# Trading factors: Heckscher-Ohlin revisited 

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#### Abstract

The Heckscher-Ohlin model of international trade is a general equilibrium model with finite numbers of goods, factors, consumers (i.e., countries) plus assumptions on consumption and production. Since only goods are traded between consumers, it is only by resorting to the factor contents of goods that one can interpret goods trading as a way of trading factors. Leontief was the first to measure the content in productive factors of US foreign trade. Vanek went on by reformulating the classical Heckscher-Ohlin theorem as a statement about the factors that a country export or import. I show in this paper that the Heckscher-Ohlin model not only can be interpreted as, but is actually equivalent to, an exchange model for productive factors. This property is established for an arbitrary number of goods, factors, consumers, with no restrictions on consumers' preferences except the usual ones made in consumer theory, and convex production subject to constant returns to scale and no-joint production of consumption goods. Vanek's reformulation of the Heckscher-Ohlin theorem is then a straightforward consequence of that equivalence. New properties of the Heckscher-Ohlin model also follow from this equivalence. They underscore the significant and complex effect of the volume of trade in factor contents resulting from specialization and globalization. At the theoretical level adopted in this paper, the only issues that are considered deal with the uniqueness of equilibrium, more generally their number, and their possible discontinuities. The equivalence of the Heckscher-Ohlin model with an exchange model and applications of that equivalence to international trade issues are proved through the equilibrium manifold and natural projection approach that I have previously developed for and applied to the general equilibrium model with no productive factors.


## 1. Introduction

The Heckscher-Ohlin model is essentially a general equilibrium model where countries are identified to consumers, consumption goods are freely traded between countries (i.e., consumers) while productive factors can be traded only within each country [11, 12, 18]. Consumption goods are outputs of productions processes where the inputs are the productive factors. Production technologies are the same across countries. They are represented by smooth concave production functions that are homogeneous of degree one (constant returns to scale). There is no-joint production of the consumption goods.

Some famous properties of the model (Heckscher-Ohlin, Rybczynski and Stolper-Samuelson theorems) hold only for the special case of two consumers, two goods and two factors. This does not reduce, however, the interest of arbitrary numbers of consumers, goods and factors. It is in that setup that Samuelson discusses the most general version of the factor price equalization theorem [19]. The impossibility of directly extending the Heckscher-Ohlin theorem about the goods that a country import or export as a function of its factor endowments to more than two goods led Vanek to compare the factors embodied in a country's production and those embodied in its consumption [22]. The factor content of trade had already been used by Leontief in a study of US foreign trade of which the main result is now known as Leontief's paradox [16]. According to Fisher [8], the main value of Vanek's contribution is to have emphasized that "trade in goods is only a
veil," that "Heckscher-Ohlin theory is really about the trade in the underlying factor services." The importance recently given to the trade content in productive factors is evidenced by the large body of literature, mostly empirical, that has been devoted during the past two decades to measuring the factor content of net trade for various countries and to apply these measures to various tests of the theory. See [21] and the references therein.

The Heckscher-Ohlin model is a general equilibrium model in its own right. It is encompassed in some of the papers that consider very general forms of the general equilibrium model as in $T$. Kehoe [14, 15], Mas-Colell [17] and Smale [20]. It follows from these papers that the following properties proved by Debreu [6] and Dierker [7] for the exchange model hold true in the HeckscherOhlin model: 1) The set of regular economies is open and dense (for suitable parameter spaces and topologies); 2) Equilibrium selections are locally unique and continuous at regular economies; 3) The number of equilibria is constant over every connected component of the set of regular economies; 4) An index number (which provides some way of "counting equilibria" and that has important invariance properties) can be defined at regular economies. These properties may not appeal very much to the international trade community. In particular, they fail to bring any real insight into possible relationships between goods and their factor content for example.

This paper is the result of my attempts at extending to the Heckscher-Ohlin model the equilibrium manifold and natural projection approach that I have developed for the exchange model in [1] and several subsequent papers. The two main results of the current paper are: 1) The equivalence of the general Heckscher-Ohlin model with convex constant returns to scale and no-joint production of goods with an exchange model for productive factors; 2) If consumers' preferences satisfy the usual assumptions of consumer theory, the equivalent exchange model satisfies the rich set of properties described in Chapters 5 to 8 of my book [4]. It follows from this equivalence and the properties of the equivalent exchange model that there exists a strong relationship between the volume of trade in factor content and the properties of the equilibria of the Heckscher-Ohlin model. For example and as in the exchange model, uniqueness, more generally, the number of equilibria and the discontinuity of equilibrium selections depend closely on the volume of trade as follows from [1]. To sum up, the two main results of this paper provide the theoretical justification that has been missing so far of the importance, theoretical and applied, of the volume of trade in factor content as a concept for analyzing international trade.

I have adopted for this article the formulation, the notation and the vocabulary that have been used rather consistently in general equilibrium theory during the past three or four decades $[3,4,5]$. I do not expect readers with international trade background to face real difficulties in recasting the developments of this paper in the terms and notation they are more familiar with.

This paper is organized as follows. The model with (consumption) goods and (production) factors, no-joint production and constant returns to scale production, i.e., the Heckscher-Ohlin model is defined in Section 2 with the level of generality adopted in this paper. In Section 3, goods prices are expressed as a function of factor prices. That relation is then exploited to define the content in productive factors of any bundle of (consumption) goods. The exchange model whose "goods" are the (productive) factors is defined in Section 4. The equivalence between that exchange model and the Heckscher-Ohlin model is proved in Section 5. Section 6 is devoted to proving that the equivalent exchange model does indeed satisfy a rich set of properties provided preferences and production in the Heckscher-Ohlin model satisfy standard assumptions that, for some of them at least, are significantly weaker than the ones usually considered in the literature. The equivalence of the two models is applied in Section 7 to describe several new properties of the Heckscher-Ohlin model. They deal in particular with the uniqueness and, more generally, the number of equilibria and the discontinuity of equilibrium selections, all important issues in comparative statics. Concluding comments make up Section 8. A few technical and mostly well-known properties of production and production functions are gathered and proved in a first appendix. A second and very short appendix is devoted to the statement and proof of a very convenient sufficient condition for a smooth map to be an embedding. Both appendices can easily be skipped in a first reading.

## 2. The Heckscher-Ohlin model

### 2.1. Goods and prices

There are two kinds of goods: 1) Output or consumption goods simply known as goods; 2) Primary or input goods also known as (productive) factors. The consumption goods are arguments of consumers' preferences (and of utility functions when preferences are represented by utility functions). The productive factors participate only to the production process; they are not arguments of consumers' preferences or utility functions. The finite numbers of consumption goods and productive factors are denoted by $\ell$ and $k$ respectively.

### 2.2. Prices

The price of the consumption good $j$ is denoted by $p_{j}$ (with $1 \leq j \leq \ell$ ) and the price of the productive factor $h$ by $q_{h}$ ( with $1 \leq h \leq k$ ). The $k$-th productive factor is taken as numeraire, i.e., $q_{k}=1$. Let $S=\mathbb{R}_{++}^{k-1} \times\{1\}$ denote the set of numeraire normalized factor prices $q=\left(q_{1}, \ldots, q_{k}\right)$. Let $X=\mathbb{R}_{++}^{\ell}$ denote the strictly positive orthant of $\mathbb{R}^{\ell}$. The set of (numeraire normalized) prices $(p, q)$ for all the goods and factors is the Cartesian product $X \times S$.

### 2.3. Consumers' preferences and endowments

There is a finite number $m$ of consumers. Consumer i's preferences, with $1 \leq i \leq m$, are defined by a utility function $u_{i}: X=\mathbb{R}_{++}^{\ell} \rightarrow \mathbb{R}$ that satisfies the following standard assumptions: 1) Smoothness; 2) Smooth monotonicity, i.e., $D u_{i}\left(x_{i}\right) \in X$ for $x_{i} \in X$ where $D u_{i}\left(x_{i}\right)$ is the gradient vector defined by the first-order derivatives of $u_{i} ; 3$ ) Smooth strict quasi-concavity, namely, the restriction of the quadratic form defined by the Hessian matrix $D^{2} u_{i}\left(x_{i}\right)$ to the tangent hyperplane to the indifference surface $\left\{y_{i} \in X \mid u_{i}\left(y_{i}\right)=u_{i}\left(x_{i}\right)\right\}$ through $x_{i}$ is negative definite; 4) The indifference surface $\left\{y_{i} \in X \mid u_{i}\left(y_{i}\right)=u_{i}\left(x_{i}\right)\right\}$ is closed in $\mathbb{R}^{\ell}$ for all $x_{i} \in X$. (See, for example, [4], Chapter 2.)

Under these assumptions, the problem of maximizing the utility $u_{i}\left(x_{i}\right)$ subject to the constraint $p \cdot x_{i} \leq w_{i}$ has a unique solution $f_{i}\left(p, w_{i}\right)$. This defines a demand function $f_{i}: X \times \mathbb{R}_{++} \rightarrow X$ that is homogenous of degree zero: $f_{i}\left(\lambda p, \lambda w_{i}\right)=f_{i}\left(p, w_{i}\right)$ for every $\lambda>0$. Note that only consumption goods are demanded and consumed by consumer $i$.

In addition, the demand function $f_{i}$ is smooth (S), satisfies Walras law (W), namely the identity $p \cdot f_{i}\left(p, w_{i}\right)=w_{i}$ for any $\left(p, w_{i}\right) \in X \times \mathbb{R}_{++}$, and the weak axiom of revealed preferences (WARP) that states that the inequality $p^{\prime} \cdot f_{i}\left(p, w_{i}\right) \leq w_{i}^{\prime}$ implies the (strict) inequality $p \cdot f_{i}\left(p^{\prime}, w_{i}^{\prime}\right)>w_{i}$. The following two properties known as desirability (A) and negative definiteness of the Slutsky matrix (ND) are also satisfied by every demand function $f_{i}$ :
(A): For any sequence $\left(p^{t}, w_{i}^{t}\right) \in X \times \mathbb{R}_{++}$converging to $\left(p^{0}, w_{i}^{0}\right) \in \mathbb{R}_{+}^{\ell} \backslash\{0\} \times \mathbb{R}_{++}$, some coordinates of $p^{0}$ being equal to zero, then $\lim \sup _{t \rightarrow \infty}\left\|f_{i}\left(p^{t}, w_{i}^{t}\right)\right\|=+\infty$.
(ND): The Slutsky matrix of the demand function $f_{i}$ at any $\left(p, w_{i}\right) \in X \times \mathbb{R}_{++}$truncated to its first $\ell-1$ rows and columns is negative definite.

Consumer i's endowments in pure production factors are represented by the vector $\omega_{i} \in \mathbb{R}_{++}^{k}$. The $m$-tuple $\omega=\left(\omega_{i}\right)$ then represents the endowment vectors of all consumers and $\Omega=\left(\mathbb{R}_{++}^{k}\right)^{m}$ is the endowment or parameter space.

### 2.4. Production and the production matrix

There is no joint production. The quantity $x^{j}$ of consumption good $j$ that is produced is a function $x^{j}=F_{j}\left(\eta_{j}^{1}, \ldots, \eta_{j}^{k}\right)$ that depends only on the inputs in productive factors $\left(\eta_{j}^{1}, \ldots, \eta_{j}^{k}\right) \in \mathbb{R}_{+}^{k}$. The
production function $F_{j}$ is assumed to be smooth, monotone (i.e., $\partial F_{j} / \partial \eta^{h}>0$ for $1 \leq h \leq k$ ), homogenous of degree one and concave, with Hessian matrix $D^{2} F_{j}(\eta)$ negative semi-definite and of rank $k-1$. In addition, I assume that $\lim _{t \rightarrow \infty} F_{j}\left(\eta^{t}\right)=0$ if $\eta^{0}=\lim _{t \rightarrow \infty} \eta^{t}$ has some coordinates equal to zero (i.e., all production factors are necessary for production). Though these properties are satisfied in the classical version of the Heckscher-Ohlin model, they could easily be weakened without impairing the main properties of the model.

## Firms' demand functions

It follows from Proposition A. 9 in Appendix A that there exists a unique combination of productive factors $\eta=b_{j}(q) \in \mathbb{R}_{++}^{k}$ that minimizes the cost of producing the quantity $\gamma_{j}=1$ of the consumption good $j$. The function $b_{j}: S \rightarrow \mathbb{R}_{++}^{k}$ is homogeneous of degree zero (Lemma A.12) and smooth by Proposition A. 11 .

Definition 1. The production matrix associated with the factor price vector $q \in S$ is the $k \times \ell$ matrix $B(q)=\left[\begin{array}{llll}b_{1}(q) & b_{2}(q) & \ldots & b_{\ell}(q)\end{array}\right]$. The production matrix function is the map $B$ : $S \rightarrow(\mathbb{R})_{++}^{k \ell}$ defined by $q \rightarrow B(q)$.

The production matrix function $B$ encapsulates all the economic properties of the production sector, which is made possible by the assumption of no-joint production and constant returns to scale.

The properties of the production matrix function play an important role in the study of the Heckscher-Ohlin model. Their derivation is given in Appendix A: See more particularly Lemma A. 19, A. 20 and A. 21.

### 2.5. The Heckscher-Ohlin model

The (general equilibrium) model with pure production factors, no-joint production, and consumers' endowments in pure production factors, in short the production model, is defined by the $m$ consumers' utility functions $u_{i}$, the (production matrix) function $B: S \rightarrow \mathbb{R}^{k \ell}$ and the endowment set $\Omega=\mathbb{R}_{++}^{k m}$. An economy is defined by a specific value of the endowment vector in pure production factors $\omega=\left(\omega_{i}\right) \in \Omega$ where $\omega_{i} \in \mathbb{R}_{++}^{k}$.

## Equilibrium

Definition 2. The 3-tuple $(p, q, \omega) \in X \times S \times \Omega$ is an equilibrium of the production model if there exists a vector $x \in X$ such that the two following equalities are satisfied:

$$
\begin{gather*}
\sum_{1 \leq i \leq m} f_{i}\left(p, q \cdot \omega_{i}\right)=x,  \tag{1}\\
B(q) x=\sum_{1 \leq i \leq m} \omega_{i} . \tag{2}
\end{gather*}
$$

The component $(p, q)$ is then an equilibrium price vector for the economy $\omega=\left(\omega_{i}\right) \in \Omega$.
The equality of the aggregate demand for consumption goods and the supply $x \in X$ of those goods by the production sector is represented by equality (1). Equality between the demand for productive factors $B(q) \times$ required for the production of consumption bundle $x \in \mathbb{R}_{++}^{\ell}$ given the factor price vector $q \in S$ and the total supply of productive factors is represented by equality (2).

## Equilibrium manifold and the natural projection

The equilibrium manifold for the Heckscher-Ohlin model is the subset $E$ of $\mathbb{R}_{++}^{\ell} \times S \times \Omega$ consisting of equilibria $(p, q, \omega)$. The natural projection $\pi: E \rightarrow \Omega$ is the restriction to the equilibrium manifold $E$ of the projection map $(p, q, \omega) \rightarrow \omega$.

The direct study of the Heckscher-Ohlin model along the lines followed for the exchange model would start with the study of the properties of the equilibrium manifold $E$ among which the local and global structures of that set stand in good place and continue with the study of the natural projection $\pi: E \rightarrow \Omega$ as a map from the equilibrium manifold $E$ into the endowment set $\Omega$. See, for example, [1] or [4]. If followed from the very beginning, that approach would face the lack of an obvious candidate for the concept of no-trade equilibrium that is so central in the exchange model. The route I am going to follow bypasses this problem.

## 3. Goods bundles and their content in productive factors

In this section, I define the content in productive factors of an arbitrary goods bundle for a given (equilibrium) price vector.

### 3.1. The factor content of a goods bundle

Proposition 3. The cost of producing the goods bundle $x \in X$ given the factor price vector $q \in S$ is strictly minimized at $y=B(q) \times$ (matrix notation).

Proof. Let $y_{j} \in R_{++}^{k}$ be a factor bundle that enables the production of the quantity $x^{j}>0$ of good $j$, with $1 \leq j \leq \ell$. The sum $y=y_{1}+\cdots+y_{\ell}$ enables the production of the goods bundle $x=\left(x^{1}, \ldots, x^{j}, \ldots, x^{\ell}\right)$. It follows from the property of no-joint production that the cost $q \cdot y=q \cdot y_{1}+\cdots+q \cdot y_{\ell}$ is minimized if each term of this sum is minimum. Each one of these minimization problems has a unique solution $y_{j}=b_{j}(q) x^{j}$ and, therefore, $y=\sum_{j} b_{j}(q) x^{j}=B(q) x$ (matrix notation).

The following definition exploits Proposition 3:
Definition 4. The factor content of the goods bundle $x \in X$ for the factor price vector $q \in S$ is the bundle $y=B(q) x \in \mathbb{R}_{++}^{k}$ made of the factors that minimize the total cost of producing $x$.

Corollary 5. The factor content $y=B(q) x$ is a linear function of the goods bundle $x$.

### 3.2. Relations between goods and factor prices at equilibrium

Equilibrium prices are obviously not arbitrary since they must satisfy the equilibrium equation. But the equilibrium equation also implies simpler equalities that can lead to helpful simplifications by giving alternative angles of attack. Here is such an example:

Proposition 6. Let $(p, q, \omega) \in X \times S \times \Omega$ be an equilibrium. Then, necessarily, $p=B(q)^{T} q$.
Proof. The production of commodity $j$ is a zero-profit operation, which implies the equality $p_{j}=$ $b_{j}(q)^{T} q$ for $1 \leq j \leq \ell$, which can be rewritten as $p=B(q)^{T} q$.

## 4. Consumer's demand function for factors

Let $q \in S$ be an arbitrary factor price vector. Define the goods price vector by $p=B(q)^{T} q \in X$. Consumer $i$ 's demand for goods given the wealth $w_{i}>0$ is equal to $f_{i}\left(p, w_{i}\right)=f_{i}\left(B(q)^{T} q, w_{i}\right)$. The factor content of this demand is equal to $B(q) f_{i}\left(B(q)^{T} q, w_{i}\right)$. This leads to the following:

Definition 7. Consumer $i$ 's demand function for factors $h_{i}: S \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}^{k}$ is the map defined by

$$
h_{i}\left(q, w_{i}\right)=B(q) f_{i}\left(B(q)^{T} q, w_{i}\right) .
$$

The question is whether this demand function is derived from the maximization of some utility function for factors.

### 4.1. The indirect and direct utility functions for factors

Let $\hat{u}_{i}\left(p, w_{i}\right)=u_{i}\left(f_{i}\left(p, w_{i}\right)\right)$ denote consumer $i$ 's indirect utility function for goods. Let $q \in S$ be a factor price vector. For $p=B(q)^{T} q$ and $w_{i}>0$, the demand for consumption goods is equal to $f_{i}\left(p, w_{i}\right)$. The factor content of that demand is equal to $B(q) f_{i}\left(B(q)^{T} q, w_{i}\right)$. It seems natural to define the indirect utility $\hat{v}_{i}\left(q, w_{i}\right)$ by the formula

$$
\hat{v}_{i}\left(q, w_{i}\right)=\hat{u}_{i}\left(B(q)^{T} q, w_{i}\right) .
$$

The question is whether this function can be the indirect utility function that generates the demand function $h_{i}\left(q, w_{i}\right)$ for factors.
Proposition 8. Consumer $i$ 's demand for factors $h_{i}: S \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}^{k}$ is the demand function associated with the indirect utility function $\hat{v}_{i}$.
Proof. It suffices to show that the demand function $h_{i}\left(q, w_{i}\right)$ is related to the (indirect utility) function $\hat{v}_{i}\left(q, w_{i}\right)$ by Roy's identity. Using the column matrix notation for $h_{i}\left(q, w_{i}\right)$ and $\partial_{q} \hat{v}_{i}\left(q, w_{i}\right)$, it suffices to prove Roy's identity, namely

$$
\begin{equation*}
\partial_{w_{i}} \hat{v}_{i}\left(q, w_{i}\right) h_{i}\left(q, w_{i}\right)=-\partial_{q} \hat{v}_{i}\left(q, w_{i}\right) . \tag{3}
\end{equation*}
$$

From the definition of $\hat{v}_{i}\left(q, w_{i}\right)=\hat{u}_{i}\left(B(q)^{T} q, w_{i}\right)$, it comes

$$
\partial_{w_{i}} \hat{v}_{i}\left(q, w_{i}\right)=\partial_{w_{i}} \hat{u}_{i}\left(B(q)^{T} q, w_{i}\right)=\partial_{w_{i}} \hat{u}_{i}\left(p, w_{i}\right) .
$$

Application of the chain rule yields

$$
\partial_{q} \hat{v}_{i}\left(q, w_{i}\right)^{T}=\partial_{p} \hat{u}_{i}\left(p, w_{i}\right)^{T} \partial_{q}\left(B(q)^{T} q\right) .
$$

The equality $\partial_{q}\left(B(q)^{T} q\right)=B(q)^{T}$ of Lemma A. 20 implies the equality

$$
\partial_{q} \hat{v}_{i}\left(q, w_{i}\right)^{T}=\partial_{p} \hat{u}_{i}\left(p, w_{i}\right)^{T} B(q)^{T}
$$

and, after taking the transpose,

$$
\partial_{q} \hat{v}_{i}\left(q, w_{i}\right)=B(q) \partial_{p} \hat{u}_{i}\left(p, w_{i}\right)
$$

Roy's identity (3) now takes the form

$$
\begin{equation*}
\partial_{w_{i}} \hat{u}_{i}\left(p, w_{i}\right) h_{i}\left(q, w_{i}\right)=-B(q) \partial_{p} \hat{u}_{i}\left(p, w_{i}\right) \tag{4}
\end{equation*}
$$

It follows from the definition of $f_{i}\left(p, w_{i}\right)$ as the demand function associated with the utility function $u_{i}$ that it satisfies Roy's identity with respect to the utility function $\hat{u}_{i}\left(p, w_{i}\right)$, which, in matrix notation, takes the form:

$$
\partial_{w_{i}} \hat{u}_{i}\left(p, w_{i}\right) f_{i}\left(p, w_{i}\right)=-\partial_{p} \hat{u}_{i}\left(p, w_{i}\right) .
$$

Left multiplication by $B(q)$ yields

$$
\partial_{w_{i}} \hat{u}_{i}\left(p, w_{i}\right) B(q) f_{i}\left(p, w_{i}\right)=-B(q) \partial_{p} \hat{u}_{i}\left(p, w_{i}\right),
$$

which, after substituting $h_{i}\left(q, w_{i}\right)=B(q) f_{i}\left(p, w_{i}\right)$, is exactly equation (4).

## The direct utility function for factors

Proposition 9. Consumer i's direct utility function $v_{i}$ for factors is defined by

$$
v_{i}\left(y_{i}\right)=\min _{q \in S} \hat{v}_{i}\left(q, q \cdot y_{i}\right) .
$$

Proof. Follows readily from the definition of the direct utility function from the indirect one.
An obvious consequence of Proposition 9 is that the demand function $h_{i}$ for production factors results from the maximization of the direct utility function $v_{i}$ subject to a budget constraint.

## 5. The factor exchange model

## Definition

The commodity space is the Euclidean space of factors $\mathbb{R}^{k}$. Good $k$ is used as numeraire. The price set is the set $S=\mathbb{R}^{k-1} \times\{1\}$ of numeraire normalized factor price vectors. The factor exchange model consists of $m$ consumers, with consumer i's utility function (with $1 \leq i \leq m$ ) the direct utility for pure production factors $v_{i}: \mathbb{R}_{++}^{k} \rightarrow \mathbb{R}$ of Proposition 9 .

An exchange economy is defined by a specific value of the endowment vector $\omega=\left(\omega_{i}\right) \in \Omega$ where $\omega_{i} \in \mathbb{R}_{++}^{k}$ represents consumer i's endowments in factors.

## Equilibrium

Definition 10. The pair $(q, \omega) \in S \times \Omega$ is an equilibrium of the factor exchange model if and only if the equilibrium equation

$$
\sum_{1 \leq i \leq m} h_{i}\left(q, q \cdot \omega_{i}\right)=\sum_{1 \leq i \leq m} \omega_{i}
$$

is satisfied. The factor price vector $q \in S$ is then an equilibrium price vector associated with the economy $\omega$.

## Equilibrium manifold and natural projection

The equilibrium manifold for the factor exchange model is the subset $\tilde{E}$ of $S \times \Omega$ consisting of the equilibria $(q, \omega)$. The natural projection $\tilde{\pi}: \tilde{E} \rightarrow \Omega$ is the restriction to the equilibrium manifold $\tilde{E}$ of the projection map $(q, \omega) \rightarrow \omega$. It is a property of the exchange model that its "equilibrium manifold" $\tilde{E}$ is indeed a smooth submanifold of $S \times \Omega$ and is diffeomorphic to a Euclidean space. (See [1] or [4], Proposition 4.9 and 5.8.)

## 6. Equivalence of the two models

### 6.1. Equivalence between equilibria

Proposition 11. The triple $(p, q, \omega) \in \mathbb{R}_{++}^{\ell} \times S \times \Omega$ is an equilibrium of the Heckscher-Ohlin model if and only if the pair $(q, \omega) \in S \times \Omega$ is an equilibrium of the factor exchange model.
Proof.
Necessity. Let $(p, q, \omega) \in X \times S \times \Omega$ that satisfies

$$
\begin{gather*}
\sum_{1 \leq i \leq m} f_{i}\left(p, q \cdot \omega_{i}\right)=x,  \tag{5}\\
B(q) x=\sum_{1 \leq i \leq m} \omega_{i} \tag{6}
\end{gather*}
$$

for some $x \in X$. Matrix multiplication of both sides of (5) by $B(q)$ yields

$$
\sum_{1 \leq i \leq m} B(q) f_{i}\left(p, q \cdot \omega_{i}\right)=B(q) x .
$$

Substituting $B(q)^{\top} q$ to $p$ and applying Equation (5) yields

$$
\begin{equation*}
\sum_{1 \leq i \leq m} B(q) f_{i}\left(B(q)^{T} q, q \cdot \omega_{i}\right)=\sum_{1 \leq i \leq m} \omega_{i}, \tag{7}
\end{equation*}
$$

which can be rewritten as

$$
\sum_{1 \leq i \leq m} h_{i}\left(q, q \cdot \omega_{i}\right)=\sum_{1 \leq i \leq m} \omega_{i} .
$$

Sufficiency. Let $p=B(q)^{T} q$ and define

$$
\begin{equation*}
x=\sum_{1 \leq i \leq m} f_{i}\left(p, q \cdot \omega_{i}\right) . \tag{8}
\end{equation*}
$$

Each vector $f_{i}\left(p, q \cdot \omega_{i}\right)$ belongs to the strictly positive orthant $X=\mathbb{R}_{++}^{\ell}$. So does the sum $x=\sum_{1 \leq i \leq m} f_{i}\left(p, q \cdot \omega_{i}\right)$. Left multiplication by $B(q)$ of equality (8) yields

$$
B(q) x=B(q) \sum_{1 \leq i \leq m} f_{i}\left(p, q \cdot \omega_{i}\right) .
$$

With $q \in S$ a solution of equation (7), the above right-hand side term is equal to $\sum_{1 \leq i \leq m} \omega_{i}$, from which follows the equality

$$
B(q) x=\sum_{1 \leq i \leq m} \omega_{i},
$$

which is equation (2) of Definition 2. The triple $(p, q, \omega)$ is then an equilibrium of the HeckscherOhlin model.

### 6.2. Equivalence of the two models

Intuitively, Proposition 11 tells us that the equilibrium equation systems of the Heckscher-Ohlin and factor exchange models have the same properties. The standard mathematical formulation of this equivalence (of equation systems) is by way of a diagram of maps (or functions) that is commutative. With these two models represented by their natural projection maps $\pi: E \rightarrow \Omega$ and $\tilde{\pi}: \tilde{E} \rightarrow \Omega$ respectively, these two models are equivalent if these two maps are equivalent in the mathematical sense: See for example [2], Definition 5.4.2. or [9], Chapter III, Definition 1.1. Since $\tilde{E}$ is a smooth manifold, equivalence means in practice that the domain $E$, the "equilibrium manifold" of the Heckscher-Ohlin model is not only a smooth manifold (and actually a smooth submanifold of $X \times S \times \Omega$ ) but that this smooth manifold is also diffeomorphic to $\tilde{E}$. The proof that these two maps are equivalent will also require identifying two smooth maps $\tilde{\alpha}: E \rightarrow \tilde{E}$ and $\tilde{\beta}: \tilde{E} \rightarrow E$ that are inverse to each other and such that the diagram of Proposition 20 is commutative.

## Definition of maps $\alpha$ and $\beta$

Beware, the maps $\alpha$ and $\beta$ are not to be confused with the maps $\tilde{\alpha}$ and $\tilde{\beta}$ respectively described just above even if those maps are closely related. They differ in particular by their domains or ranges. Some readers may consider these differences to be little more than minor mathematical technicalities. Unfortunately, these technicalities have their importance in some circumstances as, for example, here.

Definition 12. The map $\alpha: X \times S \times \Omega \rightarrow S \times \Omega$ is the projection $(p, q, \omega) \rightarrow(q, \omega)$. The map $\beta: S \times \Omega \rightarrow X \times S \times \Omega$ is defined by $\beta(q, \omega)=\left(B(q)^{T} q, q, \omega\right)$.

It follows from this definition that the maps $\alpha$ and $\beta$ are smooth. They are not the diffeomorphism we are looking for yet even if we are not that far as the following lemma suggests:

Lemma 13. $\alpha \circ \beta=\mathrm{id}_{\varsigma \times \Omega}$.
Proof. Obvious.
Note that the composition $\beta \circ \alpha$ is not the identity map of $X \times S \times \Omega$. In other words, Lemma 13 says that the map $\beta$ is a right inverse of the map $\alpha$ for the composition of maps. But the map $\alpha$ has no left inverse, which prevents the maps $\alpha$ and $\beta$ from being inverse to each other. Nevertheless, the following property of the map $\beta$ shows us how to get the diffeomorphisms we are looking for.

Lemma 14. The map $\beta: S \times \Omega \rightarrow X \times S \times \Omega$ is an embedding.
Proof. The fact that the map $\beta$ is an embedding follows readily from Lemma B. 1 of the Appendix combined with Lemma 13 above.

Corollary 15. The image $F=\beta(S \times \Omega)$ is a smooth submanifold of $X \times S \times \Omega$ that is diffeomorphic to $S \times \Omega$.

Proof. Follows readily from the property of an embedding of being a smooth map that is a diffeomorphism between its range and its domain.

Define now the map $\beta^{F}: S \times \Omega \rightarrow F$ by the same formula as the map $\beta$ except that the range is now the image $F=\beta(S \times \Omega)$ and the map $(\alpha \mid F): F \rightarrow S \times \Omega$ as the restriction of the map $\alpha$ to the submanifold $F$.

Corollary 16. The $\operatorname{map} \beta^{F}: S \times \Omega \rightarrow F$ and $(\alpha \mid F): F \rightarrow S \times \Omega$ are smooth and inverse to each other.

Proof. Obvious.
The equilibrium manifold $\tilde{E}$ being a smooth submanifold of $S \times \Omega$, the restriction of the smooth map $\beta^{F}$ to $\tilde{E}$ is therefore a smooth map $\left(\beta^{F} \mid \tilde{E}\right): \tilde{E} \rightarrow F$.
Lemma 17. We have $\left(\beta^{F} \mid \tilde{E}\right)(\tilde{E})=E$.
Proof.
Step 1: $\left(\beta^{F} \mid \tilde{E}\right)(\tilde{E}) \subset E$. Let $(q, \omega) \in \tilde{E}$. The equation of Definition 10 is satisfied. It then follows from Proposition 11 that $(p, q, \omega)=\left(\beta^{F} \mid \tilde{E}\right)(q, \omega)=\beta(q, \omega)$ satisfies the equilibrium equation of Definition 2 or, in other words, $(p, q, \omega)$ belongs to $E$.
Step 2: $E \subset\left(\beta^{F} \mid \tilde{E}\right)(\tilde{E})$. Let $(p, q, \omega) \in E$. By definition, the equation of Definition 2 is satisfied. Again by Proposition 11, the equation of Definition 10 is satisfied by $(q, \omega)$ and $p=B(q)^{T} q$. In other words, $(q, \omega)$ belongs to $\tilde{E}$ and $(p, q, \omega)=\beta(q, \omega)=\left(\beta^{F} \mid \tilde{E}\right)(q, \omega)$ belongs to $\left(\beta^{F} \mid \tilde{E}\right)(\tilde{E})$.

Proposition 18. The map $(\beta \mid \tilde{E}): \tilde{E} \rightarrow F$ is an embedding from $\tilde{E}$ into $F$ with image the set $E$. Proof. Follows readily from the previous lemmas.

Corollary 19. The "equilibrium manifold" $E$ is a smooth submanifold of $X \times S \times \Omega$ diffeomorphic to $E$.

Proof. Follows readily from the characterization of an embedding.

Define the map $\tilde{\beta}: \tilde{E} \rightarrow E$ as the map that has the same formula as $\beta$ but with domain the equilibrium manifold $\tilde{E}$ and range $E$.

Proposition 20. The maps $\tilde{\alpha}: E \rightarrow \tilde{E}$ and $\tilde{\beta}: \tilde{E} \rightarrow E$ are inverse to each other.
Proof. Follows readily from the proof of Proposition 18.
Proposition 20 can be reformulated as the commutativity of the maps $\pi$ and $\tilde{\pi}$ in the following diagram:

Corollary 21. The diagram

is commutative.
Theorem 22. The Heckscher-Ohlin model and the factor exchange model are equivalent.
Proof. First, the maps $\tilde{\alpha}$ and $\tilde{\beta}$ are inverse diffeomorphisms by Proposition 20. Second, the equality $\pi=\tilde{\pi} \circ \tilde{\alpha}$ is equivalent to $\pi(p, q, \omega)=\tilde{\pi}(\tilde{\alpha}(p, q, \omega))$ for $(p, q, \omega) \in E$. This equality is equivalent to $\pi(p, q, \omega)=\tilde{\pi}(q, \omega)$, itself equivalent to $\omega=\omega$ by the definitions of the maps $\pi$ and $\tilde{\pi}$. Third, the equality $\tilde{\pi}=\pi \circ \tilde{\beta}$ is equivalent to $\tilde{\pi}(q, \omega)=\pi\left(B(q)^{T} q, q, \omega\right)$ for $(q, \omega) \in \tilde{E}$. Again, this is equivalent to $\omega=\omega$.

## 7. Consumers' factor demand functions

The properties of the factor exchange model depend on the properties of consumers' preferences for productive factors. The richest set of properties is obtained if all demand functions $h_{i}$ satisfy smoothness (S), Walras law (W), the weak axiom of revealed preferences (WARP), and the demand function of one consumer the negative definiteness of the Slutsky matrix (ND) and desirability (A). See [4], Chapter 5 to 8 . All those properties are satisfied by demand functions $h_{i}$ (with $1 \leq i \leq m$ ) that result from the maximization subject to a budget constraint of utility functions $v_{i}$ satisfying the standard assumptions of (smooth) consumer theory.

Though it would suffice for this section at least that I show that consumer i's utility function for factors $v_{i}$ satisfies those properties, it is much easier and more general to prove directly that the factor demand function $h_{i}$ satisfies properties (S), (W), (WARP), (ND) and (A) when the respective properties are satisfied by goods demand function $f_{i}$.

Proposition 23. Consumer i's factor demand function $h_{i}$ is smooth (S) (resp. satisfies Walras law (W)) if the goods demand function $f_{i}$ satisfies (S) (resp. (W)).

Proof. Smoothness (S) for $f_{i}$ implies $(S)$ for $h_{i}$. Let $f_{i}$ be smooth. The production matrix function $q \rightarrow G(q)$ is smooth. Therefore, the demand function $h_{i}\left(q, w_{i}\right)=B(q) f_{i}\left(B(q)^{T} q, w_{i}\right)$ is smooth by composition of smooth functions, which proves (S).
Walras law (W) for $f_{i}$ implies (W) for $h_{i}$. Let $f_{i}$ satisfy (W). We then have

$$
q^{\top} h_{i}\left(q, w_{i}\right)=q^{T} B(q) f_{i}\left(p, w_{i}\right)=p^{\top} f_{i}\left(p, w_{i}\right)=w_{i} .
$$

Proposition 24. Consumer i's factor demand function $h_{i}$ satisfies the weak axiom of revealed preferences (WARP) (resp. the negative definiteness of the Slutsky matrix (ND)) is the goods demand function $f_{i}$ satisfies (WARP) (resp. (ND)).

Proof. (WARP) for $f_{i}$ implies (WARP) for $h_{i}$. Let ( $q, w_{i}$ ) and $\left(q^{\prime}, w_{i}^{\prime}\right)$ with $h_{i}\left(q, w_{i}\right) \neq h_{i}\left(q^{\prime}, w_{i}^{\prime}\right)$ be such that $\left(q^{\prime}\right)^{\top} h_{i}\left(q, w_{i}\right) \leq w_{i}^{\prime}$. This inequality can be spelled out as

$$
\begin{equation*}
\left(q^{\prime}\right)^{\top} B(q) f_{i}\left(p, w_{i}\right) \leq w_{i}^{\prime} \tag{9}
\end{equation*}
$$

Inequality

$$
\left(q^{\prime}\right)^{\top}\left(B\left(q^{\prime}\right)\right) \leq\left(q^{\prime}\right)^{T}(B(q))
$$

follows from Lemma A. 21 of Appendix A. The positivity of matrices $B(q)$ and $B\left(q^{\prime}\right)$ and of the demand vector $f_{i}\left(p, w_{i}\right)$ then leads to inequality

$$
\left(q^{\prime}\right)^{\top} B\left(q^{\prime}\right) f_{i}\left(p, w_{i}\right) \leq\left(q^{\prime}\right)^{\top} B(q) f_{i}\left(p, w_{i}\right),
$$

which implies, by (9), the inequality

$$
\begin{equation*}
\left(p^{\prime}\right)^{\top} f_{i}\left(p, w_{i}\right) \leq w_{i}^{\prime} . \tag{10}
\end{equation*}
$$

By (WARP) that is satisfied by $f_{i}$, inequality (10) implies the inequality

$$
\begin{equation*}
(p)^{T} f_{i}\left(p^{\prime}, w_{i}^{\prime}\right)>w_{i}, \tag{11}
\end{equation*}
$$

which can be rewritten as

$$
\begin{equation*}
q^{T} B(q) f_{i}\left(p^{\prime}, w_{i}^{\prime}\right)>w_{i} . \tag{12}
\end{equation*}
$$

A new application of Lemma A. 21 yields inequality $q^{\top} B(q) \leq q^{\top} B\left(q^{\prime}\right)$. The inequality

$$
q^{\top} B(q) f_{i}\left(p^{\prime}, w_{i}^{\prime}\right) \leq q^{\top} B\left(q^{\prime}\right) f_{i}\left(p^{\prime}, w_{i}^{\prime}\right) .
$$

then follows from the positivity of the demand vector $f_{i}\left(p^{\prime}, w_{i}^{\prime}\right)$. This inequality can be rewritten as

$$
p^{T} f_{i}\left(p^{\prime}, w_{i}^{\prime}\right) \leq q^{T} B\left(q^{\prime}\right) f_{i}\left(p^{\prime}, w_{i}^{\prime}\right)=q^{T} h_{i}\left(q^{\prime}, w_{i}^{\prime}\right) .
$$

Combining this inequality with the strict inequality (11) yields

$$
q^{\top} h_{i}\left(q^{\prime}, w_{i}^{\prime}\right) \geq p^{\top} f_{i}\left(p^{\prime}, w_{i}^{\prime}\right)>w_{i},
$$

which proves that (WARP) is satisfied by $h_{i}$.
(ND) for $f_{i}$ implies (ND) for $h_{i}$. Let $f_{i}$ satisfy (ND). In this part, I assume that the price vector $q \in \mathbb{R}_{++}^{k}$ is not normalized because the computation of Slutsky matrices requires computing derivatives of demand functions with respect to the price $q_{k}$ of the numeraire good (here, productive factor $k$ ). Without price normalization, the factor demand function $h_{i}\left(q, w_{i}\right)$ is homogenous of degree zero. Define $\partial_{q} h_{i}\left(q, w_{i}\right)$ as the $k \times k$ matrix of first order derivatives of $h_{i}$ with respect to the price vector $q$. Similarly, let $\partial_{p} f_{i}\left(p, w_{i}\right)$ denote the $\ell \times \ell$ matrix of partial derivatives for the demand function $f_{i}$ with respect to the price vector $p \in X$.

Step 1: Negative definiteness of the restriction of the quadratic form associated with matrix $\partial_{p} f_{i}\left(p, w_{i}\right)$ to the hyperplane $\left\{z \in \mathbb{R}^{\ell} \mid z^{\top} f_{i}\left(p, w_{i}\right)=0\right\}$. It follows from Hildenbrand and Jerison [13] that (ND) satisfied by $f_{i}$ is equivalent to the restriction of the quadratic form

$$
z \in \mathbb{R}^{\ell} \rightarrow z^{T} \partial_{p} f_{i}\left(p, w_{i}\right) z
$$

to the hyperplane $f_{i}\left(p, w_{i}\right)^{\perp}=\left\{z \in \mathbb{R}^{\ell} \mid z^{\top} f_{i}\left(p, w_{i}\right)=0\right\}$ being negative definite.
Step 2: $\partial_{q} h_{i}\left(q, w_{i}\right)=\left(\partial_{q} B(q)\right) f_{i}\left(B(q)^{T} q, w_{i}\right)+B(q)\left(\partial_{q} f_{i}\left(B(q)^{T} q, w_{i}\right)\right.$. Follows from taking the derivative of the product $h_{i}\left(q, w_{i}\right)=B(q) f_{i}\left(B(q)^{T} q, w_{i}\right)$ with respect to the price vector $q \in \mathbb{R}_{++}^{k}$. Step 3: $\partial_{q} f_{i}\left(B(q)^{T} q, w_{i}\right)=\left(\partial_{p} f_{i}\left(p, w_{i}\right)\right) B(q)^{T}$. Application of the chain rule to $f_{i}\left(B(q)^{T} q, w_{i}\right)$ yields

$$
\partial_{q} f_{i}\left(B(q)^{T} q, w_{i}\right)=\left(\partial_{p} f_{i}\left(p, w_{i}\right)\right) \partial_{q}\left(B(q)^{T} q\right) .
$$

It then suffices to apply Lemma A. 20 .
Step 4: $\left.\partial_{q} h_{i}\left(q, w_{i}\right)=\left(\partial_{q} B(q)\right) f_{i}\left(p, w_{i}\right)\right)+B(q) \partial_{p} f_{i}\left(p, w_{i}\right) B(q)^{T}$. It suffices to substitute the expression obtained in Step 3 in the formula of Step 2.
Step 5: The quadratic form defined by the matrix $\left(\partial_{q} B(q)\right) f_{i}\left(p, w_{i}\right)$ is negative semi-definite, with rank $k-1$. The column matrix $B(q) f_{i}\left(p, w_{i}\right)$ is equal to

$$
B(q) f_{i}\left(p, w_{i}\right)=\sum_{1 \leq j \leq \ell} b_{j}(q) f_{i}^{j}\left(p, w_{i}\right) .
$$

Its derivative with respect to $q$ is the $k \times k$ matrix

$$
\partial_{q} B(q) f_{i}\left(p, w_{i}\right)=\sum_{1 \leq j \leq \ell}\left(\partial_{q} b_{j}(q)\right) f_{i}^{j}\left(p, w_{i}\right) .
$$

Each square matrix $\partial_{q} b_{j}(q)$ defines a quadratic form that is negative semidefinite, with rank $k-1$ and kernel collinear with $q$ by Lemma A. 15 and A. 17 of the Appendix. The linear combination of these quadratic forms with the strictly positive coefficients $f_{i}^{j}\left(p, w_{i}\right)$, with $1 \leq j \leq \ell$, is therefore negative semidefinite and takes a value different from zero for any vector $v \in \mathbb{R}^{k}$ that is not collinear with the price vector $q \in \mathbb{R}_{++}^{k}$.
Step 6: $v \in h_{i}\left(q, w_{i}\right)^{\perp}$ implies $B(q)^{\top} v \in f_{i}\left(p, w_{i}\right)^{\perp}$. The relation $v \in h_{i}\left(q, w_{i}\right)^{\perp}$ is equivalent to $v^{\top} h_{i}\left(q, w_{i}\right)=v^{\top} B(q) f_{i}\left(p, w_{i}\right)=0$. This relation is equivalent to $z=B(q)^{\top} v$ in $f_{i}\left(p, w_{i}\right)^{\perp}$.
Step 7: $q^{\top} h_{i}\left(q, w_{i}\right) \neq 0$. Follows readily from Walras law $q^{\top} h_{i}\left(q, w_{i}\right)=w_{i} \neq 0$.
Step 8: $v^{\top} \partial_{q} h_{i}\left(q, w_{i}\right) v<z^{\top} \partial_{p} f_{i}\left(p, w_{i}\right) z<0$ for any $v \in h_{i}\left(q, w_{i}\right)^{\perp}$. By step 4, it comes

$$
v^{\top} \partial_{q} h_{i}\left(q, w_{i}\right) v=v^{\top}\left(\partial_{q} B(q)\right) f_{i}\left(p, w_{i}\right) v+v^{\top} B(q) \partial_{p} f_{i}\left(p, w_{i}\right) B(q)^{\top} v .
$$

By Step 5 , since $v \in h_{i}\left(q, w_{i}\right)^{\perp}$ is not collinear with $q$, it comes

$$
v^{\top} \partial_{q} h_{i}\left(q, w_{i}\right) v<v^{\top} B(q) \partial_{p} f_{i}\left(p, w_{i}\right) B(q)^{\top} v .
$$

The application of Steps 6 and 1 yields

$$
v^{\top} \partial_{q} h_{i}\left(q, w_{i}\right) v<z^{\top} \partial_{p} f_{i}\left(p, w_{i}\right) z<0 .
$$

Step 9: $h_{i}$ satisfies (ND). Follows readily from the negative definiteness of the restriction of the quadratic form defined by $\partial_{q} h_{i}\left(q, w_{i}\right)$ to the hyperplane $h_{i}\left(q, w_{i}\right)^{\perp}$ and its equivalence with (ND) satisfied by $h_{i}$ by [13].

## Desirability (A)

In this section again, the factor price vector $q$ is not normalized.

Proposition 25. Consumer i's factor demand function $h_{i}$ satisfies Desirability (A) if the goods demand function $f_{i}$ satisfies (A).

Proof. Let $\left(q^{t}, w_{i}^{t}\right) \in \mathbb{R}_{++}^{k}$ be a sequence of non-normalized price and income vectors converging to $\left(q^{0}, w_{i}^{0}\right) \in \mathbb{R}_{+}^{k} \times \mathbb{R}_{++}$, with some but not all coordinates of the price vector $q^{0}$ being equal to zero. It follows from Proposition A. 23 of the Appendix that, for each $j$, with $1 \leq j \leq \ell$, there is at least one coordinate $b_{j}^{k}\left(q^{t}\right)$ of $b_{j}\left(q^{t}\right)$ that tends to $+\infty$. In other words, to produce one unit of good $j$ at factor prices $q^{t}$ that tend to $q^{0}$, there is at least one production factor $k$ whose demand $b_{j}^{k}\left(q^{t}\right)$ tends to $+\infty$.

By considering if necessary a subsequence, there is no loss of generality in assuming that $p^{t}$ tends to a limit $p^{0} \in \mathbb{R}_{+}^{\ell}$, where some coordinates of $p^{0}$ may be equal to 0 . If none of these coordinates are equal to 0 , continuity implies $\lim _{t \rightarrow \infty} f_{i}\left(p^{t}, w_{i}^{t}\right)=f_{i}\left(p^{0}, w_{i}^{0}\right) \in \mathbb{R}_{++}^{\ell}$. It then follows from

$$
h_{i}\left(q^{t}, w_{i}^{t}\right)=B\left(q^{t}\right) f_{i}\left(B\left(q^{t}\right)^{T} q^{t}, w_{i}^{t}\right)=B\left(q^{t}\right) f_{i}\left(p^{t}, w_{i}^{t}\right)
$$

that $\lim \left\|h_{i}\left(q^{t}, w_{i}^{t}\right)\right\|$ is equal to $+\infty$ by the above property. If some coordinates of $p^{0}$ are equal to 0 , it follows from (A) that is satisfied by $f_{i}$ that we have $\lim \sup _{t \rightarrow \infty}\left\|f_{i}\left(p^{t}, w_{i}^{t}\right)\right\|=+\infty$, i.e., there exists some consumption good $j$ such that the demand $f_{i}^{j}\left(p^{t}, w_{i}^{t}\right)$ tends to $+\infty$. This implies that the demand for the production factor $k$ to produce the quantity $f_{i}^{j}\left(p^{t}, w_{i}^{t}\right)$ of consumption good $j$ also tends to $+\infty$.

Proposition 26. If consumer i's preferences can be represented by a utility function $u_{i}: X \rightarrow \mathbb{R}$ that satisfy the assumptions of Section 2.3 , the associated goods demand function $f_{i}: X \times \mathbb{R}_{++} \rightarrow$ $\mathbb{R}_{++}^{\ell}$ and factor demand function $h_{i}: S \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}^{k}$ satisfy (S), (W), (WARP), (ND) and (A).

Proof. For the goods demand function $f_{i}$, see for example [4], Chapter 3. For the factor demand function $h_{i}$, this follows from Propositions 23, 24 and 25.

Remark 1. The assumption that consumers' preferences are represented by continuous or, given the current context, smooth utility functions is standard. Assuming convexity and monotonicity of preferences, representability by a (continuous) utility function is roughly equivalent to transitivity of preferences. It results from the above developments that transitivity is not necessary to obtain a full set of properties for the Heckscher-Ohlin model, something that may be useful in some applications.

## 8. Application to the Heckscher-Ohlin model

This section presents a rather small sample of properties of the Heckscher-Ohlin model that are derived from its equivalence with the factor exchange model when consumers' preferences for goods satisfy the assumptions of Proposition 26, in other words when preferences are represented by smooth utility functions satisfying standard assumptions. The properties that I have chosen for the Heckscher-Ohlin model stand out for their immediate relevance for comparative statics, a major issue in many applications of that model. These are not and by large the only properties of that model. See for example [3] and [4], Chapter 5 to 8 for a list of those properties for the exchange model, their translation into properties of the Heckscher-Ohlin model being straightforward through the equivalence between the two models. All these properties are new in the setup of the HeckscherOhlin model except those described in Proposition 28.

## Global structure of the equilibrium manifold

Proposition 27. The equilibrium manifold $E$ of the Heckscher-Ohlin model is a smooth manifold diffeomorphic to $\mathbb{R}^{k m}$.

Proof. Follows from the same property of the equilibrium manifold $\tilde{E}$ of the factor exchange model. See [1]. (See also [4], Proposition 5.8 for a simpler proof in the case of a larger parameter space that allows for possibly negative endowments.)

Remark 2. The diffeomorphism property stated in Proposition 27 may not seem very appealing. In fact, it says three interesting things. First, it is possible to parameterize all the equilibria of the Heckscher-Ohlin model by simply km parameters. A closer look at these parameters shows that these parameters can be prices, wealth distribution across consumers and the coordinates that determine the factor trade vector (provided the parameters can have negative coordinates). This is coordinate system (B) of [4], Chapter 6, Section 6.3. Second, the equilibrium manifold is pathconnected. In practice, this means that any two equilibria can be linked by a continuous path in the equilibrium manifold. Third, the natural projection is a smooth map: using a suitable coordinate system, that map can be expressed as a map from $\mathbb{R}^{k m}$ into itself or, in other words, by just a set of km real-valued functions depending on km parameters. No more need even for the most elementary aspects of differential topology!

## Regular economies

The natural projection $\pi: E \rightarrow \Omega$ being a smooth map, one defines a regular (resp. critical) point of that map as an element $x$ of $E$ (i.e., $x$ is an equilibrium) that is such that the derivative $D_{x} \pi: T_{x}(E) \rightarrow T_{\pi(x)}(\Omega)$ is a bijection (resp. is not). The spaces $T_{x}(E)$ and $T_{\pi(x)}(\Omega)$ are the tangent spaces to $X$ and $\Omega$ at $x \in E$ and $\pi(x) \in \Omega$ respectively. By definition a singular value $\omega \in \Omega$ of the map $\pi: E \rightarrow \Omega$ is the image of a critical point, i.e., there exists $x \in E$ that is a critical point and such that $\pi(x)=\omega \in \Omega$. It is easy to see that the set of critical points $\Sigma$ is closed in the equilibrium manifold $E$. If the natural projection $\pi: E \rightarrow \Omega$ is proper, then the image of every closed set is also closed. By Sard's theorem, the set $\Sigma$ has also measure zero in $\Omega$. Last but not least, the element $\omega \in \Omega$ is by definition a regular value of the map $\pi: E \rightarrow \Omega$ if it is not a singular value. The set of regular values $\mathcal{R}$ of the map $\pi$ is therefore the complement $\Omega \backslash \Sigma$. This set is open dense in $\Omega$. For details, see [4], Chapter 7.

The following four properties of the Heckscher-Ohlin model have already been mentioned in the introduction.

Proposition 28. 1) The set of regular economies $\mathcal{R}$ is an open and dense subset of the parameter space $\Omega$; 2) Equilibrium selections are locally unique and continuous in sufficiently small neighborhoods of $\omega \in \mathcal{R}$; 3) The number of equilibria $\# \pi^{-1}(\omega)$ is constant for $\omega$ in a connected component of the set of regular economies $\mathcal{R}$; 4) The modulo 2 degree of the natural projection $\pi: E \rightarrow \Omega$ is equal to 1.

Proof. They were proved for the exchange model by Debreu [6] and Dierker [7]. It then suffices to apply the equivalence between the Heckscher-Ohlin model and its factor exchange model.

Remark 3. In the exchange model, the properties stated under Proposition 28 follow readily from the smoothness and properness of the natural projection. See [4], Chapter 7. It is also obvious by the equivalence property that the natural projection in the Heckscher-Ohlin model is also proper and smooth.
Remark 4. Dierker's degree concept and its extension by Kehoe to regular production economies and, henceforth, to the Heckscher-Ohlin model correspond to the topological or Brouwer degree of the natural projection instead of the modulo 2 degree. See Section 7.6.1 of [4].

## Uniqueness of equilibrium

Let $P$ denote the set of Pareto optima of the factor exchange model. This set is a subset of $\Omega$ and consists of the allocations (or endowments) $\omega=\left(\omega_{i}\right)$ that are Pareto optima of the factor exchange
model. In the case of $m=2$ consumers and $k=2$ two factors, this set coincides with the contract curve of the Edgeworth-Bowley box. It follows from the two theorems of welfare economics (for the exchange model) that the set of Pareto optima $P$ for the factor exchange model coincides with the set of factor contents of the equilibrium allocations of the Heckscher-Ohlin model.

Proposition 29. The set of Pareto optima $P$ is a pathconnected smooth submanifold of the endowment space $\Omega=\left(\mathbb{R}_{++}^{k}\right)^{m}$.

Proof. Follows from [4] Corollary 8.5 applied to the factor exchange.
Remark 5. The most important part of Proposition 29 is the pathconnectedness. Note that, in fact, the set of Pareto optima $P$ is diffeomorphic to $\mathbb{R}^{\ell+m-1}$.

The set of regular economies $\mathcal{R}$ is open. As such, it is partitioned into a collection of connected components. These are open subsets. By Proposition 28, the number of equilibria is constant for $\omega$ in every pathconnected component of $\mathcal{R}$ : [6] and [4], Proposition 7.7. The following property is therefore quite remarkable:

Proposition 30. The set of Pareto optima $P$ is contained in a unique connected component of the set of regular economies $\mathcal{R}$.

Proof. See [4], Proposition 8.8.
Let this component be denoted by $\mathcal{R}_{1}$. In Figure 1 , the component $\mathcal{R}_{1}$ is the subset of $\Omega$ bounded by the curve $\Sigma$ (the set of singular economies) and containing the curve $P$ (the set of Pareto optima). The grey area in $\Omega$ consists of the regular economies $\omega$ with three equilibria.


Figure 1: The equilibrium manifold and the natural projection

Proposition 31. Equilibrium is unique for all endowment vectors $\omega$ in the component $\mathcal{R}_{1}$.
Proof. See [4], Proposition 8.8.
In practice, the endowment vector $\omega$ belongs to the connected component $\mathcal{R}_{1}$ if the vector of net trades in factor contents $\left(h_{i}\left(p, p \cdot \omega_{i}\right)-\omega_{i}\right)$ is sufficiently small. This is equivalent to the volume of trade represented by the distance between the vector of factor endowments $\omega$ and the set of factor contents of equilibrium allocations $P$ being sufficiently small.

Remark 6. The first mention and proof that the set of Pareto optima is contained in a unique pathconnected component of the set of regular economies $\mathcal{R}$ and that equilibrium is unique for endowments in that component are due to Balasko [1].

## Multiplicity of equilibria

The connected component $U$ of $\mathcal{R}$ is said to be adjacent to the connected component $\mathcal{R}_{1}$ if the boundaries or these two open sets have a non-empty intersection: $\partial \mathcal{R}_{1} \cap \partial U \neq \emptyset$.

Proposition 32. The number of equilibria is larger than or equal to three for all factor endowment $\omega$ in the connected components of the set of regular economies $\mathcal{R}$ that are adjacent to $\mathcal{R}_{1}$.

Proof. Here is a brief outline of the proof. Let $\omega \in U$, with $U$ is connected component of $\mathcal{R}$ adjacent to $\mathcal{R}_{1}$. Let $\omega^{\prime} \in \mathcal{R}_{1}$. By the definition of $U$, there exists a continuous path going from $\omega^{\prime}$ to $\omega$ that intersects the set of singular economies $\Sigma$ just once at some point $\omega^{\prime \prime}$. When following that path, the number of equilibria jumps from one to at least three when it goes through $\Sigma$ at $\omega^{\prime \prime}$. For more details, see for example [2], p. 144 and the developments in its Chapter 7.

Remark 7. The location and size of the uniqueness component $\mathcal{R}_{1}$ are therefore critical for issues of comparative statics. The location is determined by the location of the set of Pareto optima, i.e., of the set of factor contents of equilibrium allocations. The size can be inferred from the distance of the set of singular economies $\Sigma$, to the set of Pareto optima $P$. It follows from [4], Chapter 6, Proposition 6.6. that this distance is inversely related to the size of the polyhedral cone defined by the $m$ vectors $\frac{\partial f_{1}}{\partial w_{1}}\left(q, w_{1}\right), \frac{\partial f_{2}}{\partial w_{2}}\left(q, w_{2}\right), \ldots, \frac{\partial f_{m}}{\partial w_{m}}\left(q, w_{m}\right)$ in $\mathbb{R}^{k}$ that represent the wealth effects on the factor demands of each consumer. When all these vectors are collinear, then all these factor demand functions are identical and, furthermore, satisfy the Gorman-Nataf perfect aggregation condition: [10] and [2], Theorem 7.Ann.3. No endowment vector $\omega \in \Omega$ is then singular, and equilibrium is always unique.
Remark 8. Very roughly speaking, it follows from the previous remark that the more consumers have different preferences and the smaller is the uniqueness domain $\mathcal{R}_{1}$ and the more likely it is to observe multiple equilibria and discontinuities of equilibrium selections for sufficiently large volume of trade in factor contents.

## 9. Concluding comments

The main conclusion that emerges from this paper is the importance of the volume of trade in factor contents in issues of comparative statics within the setup of the Heckscher-Ohlin model. The accelerating trend towards more specialization and globalization that has been going on in the world economy since a few decades has led to increasingly large volumes of trade in the factor contents of goods. When reformulated within the Heckscher-Ohlin model, this phenomenon implies that the world economy may very well be hovering over domains of multiple equilibria, and the latter are also associated with discontinuities of equilibrium selections. This conclusion requires nothing more than widely accepted economic assumptions on preferences and production. This deeply ingrained property of the Heckscher-Ohlin model (and also of the exchange model) holds without more or less artificial additions of complicating elements for the sake of realism.

It may very well be that these discontinuities of and jumps between equilibrium selections have not occurred in the world economy yet. Nevertheless, it is hard not to think that they have not already occurred but haven't been properly identified and understood. And, even if this has not been the case yet, the fact that these phenomena are so deeply enrooted in the equilibrium equation of a simple and beautiful model implies that these phenomena will occur sooner or later. I bet that Dirac would not disagree with me. Equations ought to be taken seriously.

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## A. The production sector

The main properties of the production sector follow from the lack of joint production and from the concavity of the production functions $F_{j}$. These properties are well-known, at least in the special case of the Heckscher-Ohlin model. They are gathered in this appendix for lack of convenient references at this level of generality.

## A.1. Production functions

The production of the quantity $x^{j}$ of consumption good $j$ is a function $x^{j}=F_{j}\left(\eta_{j}^{1}, \ldots, \eta_{j}^{k}\right)$ of the pure production factors $\left(\eta_{j}^{1}, \ldots, \eta_{j}^{k}\right) \in \mathbb{R}_{+}^{k}$. The production function $F_{j}$ is smooth, monotone (i.e., $\partial F_{j} / \partial \eta^{h}>0$ for $1 \leq h \leq k$ ), homogenous of degree one and concave, with Hessian matrix $D^{2} F_{j}(\eta)$ negative semi-definite and of rank $k-1$. In addition, all production factors are necessary for production, i.e., $\lim _{t \rightarrow \infty} F_{j}\left(\eta^{t}\right)=0$ if $\eta^{0}=\lim _{t \rightarrow \infty} \eta^{t}$ has some coordinates equal to zero.

Lemma A.1. $D F_{j}(\eta)^{T} \eta=F_{j}(\eta)$.
Proof. By homogeneity of degree one, it comes $F_{j}(\lambda \eta)=\lambda F_{j}(\eta)$ with $\lambda \in \mathbb{R}$. It then suffices to take the derivative with respect to $\lambda$ (Euler's identity).

Corollary A.2. $D F_{j}(\eta)^{T} \eta \neq 0$ for $\eta \in \mathbb{R}_{++}^{k}$.

## The Hessian matrix $D^{2} F_{j}$

It follows from the concavity of the production function $F_{j}$ that its Hessian matrix $D^{2} F_{j}$ is negative semidefinite. It is not negative definite as follows from:

## Lemma A.3.

$$
\eta^{T} D^{2} F_{j}(\eta)=D^{2} F_{j}(\eta) \eta=0
$$

Proof. It follows from the homogeneity of degree one of $F_{j}$ that the first order partial derivatives of $F_{j}$ are homogenous of degree zero. It then suffices to apply Euler's identity to these partial derivatives.

Lemma A.4. The kernel of matrix $D^{2} F_{j}(\eta)$ is collinear with $\eta \in \mathbb{R}_{++}^{k}$
Proof. The rank of matrix $D^{2} F_{j}(\eta)$ being equal to $k-1$, its kernel is one dimensional. It also contains the vector $\eta=\left(\eta^{1}, \ldots, \eta^{k}\right) \neq 0$.

The Hessian matrix $D^{2} F_{j}$ is as close as possible to being negative definite as follows from:
Lemma A.5. Let $\eta \in \mathbb{R}_{++}^{k}$. The strict inequality $z^{T} D^{2} F_{j}(\eta) z<0$ is satisfied for any non-zero vector $z \in \mathbb{R}^{k}$ that is not collinear with $\eta$.

Proof. All the $k$ eigenvalues of the symmetric matrix $D^{2} F_{j}(\eta)$ are real and there is a set of $k$ two by two orthogonal eigenvectors. One of these eigenvectors can be $\eta$, an eigenvector associated with the eigenvalue 0 . The $k-1$ remaining eigenvalues are then strictly negative because of the rank assumption. The $k-1$ associated eigenvectors generate a hyperplane that is orthogonal to the vector $\eta$. The restriction of the quadratic form associated with $D^{2} F_{j}(\eta)$ to that hyperplane is therefore negative definite.

The vector $z \in \mathbb{R}^{k}$ is the sum $z=z^{\prime}+z^{\prime \prime}$ of its orthogonal projections $z^{\prime}$ and $z^{\prime \prime}$, with $z^{\prime}$ in the vector space generated by $\eta$ and $z^{\prime \prime}$ in the hyperplane orthogonal to $\eta$. It comes $z^{\top} D^{2} F_{j}(\eta) z=\left(z^{\prime}\right)^{\top} D^{2} F_{j}(\eta) z^{\prime}+\left(z^{\prime \prime}\right)^{\top} D^{2} F_{j}(\eta) z^{\prime \prime}=$ $\left(z^{\prime \prime}\right)^{\top} D^{2} F_{j}(\eta) z^{\prime \prime}$. The strict inequality $\left(z^{\prime \prime}\right)^{\top} D^{2} F_{j}(\eta) z^{\prime \prime}<0$ follows from $z^{\prime \prime} \neq 0$ for $z$ not collinear with $\eta$.

Lemma A.6. The bordered Hessian matrix $\left[\begin{array}{cc}D^{2} F_{j}(\eta) \\ D F_{j}(\eta)^{T} & D F_{j}(\eta) \\ 0\end{array}\right]$ is invertible.

Proof. Assume the contrary. There exists a vector $z=\left(\bar{z}, z^{k+1}\right) \neq 0 \in \mathbb{R}^{k} \times \mathbb{R}$ such that

$$
\left[\begin{array}{cc}
D^{2} F_{j}(\eta) & D F_{j}(\eta) \\
D F_{j}(\eta)^{T} & 0
\end{array}\right]\left[\begin{array}{c}
\bar{z} \\
z^{k+1}
\end{array}\right]=0
$$

which can be rewritten as

$$
\begin{gather*}
D^{2} F_{j}(\eta) \bar{z}+z^{k+1} D F_{j}(\eta)=0  \tag{13}\\
\bar{z}^{T} D F_{j}(\eta)=0 \tag{14}
\end{gather*}
$$

Left multiplication of (13) by $\bar{z}^{\top}$ yields, given (14),

$$
\bar{z}^{T} D^{2} F_{j}(\eta) \bar{z}=0 .
$$

By Lemma A.5, the vector $\bar{z}$ is collinear with $\eta$. By Lemma A.3, $D^{2} F_{j}(\eta) \bar{z}=0$, which implies $z^{k+1} D F_{j}(\eta)=0$ in (13). Therefore, it comes $z=\lambda(\eta, 0) \in \mathbb{R}^{k+1}$ with $\lambda \in \mathbb{R}$. Equation (13) becomes $\lambda D^{2} F_{j}(\eta) \eta=0$, which implies $\lambda=0$ by Corollary A.2, a contradiction with $z \neq 0$.

## Strict convexity of isoquants

The set $\left\{\eta_{j} \in \mathbb{R}_{++}^{k} \mid F_{j}\left(\eta_{j}\right)=1\right\}$ is the analog in our setup with $k$ production factors of the isoquant curve for two factors.
Lemma A.7. The set $\left\{\eta \in \mathbb{R}_{++}^{k} \mid F_{j}(\eta) \geq 1\right\}$ is strictly convex.
Proof. Let $\eta$ and $\eta^{\prime}$ in $\mathbb{R}_{++}^{k}$ be such that $F(\eta)=F\left(\eta^{\prime}\right)=1$. The vector $\eta$ and $\eta^{\prime}$ are not collinear. Otherwise, assume $\eta^{\prime}=\lambda \eta$ with $\lambda \neq 1$. Then, we would have $1=F_{j}\left(\eta^{\prime}\right)=F_{j}(\lambda \eta)=\lambda F(\eta)=\lambda$, a contradiction.

The second derivative of the function $t \in[0,1] \rightarrow F_{j}\left((1-t) \eta+t \eta^{\prime}\right)$ is equal to $\left(\eta^{\prime}-\eta\right)^{\top} D^{2} F_{j}\left((1-t) \eta+t \eta^{\prime}\right)\left(\eta^{\prime}-\eta\right)$ and is strictly negative by Lemma A. 5 because $\eta^{\prime}-\eta$ is not collinear with $\eta$. This implies the strict concavity of the function $t \in[0,1] \rightarrow F_{j}\left((1-t) \eta+t \eta^{\prime}\right)$, hence the strict inequality $F_{j}\left((1-t) \eta+t \eta^{\prime}\right)>1$ for $t \in(0,1)$ and, therefore, the strict convexity of the set $\left\{\eta \in \mathbb{R}_{++}^{k} \mid F_{j}(\eta) \geq 1\right\}$.
Lemma A.8. The recession cone of the strictly convex set $\left\{\eta_{j} \in \mathbb{R}_{++}^{k} \mid F_{j}\left(\eta_{j}\right) \geq 1\right\}$ is the non-negative orthant $\mathbb{R}_{+}^{k}$. Proof. The vector $d \in \mathbb{R}^{k}$ defines a direction of recession for the set $\left\{\eta_{j} \in \mathbb{R}_{++}^{k} \mid F_{j}\left(\eta_{j}\right) \geq 1\right\}$ if, for some $\eta^{*}$ given in that set, the set $\left\{\eta^{*}+\alpha d \mid \alpha \geq 0\right\}$ is also contained in that set. This is equivalent to having $F_{j}\left(\eta^{*}+\alpha d\right) \geq 1$ for $\alpha \geq 0$. This is obviously satisfied by the monotonicity of $F_{j}$ for $d \in \mathbb{R}_{+}^{k}$, which proves that the recession cone contains the non-negative orthant.

Conversely, let $d \in \mathbb{R}^{k}$ with at least one strictly negative coordinate which, without loss of generality, can be assumed to be $d^{1}<0$. Let $\alpha$ be defined by $\eta^{* 1}+\alpha d^{1}=0$. From $\eta^{*}>0$ and $d^{1}<0$ results $\alpha>0$. In addition, we have $F_{j}\left(\eta^{*}+\alpha d^{1}\right)=0$, which implies that $d$ cannot be a direction of recession.

## A.2. Firms' factor demand functions

## The cost minimization problem

Proposition A.9. For every factor price vector $q \in S$, there exists a unique combination of inputs consisting of pure production factors $\eta=b_{j}(q) \in \mathbb{R}_{++}^{k}$ that minimizes the total cost of producing the quantity $\gamma_{j}=1$ of the consumption good $j$.
Proof. Let $\eta^{*} \in \mathbb{R}_{++}^{k}$ be such that $F_{j}\left(\eta^{*}\right) \geq 1$. It is the same problem to minimize the cost $q \cdot \eta$ subject to the constraint $F_{j}(\eta) \geq 1$ or the two constraints $F_{j}(\eta) \geq 1$ and $q \cdot \eta \geq q \cdot \eta^{*}$.

The set defined by these two constraints is closed as the intersection of two closed sets. It is also bounded. Obviously, we have $\eta \geq 0$. It follows from $\eta^{h}>0$ for every $h$ with $1 \leq h \leq k$ that the inequality $q \cdot \eta * \geq q \cdot \eta$ implies $q \cdot \eta *>q^{h} \eta^{h}$, from which follows $\eta^{h}<\frac{q \cdot \eta^{*}}{q^{h}}$ for $1 \leq h \leq k$. It follows from the compactness of that set and the continuity of the objective function that a solution exists to the above maximization problem.

The constraint $F_{j}(\eta) \geq 1$ is obviously binding by the monotonicity of $F_{j}$. The proof that the solution is unique is standard and proceeds by contradiction. Let $\eta \neq \eta^{\prime}$ be two different solutions. By definition, $q \cdot \eta=q \cdot \eta^{\prime}$ while $F_{j}(\eta)=F_{j}\left(\eta^{\prime}\right)=1$. Let $\eta^{\prime \prime}=\left(\eta+\eta^{\prime}\right) / 2$. It follows from Lemma A. 7 that the strict inequality $F_{j}\left(\eta^{\prime \prime}\right)>1$ is satisfied while $q \cdot \eta^{\prime \prime}=q \cdot \eta=q \cdot \eta^{\prime}$.

Proposition A.10. The vector $\eta \in \mathbb{R}_{++}^{k}$ is a solution to the minimization problem of Proposition A. 9 if and only if there exists a real number $\mu>0$ such that the following equation system is satisfied:

$$
\left\{\begin{array}{l}
D F_{j}(\eta)-\mu q=0 \\
F_{j}(\eta)-1=0
\end{array}\right.
$$

Proof. These are the first order conditions for the minimization problem of Proposition maximization problem of the firm with respect to production factors. These conditions are are necessary and sufficient here.

## Firms' factor demand functions

Let $q \in S$ be a (normalized) factor price vector. We denote by $b_{j}(q)$ the bundles of pure production factors $\eta \in \mathbb{R}_{++}^{k}$ that solves firm $j$ 's cost minimization problem of Proposition A.9. All $\ell$ firms' demand functions are represented by the matrix $B(q)=\left[\begin{array}{lllll}b_{1}(q) & \cdots & b_{j}(q) & \cdots & b_{\ell}(q)\end{array}\right]$,
Proposition A.11. The function $b_{j}: S \rightarrow \mathbb{R}_{++}^{k}$ is smooth.
Proof. The idea is to apply the implicit function theorem to the first order conditions of Proposition A.10. This follows from Lemma A.6.

Lemma A.12. The function $b_{j}(q)$ is homogenous of degree zero for non normalized factor price vectors $q \in \mathbb{R}_{++}^{k}$. Proof. Obvious.

Lemma A.13. $\left(\partial_{q} b_{j}(q)\right)^{T} q=0$.
Proof. Follows readily from Euler's identity applied to $b_{j}(q)$, homogenous function of degree zero.
Lemma A.14. Let $q$ and $q^{\prime}$ in S. Then $q \cdot b_{j}(q) \leq q \cdot b_{j}\left(q^{\prime}\right)$.
Proof. Follows readily from the definition of $b_{j}(q)$ as minimizing the cost $q \cdot \eta$ subject to the constraint $F_{j}(\eta) \geq 1$.
(Note that the inequality in Lemma A. 14 is strict for $q \neq q^{\prime}$.)
Lemma A.15. Let $q$ and $q^{\prime}$ in $S$. Then, $\left(q-q^{\prime}\right) \cdot\left(b_{j}(q)-b_{j}\left(q^{\prime}\right)\right) \leq 0$.
Proof. The inequalities $q \cdot\left(b_{j}(q)-b_{j}\left(q^{\prime}\right)\right) \leq 0$ and $q^{\prime} \cdot\left(b_{j}\left(q^{\prime}\right)-b_{j}(q)\right) \leq 0$ follow from Lemma A.14. It then suffices to add up these two inequalities.
(The property stated in Lemma A. 15 is known as the monotonicity of the function $b_{j}$.)
Lemma A.16. The Jacobian matrix $\partial_{q} b_{j}(q)$ defines a negative semidefinite quadratic form
Proof. The derivation of that property from Lemma A. 15 is standard.
Lemma A.17. The Jacobian matrix $\partial_{q} b_{j}(q)$ has rank $k-1$ and its kernel is collinear with the factor price vector $q$.
Proof. The idea of the proof is to show that any vector $v \neq 0 \in \mathbb{R}^{k}$ in the kernel of $\partial_{q} g_{j}(q)$ is collinear to the factor price vector $q \in \mathbb{R}_{++}^{k}$. From the first order conditions of Proposition A.10, it comes $\partial_{q} F_{j}\left(b_{j}(q)\right)-\mu(q) q=0$ where $\mu(q) \neq 0$.

Taking the derivative of this equality with respect to the price vector $q$ yields

$$
D^{2} F_{j}\left(b_{j}(q)\right) \partial_{q} b_{j}(q)=\mu(q) I+q\left(\partial_{q} \mu\right)^{T}
$$

with I being the $k \times k$ identity matrix. Right multiplication of this equality by $v \neq 0$ (in the kernel of $\left.\partial_{q} b_{j}(q)\right)$ yields

$$
\mu(q) v=-\left(\partial_{q} \mu \cdot v\right) q
$$

from which follows that $v \neq 0$ is necessary collinear with the factor price vector $q$.
Corollary A.18. $\operatorname{rank} \partial_{q} b_{j}(q)=k-1$.

## Some useful properties of the production matrix function $B(q)$

Lemma A.19. $\left(\partial_{q} B(q)\right)^{T} q=0$.
Proof. Follows readily from Lemma A. 13 .
Lemma A.20. $\partial_{q}\left(B(q)^{T} q\right)=B(q)^{T}$.
Proof. The derivative of the matrix product $B(q)^{T} q$ is equal to

$$
\partial_{q}\left(B(q)^{T} q\right)=B(q)^{T}+\left(\partial_{q} B(q)\right)^{T} q .
$$

One concludes by applying Lemma A.19.
Lemma A.21. Let $q$ and $q^{\prime}$ in $S$ two (numeraire normalized) factor price vectors. The following (vector) inequality is then satisfied:

$$
q^{\top} B(q) \leq q^{\top} B\left(q^{\prime}\right) .
$$

Proof. Follows from Lemma A. 14 applied to each $b_{j}(q)$ amd $b_{j}\left(q^{\prime}\right)$.
Remark 9. For $q \neq q^{\prime}$, it results from the uniqueness of the cost minimizing input in pure production factors in Proposition 3 that the inequality in Lemma A. 21 is then strict.

## Asymptotic behavior when some (non-normalized) prices tend to zero

Lemma A.22. Let $A>0$. The set $\left\{y \in \mathbb{R}_{++}^{k} \mid F_{j}(y)=1\right.$ and $\left.y \leq(A, \ldots, A)\right\}$ is bounded away from zero.
Proof. For $1 \leq h \leq k$, the function $y^{h} \rightarrow F_{j}\left(A, \ldots, A, y^{h}, A, \ldots, A\right)$ is equal to 0 for $y^{h}=0$, is increasing with $y^{h}$, and tends to $+\infty$ as $y^{h}$ tends to $+\infty$.

Let $y=\left(y^{1}, \ldots, y^{k}\right) \in \mathbb{R}_{++}^{k}$ such that $F_{j}(y)=1$ and $y \leq(A, \ldots, A)$. It follows from the monotonicity of the production function $F_{j}$ that the inequality

$$
1=F_{j}\left(y^{1}, \ldots, y^{h-1}, y^{h}, y^{h+1}, \ldots, y^{k}\right) \leq F_{j}\left(A, \ldots, A, y^{h}, A, \ldots, A\right)
$$

is satisfied. Let $y_{A}^{h}$ be such that $F_{j}\left(A, \ldots, A, y_{A}^{h}, A, \ldots, A\right)=1$
It follows from the monotonicity of the production function $F_{j}$ that the inequality $0<y_{A}^{h} \leq y^{h}$ is satisfied for $1 \leq h \leq k$, which implies the inequality $y_{A} \leq y$.

Lemma A.23. Let $\left(q^{t}\right) \in \mathbb{R}_{++}^{k}$ be a sequence of non-normalized price vectors (for pure production factors) converging to some limit $q^{0} \in \mathbb{R}_{+}^{k}$, with some coordinates of $q^{0} \neq 0$ equal to zero, then $\lim \sup _{t \rightarrow \infty}\left\|b_{j}\left(q^{t}\right)\right\|=+\infty$.
Proof. The proof proceeds by contradiction. Let us write $y^{t}=g_{j}\left(q^{t}\right)$ and assume $\lim \sup _{t \rightarrow \infty}\left\|y^{t}\right\|<+\infty$. This is equivalent to the sequence $\left\|y^{t}\right\|$ being bounded. There exists a real number $A>0$ such that the inequalities $0 \leq y^{t} \leq(A, A, \ldots, A)$ are satisfied for all $t$. Recall that $F\left(y^{t}\right)=1$. Therefore, there exists by Lemma A. 22 $y_{A} \in \mathbb{R}_{++}^{k}$ such that $y_{A} \leq y^{t} \leq A$ for all $t$. By considering if necessary a subsequence, we can assume that the sequence $y^{t}$ converges to some $y^{0}$ that satisfies the inequalities $y_{A} \leq y^{0} \leq A$. By continuity, it comes $F_{j}\left(y^{0}\right)=1$. In addition, the price vector $q^{t}$ is collinear with the gradient vector $D F_{j}\left(y^{t}\right)$, i.e., there exists $\lambda^{t}>0$ such that $q^{t}=\lambda^{t} D F_{j}\left(y^{t}\right)$. The sequences $q^{t}$ and $D F_{j}\left(y^{t}\right)$ are bounded from above and bounded away from zero: the sequence $\lambda^{t}$ is therefore bounded from above and away from zero. Again considering if necessary a subsequence, we can assume that the sequence $\lambda^{t}$ converges to some $\lambda^{0}>0$. It then follows from the continuity of $D F_{j}$ that, at the limit, it comes $q^{0}=\lambda^{0} D F_{j}\left(y^{0}\right)$. The contradiction comes from the fact that some coordinates of $q^{0}$ are equal to zero while each partial derivative of the production function $F_{j}$ is different from zero.

## B. A helpful lemma about embeddings

An embedding $\phi: X \rightarrow Y$ is a smooth map between two smooth manifolds $X$ and $Y$ that is an immersion (its derivative map $D \phi(x): T_{x} X \rightarrow T_{f(x)} Y$ between the tangent spaces $T_{x} X$ and $T_{f(x)} Y$ is into, i.e., an injection) and also a homeomorphism between its domain $X$ and its image $\phi(X)$. A very nice feature of embeddings is that the image $\phi(X)$ is then also a smooth submanifold of the range $Y$. Embeddings provide a very convenient way of proving that some subset $\phi(X) \subset Y$ is actually a smooth submanifold of $Y$. The global structure of the smooth submanifold $\phi(X)$ as homeomorphic to $X$ then comes as a courtesy. The application of the following lemma requires little more than the computation of derivatives (i.e., Jacobian matrices).

Lemma B.1. Let $\phi: X \rightarrow Y$ and $\psi: Y \rightarrow X$ be two smooth mappings between smooth manifolds with: 1) The map $\psi: Y \rightarrow X$ is onto (i.e., a surjection); 2) The composition $\phi \circ \psi: Y \rightarrow Y$ is the identity map id $Y: Y \rightarrow Y$. Then, the set $Z=\phi(X)$ ), the image of $\phi$, is a smooth submanifold of $Y$ diffeomorphic to $X$.

Proof. The strategy is to show that the smooth map $\phi: X \rightarrow Y$ is an embedding, which therefore implies that its image $Z=\phi(X)$ is a submanifold of $Y$ diffeomorphic to $X$.

To prove the homeomorphism part, we first remark that $\phi$, viewed as a map from $X$ to $\phi(X)$, is a surjection. To prove that $\phi$ is an injection, assume $\phi(x)=\phi\left(x^{\prime}\right)$. Since $\psi: Y \rightarrow X$ is onto, there exist $y$ and $y^{\prime}$ with $x=\psi(y)$ and $x^{\prime}=\psi\left(y^{\prime}\right)$. It comes $\phi(x)=\phi \circ \psi(y)=y$ and $\phi\left(x^{\prime}\right)=\phi \circ \psi\left(y^{\prime}\right)=y^{\prime}$, hence $y=y^{\prime}$.

Let $\psi \mid Z$ denote the restriction of the map $\psi$ to the subset $Z$ of $Y$. The relation $\psi \circ \phi=$ id implies $(\psi \mid Z) \circ \phi=$ id $_{Y}$; with $\phi$ being a bijection between $X$ and $Z$, its inverse map is therefore equal to $\psi \mid Z$. The maps $\phi: X \rightarrow Y$ and $\psi: Y \rightarrow X$ are continuous (in fact, smooth). It follows readily from the definition of the induced topology of $Z$ that the restriction $\psi \mid Z: Z \rightarrow X$ is continuous as well as the map still denoted by $\phi: X \rightarrow Z=\phi(X)$. (Note that the fact that $Z$ is simply a subset of $Y$ equipped with the induced topology does not make it a "nice" subset of $Y$ yet, which prevents us from using the above argument to infer that $\psi \mid Z: Z \rightarrow X$ and $\phi: X \rightarrow Z$ are smooth mappings.) At the moment, these maps are just continuous and define a homeomorphism between $X$ and $Z$.

To prove the immersion part, take $y \in Y$. Let $x=\psi(y)$. The relation $\phi \circ \psi=i d Y$ yields, by taking the derivatives, the relation

$$
D \phi(x) \circ D \psi(y)=\operatorname{id}_{T_{y}(Y)}
$$

where $T_{y}(Y)$ denotes the tangent space to the manifold $Y$ at $y$. This relation implies that the linear map between tangent spaces $D \psi(y): T_{y}(Y) \rightarrow T_{x}(X)$ is an injection. The map $\phi: X \rightarrow Y$ is therefore an immersion. In combination with the homeomorphism part above, this proves that the map $\phi: X \rightarrow Y$ is an embedding.

