Then,*

$$\|g - \tilde{g}_n\| \leq \left[\frac{4}{\eta_1} \|v - \tilde{v}_n\| + \tau \right]^{1/2}.$$

In particular,

$$\|g - \tilde{g}_n\| \leq \left[\frac{4}{\eta_1} \left(\frac{\varepsilon}{1-\beta} + \beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \right) + \tau \right]^{1/2}.$$

Remark Theorem 4.2 says that if n is such that

$$\beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \ll \frac{\varepsilon}{1-\beta} + \frac{\eta_1 \tau}{4}$$

then more than n iterations may not appreciably improve the accuracy of the estimated policy.

5 Conclusions

In this paper we show that the Bellman contraction method provides accurate estimates for the optimal policy function of the stochastic dynamic programming problem. An error bound for the estimate using the n-th iteration is explicitly constructed. It only depends on the norm and the modulus of strong concavity of the return function and the discount factor. Neither differentiability of the return function nor interiority of the solution are required.

The case where inexact computations are performed is also considered. We prove the stability of the method under small numerical errors.

The error bounds presented in this paper can be used for evaluating *a priori* the number of iterations needed in practical computations of the Bellman method. For example, following the notation of Section 4, let ε be the error on the Bellman operator and τ be the error on the maximization procedure used to compute the associated policy. If n is such that

$$\beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \ll \frac{\varepsilon}{1-\beta} + \frac{\eta_1 \tau}{4}$$

then more than n iterations may not appreciably improve the accuracy of the estimated policy.

A Appendix

To prove Lemma 3.1, let us show the following:

Lemma A.1 *$F(x,y,z) + (\eta/2)|x|^2$ is a concave function in (x,y) if and only if for all $(x_i, y_i, z) \in \Omega$, $i = 1,2$ and for all $\alpha \in [0,1]$ it holds that:*

$$F(x^\alpha, y^\alpha, z) \geq \alpha F(x_1, y_1, z) + (1-\alpha) F(x_2, y_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2,$$

where $x^\alpha = \alpha x_1 + (1-\alpha)x_2$ and $y^\alpha = \alpha y_1 + (1-\alpha) y_2$.

Proof:
($\Rightarrow$) By hypothesis:

$$F(x^\alpha, y^\alpha, z) + \frac{\eta}{2}|x^\alpha|^2 \geq \alpha[F(x_1, y_1, z) + \frac{\eta}{2}|x_1|^2] + (1-\alpha)[F(x_2, y_2, z) + \frac{\eta}{2}|x_2|^2],$$

expanding the square of the left side and simplifying it results in:

$$F(x^\alpha, y^\alpha, z) \geq \alpha F(x_1, y_1, z) + (1-\alpha) F(x_2, y_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2.$$

($\Leftarrow$) Completely analogous.

Proof of Lemma 3.1 It will be sufficient to prove that if $V(.,z)$ is a concave function then $TV(.,z)$ is a strongly concave function with modulus η_1. Let $x_1, x_2 \in X$, $\alpha \in [0,1]$, $x^\alpha = \alpha x_1 + (1-\alpha)x_2$ and for $i = 1,2$ let $y_i \in \Gamma(x_i, z)$ be such that:

$$TV(x_i, z) = F(x_i, y_i, z) + \beta \int_Z V(y_i, z')\, Q(z, dz').$$

Then, using hypotheses 1, 2 and Lemma A.1 we have that:

$$\begin{aligned} TV(x^\alpha) &\geq F(x^\alpha, \alpha y_1 + (1-\alpha)y_2, z) + \beta \int_Z V(\alpha y_1 + (1-\alpha)y_2, z')Q(z, dz') \\ &\geq \alpha F(x_1, y_1, z) + (1-\alpha)F(x_2, y_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2 + \\ &\quad \beta \int_Z [\alpha V(y_1, z') + (1-\alpha)V(y_2, z')]\, Q(z, dz') \\ &= \alpha TV(x_1, z) + (1-\alpha)TV(x_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2. \end{aligned}$$

Since the set of strongly concave functions is a closed set with the topology induced by the sup norm it follows that the fixed point of T is strongly concave.

To prove the main theorem, we will need the following lemmata.

Lemma A.2 *$f : C \subset \mathbb{R}^n \to \mathbb{R}$ (C is a convex set) is strongly concave with modulus η if and only if for all $x_1, x_2 \in C$ it holds that:*

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2) + (\eta/2)\alpha(1-\alpha)|x_1 - x_2|^2.$$

Proof: Analogous to the proof of lemma A.1.

Lemma A.3 *Let $f : C \subset \mathbb{R}^n \to \mathbb{R}$ (C is a convex set) be a strongly concave function with modulus η. If $x^* = Argmax_{x \in C} f(x)$ then*

$$f(x) \leq f(x^*) - \frac{\eta}{2}|x - x^*|^2, \quad \forall x \in C.$$

Proof: Let $x \in C$ and $\alpha \in (0,1)$. By definition of x^* and the characterization of strong concavity given in Lemma A.2 we have:

$$f(x^*) \geq f(\alpha x^* + (1-\alpha)x) \geq \alpha f(x^*) + (1-\alpha)f(x) + \tfrac{\eta}{2}\alpha(1-\alpha)|x - x^*|^2$$

$$\Rightarrow f(x^*) \geq f(x) + \tfrac{\eta}{2}\alpha|x - x^*|^2,$$

making $\alpha \to 1$ we obtain the result.

Proof of the main theorem.

Let us fix some notations. Let $v_n \in C(X \times Z)$ be the n-th iteration ($n \geq 1$) of Bellman's operator from some initial concave function $v_0 \in C(X \times Z)$. Let

$$g_n(x,z) = \text{Argmax}_{\{y \in X;\,(x,y,z) \in \Omega\}} F(x,y,z) + \beta \int_Z v_n(y,z')\, Q(z,dz'),$$

$$\phi_n(x,y,z) = F(x,y,z) + \beta \int_Z v_n(y,z',)\, Q(z,dz'),$$

$$\phi(x,y,z) = F(x,y,z) + \beta \int_Z v(y,z')\, Q(z,dz').$$

By lemma 3.1 the functions $\phi(x,.,z)$ and $\phi_n(x,.,z)$ are strongly concave with modulus $\beta\eta_1$. Then by lemma A.3 we have that:

$$\phi(x,g(x,z),z) \geq \phi(x,g_n(x,z),z) + \frac{\beta\eta_1}{2}|g(x,z) - g_n(x,z)|^2,$$

$$\phi_n(x,g_n(x,z),z) \geq \phi_n(x,g(x,z),z) + \frac{\beta\eta_1}{2}|g(x,z) - g_n(x,z)|^2.$$

Summing up the above inequalities we obtain that:

$$\beta\{\int_Z [(v_n - v)(g_n(x,z),z') + (v - v_n)(g(x,z),z')]\, Q(z,dz')\} \geq \beta\eta_1|g(x,z) - g_n(x,z)|^2$$

$$\Rightarrow 2\|v - v_n\| \geq \eta_1|g(x,z) - g_n(x,z)|^2$$

this inequality holds for all $(x,z) \in X \times Z$, so we conclude:

$$\|g - g_n\| \leq [\frac{2}{\eta_1}\|v - v_n\|]^{1/2}.$$

Finally, using that T is a contraction it results that $\|v - v_n\| \leq \beta^n\|v - v_0\|$. The bound is obtained since $\|v\| \leq \|F\|/(1-\beta)$.

Proof of theorem 3.2

Under hypothesis 3, $\phi_n(x,.,z)$ and $\phi(x,.,z)$ are strongly concave with modulus η_2. Then by lemma A.3 we have that:

$$\phi(x,g(x,z),z) \geq \phi(x,g_n(x,z),z) + \frac{\eta_2}{2}|g(x,z) - g_n(x,z)|^2,$$

$$\phi_n(x, g_n(x,z), z) \geq \phi_n(x, g(x,z), z) + \frac{\eta_2}{2}|g(x,z) - g_n(x,z)|^2.$$

Using the same reasoning as in the proof of the main theorem we conclude that:

$$\|g - g_n\| \leq [\frac{2\beta}{\eta_2}\|v - v_n\|]^{1/2} \leq \left[\frac{2}{\eta_2}\left(\frac{\|F\|}{1-\beta} + \|v_0\|\right)\right]^{1/2} \beta^{(n+1)/2}.$$

Proof of Proposition 4.1

By our assumption, T is a β-contraction on $C(X \times Z)$ with fixed point v. Using also the definition of the sequence $(\tilde{v}_n)_{n\geq 0}$, the triangular inequality and Hypothesis 4 we get

$$\begin{aligned} \|\tilde{v}_{n+1} - v\| &= \|\widetilde{T}(\tilde{v}_n) - v\| \\ &\leq \|\widetilde{T}(\tilde{v}_n) - T(\tilde{v}_n)\| + \|T(\tilde{v}_n) - v\| \\ &\leq \varepsilon + \beta\|\tilde{v}_n - v\|. \end{aligned}$$

Hence

$$\|\tilde{v}_n - v\| \leq \sum_{j=0}^{n-1} \varepsilon\beta^j + \beta^n\|\tilde{v}_0 - v\| \leq \varepsilon/(1-\beta) + \beta^n\|\tilde{v}_0 - v\|.$$

Proof of theorem 4.2

Let

$$\tilde{\phi}_n(x,y,z) = F(x,y,z) + \beta\int_Z \tilde{v}_n(y,z')\, Q(z,dz').$$

Then

$$\tilde{\phi}_n(x, \tilde{g}_n(x,z), z) \geq \tilde{\phi}_n(x, g(x,z), z) - \tau.$$

With hypotheses 1 and 2 we have:

$$\phi(x, g(x,z), z) \geq \phi(x, \tilde{g}_n(x,z), z) + \frac{\beta\eta_1}{2}|g(x,z) - \tilde{g}_n(x,z)|^2.$$

Adding up these inequalities and following the same procedure as in the proof of the main theorem we will obtain

$$\|g - \tilde{g}_n\| \leq \left[\frac{4}{\eta_1}\|v - \tilde{v}_n\| + \tau\right]^{1/2}.$$

Finally, using Proposition 4.1 it will result that

$$\|g - \tilde{g}_n\| \leq \left[\frac{4}{\eta_1}\left(\frac{\varepsilon}{1-\beta} + \beta^n\left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\|\right)\right) + \tau\right]^{1/2}.$$

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