Risk-sharing and contagion in networks*

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Abstract

The aim of this paper is to investigate how the capacity of an economic system to absorb shocks depends on the specific pattern of interconnections established among financial firms. The key trade-off at work is between the risk-sharing gains enjoyed by firms when they become more interconnected and the large-scale costs resulting from an increased risk exposure. We focus on two dimensions of the network structure: the size of the (disjoint) components into which the network is divided, and the "relative *density*" of connections within each component. We find that when the distribution of the shocks displays "fat" tails extreme segmentation is optimal, while minimal segmentation and high density are optimal when the distribution exhibits "thin" tails. For other, less regular distributions intermediate degrees of segmentation and sparser connections are also optimal. We also find that there is typically a conflict between efficiency and pairwise stability, due to a "size externality" that is not internalized by firms who belong to components that have reached an individually optimal size. Finally, optimality requires perfect assortativity for firms in a component.

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1 Introduction

Recent economic events have made it clear that looking at financial entities in isolation, abstracting from their linkages, gives an incomplete, and possibly very misleading, impression of the potential impact of shocks to the financial system. In the words of Acharya *et al.* (2010) "current financial regulations, such as Basel I and Basel II, are designed to limit each institution's risk seen in isolation; they are not sufficiently focused on systemic risk even though systemic risk is often the rationale provided for such regulation." The aim of this paper is precisely to investigate how the capacity of the system to absorb shocks depends on the pattern of interconnections established among financial firms, say banks.

More specifically, we intend to study the extent to which the risk-sharing benefits to firms of becoming more highly interconnected (which provides some insurance against relatively small shocks, absorbable within the whole system) may be offset by the large-scale costs resulting from an increased risk exposure (which, for large shocks, could entail a large wave of induced bankruptcies). That is, we want to analyze the trade-off between risk-sharing and contagion. Clearly, this trade-off must be at the center of any regulatory efforts of the financial world that takes a truly systemic view of the problem. This paper highlights some of the considerations that should play a key role in this endeavor. In particular, by formulating the problem in a stylized and analytically tractable framework, it examines how the segmentation of the system into separate components, the density of the connections within each component as well as asymmetries in the pattern of connections should be tailored to the underlying shock structure. It also sheds light on the key issue of whether the normative prescriptions on the optimal pattern of linkages are consistent with the individual incentives to form or remove links.

We analyze a model in which there is a network consisting of N nodes, each of them interpreted as a firm. For simplicity, in most of the paper we shall consider the case where all firms are *ex ante* identical and are endowed with the same level of assets and liabilities. But *ex post* they will be different since we assume that, with some probability, a shock hits a randomly selected firm. The first *direct* effect of such a shock is to decrease the income generated by the firm's assets, thus possibly leading to the default of the firm if its resulting income falls short of its liabilities. But if this firm has links to other firms, the latter will also be affected. To be specific, let us think of the presence of a link between two firms as reflecting an exchange of the assets they are endowed with. Then, the overall network of connections generates patterns of mutual exposure between any pair of directly or indirectly connected firms, the magnitude of such exposure decreasing with their respective network distance as well as on the degree (the number of links) of each firm. Thus, when a shock hits a firm, any other firm in the same network component[1] becomes affected in proportion

[1] As usual, a component of the network is defined as a maximal set of nodes (i.e. firms) that are directly or indirectly connected.

to its exposure to that firm and has to default when its share of the shock exceeds the value of the firm's assets, net of its liabilities. So, in the end, it is the overall 'network' structure that determines how any given shock affects different firms and what is its overall aggregate impact on the whole system.

In order to highlight our basic trade-off – insurance versus contagion – we focus most of the analysis on just two dimensions of the network structure. One is the size of the (disjoint) components into which the network is divided, i.e. the degree of *segmentation* of the system. The other dimension is the "relative *density* " of connections within each component, as measured by the average network distance between a firm and any other firm in a component.

Clearly, the maximum extent of risk sharing obtains when all firms belong to a single and fully connected network. This configuration, however, exhibits the widest exposure of firms in the system to shocks and a large shock, which would affect all firms in the system, could lead to extensive default. There are two alternative (and in some cases complementary) ways of reducing such exposure. One is by segmentation, which isolates the firms in each component from the shocks that hit any other component. The second route to lowering exposure is by reducing the density of connections in each component. This buffers the network-mediated propagation of any shock that hits one of the firms in the component by increasing its distance to other firms (which in turn reduces their exposure). To obtain a sharp understanding of the effect of network density on contagion, our analysis will contrast two polar cases: (i) completely connected components, where there is a direct link between any pair of firms in each component; (ii) minimally connected (symmetric) components, where each firm has the minimum number of links (i.e. two) required to obtain indirect connectivity to every other firm in the component, thus firms are arranged in a ring. In the first case the mutual exposure between any pair of firms in the same component is exactly the same, while in the second case the reciprocal exposure between firms is heterogeneous, falling with network distance.

The key objective of the paper will be to identify the architecture of the system that best tackles the trade-off between risk sharing and contagion, hence minimizing the expected number of defaults in the system. Our model, despite being very stylized, captures the essence of the problem and allows the study of such a trade-off under a fairly general structure of the random shocks. As this shock structure changes, our analysis will deliver a precise determination of how the *optimal degree of segmentation* as well as *optimal link density* correspondingly adapt.

A first set of our results can be summarized as follows. We find that when the probability distribution of the shocks exhibits "fat tails" (i.e. attributes a high mass to large shocks), the optimal configuration involves a maximum degree of segmentation – that is, components should be of the minimum possible size. This reflects a situation where the priority is to minimize contagion. Instead, in the opposite case where the probability distribution places

high enough mass on relatively small shocks ("thin tails), the best configuration has all firms arranged in a single component. The main aim in this latter case is to achieve the highest level of risk sharing. These two polar cases, however, do not exhaust all possibilities. For we also find that for other, more complex specifications of the shock structure (e.g. mixtures of fat and thin tails) intermediate arrangements are optimal, i.e. the optimal degree of segmentation involves medium-sized components.

It is interesting to note that all of the previous conclusions apply irrespectively of whether one considers structures involving completely connected components or, alternatively, minimally connected ones. As explained, the potential advantage of minimally connected structures is that the exposure between firms in a component is not uniform but decays with network distance, which limits the number of defaults induced when the shocks are large. A natural implication of this feature is that, among minimally connected structures, the optimal degree of segmentation is lower (or, equivalently, the component size is larger) than for completely connected ones. This conclusion notwithstanding, we find that when segmentation in complete components is implemented optimally, it always dominates segmentation in minimally connected structures if the shock distributions are as described in the previous paragraph. It is however important to emphasize that for other, less regular shock distributions a minimally connected structure is optimal (i.e. better than any segmentation in complete components). This happens, for instance, when the shock distribution assigns a high probability mass on a small range of large shocks as well as on a wide range of small and medium size shocks. In those cases, we find that low density is preferred to more segmentation as the mechanism for limiting contagion when the shocks are large.

The paper also addresses the issue of whether the requirements for the optimality of the network structure of the system are compatible with the incentives of individual firms to establish links. Formally, we analyze this issue by examining the *Coalition-Proof Equilibria* (CPE) of a network formation game, where any group (i.e. coalition) of firms can jointly deviate, to avoid the multiplicity issues associated to the coordination problems present in these games. Our main conclusion is that there is typically a conflict between efficiency and individual incentives. This conflict derives from the fact that CPE typically exhibit heterogeneities in component size and these are inconsistent with efficiency. There are, therefore, positive externalities associated to displaying a uniform size for all components that are not internalized at CPE – in general, the equilibria have some firms that lie in a component that is inefficiently small.

The results summarized so far refer to environments where all firms are *ex ante* identical and thus operate at the same scale and face the same shock distribution. The paper, however, also studies the case where firms may be different, either with regard to their size or to their shock distribution. Under these circumstances, our main conclusion is particularly sharp: if we focus on completely connected components, optimality requires perfect assortativity. That is, any optimal configuration must have all firms arranged in

homogeneous components (where within each of them firms are identical both in size and in shock distributions). We also consider the implications of allowing for asymmetric network structures, where the network position of firms within a component matters. In particular, we contrast two situations: the symmetric configuration where components are completely connected (and hence all firms are in a symmetric position), and the case where firms are arranged in star networks (thus there is a large central firm connected to the smaller peripheral ones). We find that the symmetric structure is optimal when the shocks are not too large (because this maximizes risk-sharing possibilities) while the star structure is optimal for bigger shocks. The latter reflects the fact that star networks limit contagion when shocks are large, by restricting overall connectivity and having the central larger nodes act as buffers.

To sum up, our analysis highlights that the efficient configuration of the pattern of connections among financial firms – concerning, in particular, its segmentation, density, and the handling of size asymmetries – crucially depends on the nature of the shocks faced by the system. Since, as explained, one cannot generally expect that social and individual incentives be aligned, an important role for policy opens up. Our model, of course, is too stylized to allow for the formulation of concrete policy advice. It provides, however, a theoretical framework that is useful to understand the core issues and trade-offs involved, thus setting up the basis for the derivation of more specific policy implications.

We end this introduction with a brief review of the related literature. The research on financial contagion and systemic risk is quite diverse and also fast-growing. Hence we shall provide here only a brief summary of some of the more closely related papers.[2]

Allen and Gale (2000) pioneered the study of the stability of interconnected financial systems. They analyze a model in the Diamond and Dybvig (1983) tradition, where a single, completely connected component is *always* the efficient network structure, i.e. the one that minimizes the extent of default. Our model, in contrast, shows that a richer shock structure can generate a genuine trade-off between risk-sharing and contagion and that in some circumstances a certain degree of segmentation and/or low density may turn out to be efficient.

Freixas *et al.* (2000) consider an environment where a *lower density* of interaction, even though it limits risk sharing, has the positive consequence of reducing the incentives for deposit withdrawal. Concerning *segmentation*, on the other hand, a positive role for it is obtained by Leitner (2005) in a model where, unlike ours, no role is played by the interaction among agents in a component, since risk is always shared completely within each component.

[2]The reader is referred to Allen and Babus (2009) for a recent survey on how risk sharing in financial contexts can lead, through contagion, to large systemic effects. There is also a large body of literature that studies the general problem of risk sharing in non financial contexts, largely motivated by its application to consumption sharing in poor economies that lack formal insurance mechanisms. Paradigmatic examples are the papers by Bramoullé and Kranton (2007), Bloch *et al.* (2008), and Ambrus *et al.* (2011).

More in line with our approach, Allen et al. (2011) consider a six-firm environment where each firm needs funds for its investment. Since these investments are risky, firms may gain from risk diversification, which is achieved by exchanging shares with other firms. This gives rise to a financial network for which two possibilities are considered: a segmented and an unsegmented structure. The paper then analyses the different effects in these two structures of the arrival of a signal indicating that *some* firm in the system will have to default. Depending on parameters, either of these structures can dominate the other.

Two more recent, related papers are Elliott et al. (2012) and Acemoglu et al. (2013). In the first one financial contagion is modelled in a similar way to ours, as the consequence of cross-holdings among firms. Such cross-holdings, however, embody cross ownership of equity shares among firms, rather than being the result of the securitization and exchange of assets as here. This, together with the assumption that default brings about some fixed bankruptcy costs, imply that the default of any given firm induces a discontinuous drop in the revenue of all the firms owning its equity and may thus trigger a default chain. No such discontinuities occur in our setup, where contagion arises only from the correlation in returns that is induced by the firms' exchanges of assets (i.e. by their joint *ownership of assets*). Another difference with our paper concerns the main focus of the analysis. While the aim of the aforementioned paper is to characterize conditions on the structure of the network under which default cascades occur, the primary objective of ours is to characterize the optimal financial structures in diverse scenarios and their consistency with individual incentives.

The second paper, Acemoglu *et al* (2013), shares with the present one its concern with the optimality and incentive-compatibility of financial networks but displays important modelling differences in some key respects. On the one hand, the setup is one where firms are required to engage in credit-borrowing relationships, and hence the risk-sharing considerations in the model are only implicit. This feature also implies that the contagion is channelled throughout the system in a manner that is different from our model. One of the main results of their paper is that, depending on whether the magnitude of shocks is small or large, the optimal network is either complete or has components almost isolated from the rest. In contrast we characterize the optimal network structure for a non trivial distribution function of the shocks, finding a richer pattern of optimal structures, which may exhibit intermediate degrees of segmentation or different levels of link density. Finally, both approaches find that a strategically stable configuration is not optimal, as agents do *not* internalize the global network-based externalities imposed on others. The reasons, however, are different in each case. While in Acemoglu *et al* (2013), the indicated externalities pertain to the enhanced spreading of risk brought about by individual connections, in our case it involves a distortion in the size of risk-sharing components.

There is also a complementary line of literature that, in contrast with the papers just mentioned, studies the issue of contagion and systemic risk in the context of large networks (usually, randomly generated). In many of these papers, the approach is numerical, based

on large-scale simulations (see e.g. Nier *et al.* (2007)). A notable exception is the recent paper by Blume *et al.* (2011), which integrates the mathematical theory of random networks with the strategic analysis of network formation. Its primary aim is to address the question of what is the socially optimal connectivity of the system (i.e. the average degree of the network) and whether such optimal connectivity is consistent with agents' incentives to create or destroy links. They find that social optimality is attained around the threshold where a large component emerges, but individuals will generally want to connect beyond this point. As in our paper, this induces a conflict between social and individual optimality, which in their case is due to the fact that agents do not internalize the effect on others of adding new channels (i.e. links) that facilitate the spread of the effects of the shocks.

Finally, we should refer to the large empirical and policy-oriented literature whose main objective is to devise summary measures of the network of inter-firm (mostly banks) relationships that is able to anticipate systemic failures. For example, Battiston *et al.* (2012) propose a measure of centrality (*Debtrank*), inspired on the measure of *Pagerank* centrality that Google uses to rank webpages). They take this measure, which takes into account the feedback effect that failures can have on neighbors at different distances, and estimate its value for US data. Similarly, Denbee *et al.* (2011) propose a measure of centrality á la Katz-Bonacich, following Ballester, Calvó-Armengol and Zenou (2006), and apply it to English data. Of particular interest in this respect is Elsinger, Lehar and Summer (2011) who, using Austrian data, show that correlation in banks' asset portfolios is the main source of systemic risk. This is very much in line with our model, which precisely highlights portfolio correlation as the key driver of default risk.

The rest of the paper is organized as follows. Section 2 presents the model, including the continuum approximation that proves very convenient in the analysis. Section 3 characterizes optimal financial structures for various properties of the shock distribution. Section 4 addresses the network formation problem and studies the tension between strategic stability and optimality. Section 5 considers different extensions to environments with asymmetries, in particular concerning the size of firms, the distribution of their shocks, or the underlying network architecture. Finally, Section 6 concludes with a summary and a short outline of pending research issues. For convenience, all formal proofs of our results are relegated to an Appendix.

2 The Model

2.1 The Environment

We consider an environment with N ex ante identical, risk-neutral financial firms and a continuum of small investors. At any given point in time, each firm has an investment opportunity - a project - which requires an initial payment $I = 1$ and yields a random gross return $\widetilde{R}$ at the end of the period. The resources needed to undertake the project

are obtained by issuing liabilities (e.g. deposits or bonds) on which a deterministic rate of return must be paid.

The gross return of the project is random, as with some probability q the firm is hit by a negative shock. If no shock hits, the return equals some normal level R. The shock can in turn be small (which we label as s) or large (labelled b), with respective conditional probabilities π_s and π_b. When a small shock hits, the firm experiences a loss of some fixed size L_s, so its gross return is $R - L_s$. When the shock is large, the loss $\widetilde{L}_b$ is a random variable, with support $[L_s, \infty)$ and cumulative distribution function $\Phi(L_b)$. Summarizing, the gross return on a firm's project is:

$$\widetilde{R} = \begin{cases} R \text{ with prob. } 1-q \\ R - L_s \text{ with prob. } q\,\pi_s \\ R - \widetilde{L}_b \text{ with prob. } q\,\pi_b \end{cases} \tag{1}$$

Since the return on a firm's investment is subject to shocks, while the return promised to its creditors is deterministic, when the firm is hit by a shock it may be unable to meet the required payments on its liabilities, in which case it must *default*. Default entails two types of costs. First, there are the liquidations costs – for simplicity, we assume that these costs leave no resources available to make any payment to creditors at the time of default. In addition, there are additional costs deriving from the fact that a defaulting firm stops operating and hence loses any future earnings possibility. These costs are assumed to be substantial, so that the value of a firm is maximized when its probability of default at any point in time is minimized.

There is a large set of investors, who are the source of the supply of funds to firms. Investors are risk neutral and require an expected gross rate of return equal to r in order to lend their funds in any given period. Since firms may default, in which case creditors receive a payment equal to zero, the nominal gross rate of return M on the deposits to the firms must be greater or equal than r. Specifically, if we denote by φ the *ex ante* probability that any given firm defaults (an endogenous variable), we must have:

$$M = \frac{r}{1-\varphi}. \tag{2}$$

For the risk of default to be an issue, we assume:

A1. (i) $R\,(1-q) > r$
(ii) $R - L_s < r$.

The first inequality ensures that a firm's project is viable, that is, its expected return exceeds what must be paid to lenders. On the other hand, the second inequality implies (since $r \leq M$ and $\widetilde{L}_b \geq L_s$) that if a firm can only draw on the revenue generated by its project, it is surely unable to pay depositors (and hence must default) whenever a shock (whether s or b) hits its return.

Since, as stated above, default entails a significant cost for a firm, a firm may benefit from entering *risk sharing arrangements* with other firms and hence diversify risks. Here we consider the case where these arrangements take the form of swaps of assets between firms, that is, exchanges of claims to the yields of the firms' investments, prior to the realization of the uncertainty (similarly to Allen, Babus and Carletti (2011)). The possibly iterative procedure through which each firm exchanges shares on its whole array of asset holdings can be viewed as a *securitization* process of the firms' claims.

More precisely, let us posit that each firm exchanges a fraction $1-\theta$ of its standing shares, giving rights to the return on its investments, for shares held by other firms. Such an exchange entitles the firm to a fraction of the returns on the other firms' investments. Note that, due to the *ex ante* symmetry of firms, this exchange takes place on a one for one basis, as it involves assets of equal expected return. The specific pattern of exchanges among firms is formalized by a network, where a direct linkage between two firms reflects the fact that they undertake a *direct* exchange of their assets. We allow for these asset swaps to occur repeatedly. *Indirect* connections are then also formed, whereby a firm ends up having claims on the returns of projects of firms who swapped assets with the firms it exchanges assets with, and so on. As a consequence a pair of firms lying at a certain distance in the network will have some reciprocal exposure to the yields of each other's projects provided the number of exchange rounds is high enough – in particular, as high as their network distance.

As a result of this process of asset exchanges, the return on a firm's assets becomes a weighted average of the yield of its own project and the yields of the projects of the firms it exchanged assets with, directly or indirectly. A convenient way of representing a network structure is through a matrix A of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \tag{3}$$

where, for each i, j, $(i \neq j)$, $a_{ij} \geq 0$ denotes the fraction of shares in the investment project run by firm i that is owned by firm j. By construction, the following adding-up constraints must then hold:

$$\sum_{j=i}^{N} a_{ij} = 1 \qquad\qquad i = 1, 2...., N \tag{4}$$

In addition, given that, for the moment, all firms are assumed ex-ante symmetric it is natural to focus our attention on the case where the pattern of asset exchanges is also symmetric across firms. This, together with the fact that all portfolio swaps are conducted on a one-for-one basis imply that A is symmetric, i.e.

$$a_{ij} = a_{ji} \quad \text{for all} \quad i, j = 1, 2, ..., N.$$

In what follows we shall compare different network configurations in terms of social welfare. Our assumptions that default costs are very significant and that all firms are risk neutral and *ex ante* identical imply that social welfare is maximized when the expected number of defaults in the system is minimized.[3] It can be easily checked that this criterion is equivalent to that of minimizing the individual probability that any single firm defaults.[4] Thus, from an *ex ante* viewpoint, we have that social and individual objectives are aligned.

We shall assume that all network structures involve the same "externalization of risk" by each firm. That is, each firm, after all rounds of asset exchanges have been completed, holds a claim to the same fraction α of the yield of its own project, and the residual fraction $1-\alpha$ of claims to other firms' projects. Formally, this amount to making $a_{ii} = \alpha$ for each $i = 1, .., N$ and *every* network configuration. Alternative structures, therefore, differ only in terms of how the fraction $1-\alpha$ of the risk that is externalized is distributed among the rest of the firms. This allows a meaningful comparison across different network structures.

The value of α, a parameter of the model, is taken to be at least as large as $1/2$. Such a lower bound on α may be motivated by considerations of moral hazard.[5] Indeed, if in order to operate one's project some costly effort needs to be exerted, a standard way to induce a high enough effort level is to make sure that the firm retains some minimum share of the project's returns.[6] On the other hand, an upper bound on the fraction of claims retained to its own project is explained by the benefits of risk sharing and the fact that, as we will see below, a situation with all firms in isolation is never optimal.

We assume that shocks are rare and thus each period at most one firm is hit by a shock. Given (1), this can be motivated by postulating that, even if shocks hit firms in a stochastically independent manner, the probability q that a shock hits any given firm is so low that the probability that two or more shocks arrive in a single period is of an order of

[3] Beale et al. (2011) propose a similar criterion to evaluate and compare different financial systems, based on the minimization of a "systemic cost function" defined as the expectation of a convex function of the number of defaults in the system. Thus, in this case, not only the expected number of defaults matters, but also its variability.

[4] To see this, note that the *ex ante* probability of default of an individual firm φ is equal to $\sum_m \rho(m)\varphi(m)$, where $\rho(m)$ stands for the probability that m firms default and $\varphi(m)$ for the conditional probability that any particular firm defaults when there are a total of m defaults in the system. Then, since

$$\varphi(m) = \frac{1}{N} + (1 - \frac{1}{N})\frac{1}{N-1} + \cdots + (1 - \frac{1}{N} - \cdots - \frac{1}{N})\frac{1}{N-m+1} = \frac{m}{N}$$

we obtain that the expected number of defaults in the system is $\sum_m \rho(m)\, m = \varphi N$, i.e. is proportional to the individual default probability.

[5] This requirement is analogous, for example, to a well-known provision in the recent Dodd-Frank act passed in the USA to strengthen the regulation of the financial system. By virtue of this new legislation, under certain circumstances "a securitizer is required to retain not less than 5 percent of the credit risk..." (see http://www.sec.gov/about/laws/wallstreetreform-cpa.pdf).

[6] In particular, the lower bound of $1/2$ ensures that a firm always retains a larger share of claims on its project than any other firm.

magnitude that can be ignored.[7]

When a shock of size L hits the return on the project run by some firm i, the exposure to it by each one of the firms in the system is given by the vector Ae_iL, where e_i is the i-th unit vector $[0,..,1,..0]^\top$. Hence firm i defaults in response to such a shock when

$$\alpha(R-L)+\sum_{j\neq i} a_{ij}R < M, \quad \text{that is,} \quad \alpha L > R-M,$$

while any firm $k \neq i$ defaults whenever

$$\left(\alpha + \sum_{i\neq j\neq k} a_{kj}\right) R \; + \; a_{ki}(R-L) < M.$$

We readily see from the above expressions that when a firm exchanges its assets with other firms, the firm reduces its exposure to the shocks hitting its own project but at the same time becomes exposed to the shocks affecting the projects of those firms with which the firm in question is directly or indirectly connected.

On the nature of those shocks, we make the following key assumption:

A2. (i) $\pi_s > N\pi_b$.
(ii) $\alpha(R-L_s)+(1-\alpha)R \geq \frac{r}{1-Nq\pi_b}$.

Part (i) of the above assumption says that s (small) shocks are significantly more likely than b (big) ones. In particular, it is more likely that a given firm is hit directly by a s shock than a b shock hits any of the firms. Part (ii) then ensures that, by exchanging a fraction $1-\alpha$ of its shares with shares of other firms, no firm defaults when a s shock hits any firm in the environment. The term on the right-hand side of A2(ii) constitutes an upper bound on the gross rate of return M on deposits when no default occurs with an s shock, since the probability of default φ of a firm is no larger than $Nq\pi_b$, which is the probability that a b shock hits anywhere in the system. On the other hand, the term on the left-hand side of the inequality in A2(ii) constitutes the gross return of a firm that swapped a fraction $1-\alpha$ of its shares in the event where the firm is hit by a s shock. Since $\alpha \geq 1/2$ this term is also a lower bound on the firm's revenue when any other firm in the system is hit by a s shock.

The inequality in A2(ii), therefore, implies that asset exchange and securitization – i.e. the establishment of links – allows firms to fully insure against the s shocks. But, on the other hand, diversification also exposes a firm to the risk of contagion when a b shock hits any of the firms directly or indirectly linked to the firm. Recall, however, that Assumptions A1(ii) and A2(i) imply that a firm in isolation always defaults when hit by a s shock and that s shocks are much more likely than b shocks. This implies that the probability of

[7] Or, as an extreme formalization of this idea, we could model time continuously and assume that the arrivals of small and big shocks to each firm are governed by independent Poisson processes with fixed rates π_s and π_b, respectively.

default of a firm is always lower when it exchanges assets with other firms than when it is in autarky, and therefore guarantees that every firm will always want to display *some* links to others.

While *any* pattern of asset exchanges allows firms to attain full insurance against s shocks, the default performance in the event of b shocks will typically not be the same across different financial network structures. In general, that is, these structures will be markedly different in terms of the extent of contagion they induce when big shocks hit the system. Indeed, to understand which configurations minimize those detrimental side effects of risk-sharing is one of the primary aims of this paper.

Remark 1 *We should point out that in the environment considered the mutual exposure between firms comes from the cross-ownership of their shares, not from mutual lending relationships. Hence the default of one firm has no direct implication for the solvency of other firms, the possibility of contagion of a large shock hitting a firm only coming from the correlation of their portfolio returns.*

2.2 Financial Structures

We shall consider financial structures that differ along two dimensions: segmentation and network density. For the moment, we shall also restrict our analysis to symmetric configurations – which means, in particular, that all components of the network are of identical size and have the same network density. This, however, implies no loss of generality for our normative analysis since, as we will show, the welfare maximizing configurations are symmetric. In contrast, in Section 4, where the implications of strategic stability are examined, we will allow for asymmetries in component sizes.

By *segmentation* we mean the partition of the N firms into disjoint components. Each component is formed by firms that are either directly or indirectly linked by the exchange of assets (and hence the cross-ownership of shares), while there is no exchange across components. The measure of the segmentation is given by the number C of equal-sized components in which the set of firms is divided, or equivalently by the number $K \equiv \frac{N}{C} - 1$ of other firms to whom every firm is linked.[8] In terms of the matrix A, this amounts to having a block diagonal structure with C blocks along the diagonal. The larger the segmentation, the lower the number K of firms indirectly affected by a given shock but, *ceteris paribus*, the larger their mutual exposure and hence the probability of default if a b shock hits them. At the two extremes of segmentation, we have the case $K = N - 1$, where all firms are connected (directly or indirectly), and $K = 1$, where each firm only engages in trade with a single other firm. We shall allow for all possible values of K between 1 and $N - 1$ and explore their welfare implications.

[8] We ignore for now all possible integer problems regarding these magnitudes.

By *network density* we refer to the proportion of direct versus indirect linkages within each component. As we said in the previous section, a firm is directly linked to another one if the two are involved in a direct exchange of assets. Instead, two firms are only indirectly linked if they are connected by a multiple-link path in the financial network. In this case, they will end up holding claims to each other's projects once there has been a repeated exchange of assets (repeated rounds of securitization). In general, we can have different levels of density, ranging from 1 (all linkages are direct) to the case where the amount of direct connectivity is minimal (i.e. $2/K$, which is attained in the ring, as explained below). For simplicity, our analysis will just compare these two extremes, which are described in more detail in what follows.

For the choice of the financial structure to be non trivial, we assume that the number of firms in the system is not too small, that is:

A3. $N > 4$.

Minimally Connected Structures In this case firms display the minimum number of links required to be connected, directly or indirectly (i.e. through a path of some length), to all other firms in the component. In a symmetric component, it must then be that all firms have exactly two links and the architecture is that of a ring. Let K (the number of "other" firms in a component) parametrize the degree of segmentation of the system. Then, for some suitable labelling of the firms, the pattern of *direct* exchanges of assets among the firms in any given component is described by a $(K+1) \times (K+1)$ matrix of the form

$$B_K = \begin{pmatrix} \theta & (1-\theta)/2 & 0 & \cdots & 0 & (1-\theta)/2 \\ (1-\theta)/2 & \theta & (1-\theta)/2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (1-\theta)/2 & 0 & 0 & \cdots & (1-\theta)/2 & \theta \end{pmatrix}, \tag{5}$$

where θ is the fraction of assets that each firm retains for itself and the rest is exchanged with its two trading partners. These asset exchanges are iterated m times in a repeated process of securitization. In the second round, each firm trades the composite asset given by claims on its project and on those of its two neighbors as obtained in the first round. By exchanging these assets the firm acquires claims on the projects of firms that are at network distance two, i.e. the neighbors of its neighbors. And so on. The pattern of exposures among firms in the component at the end of this process is described by the matrix A_K, obtained by repeated composition of the matrix B_K with itself m times, i.e.

$$A_K = (B_K)^m. \tag{6}$$

We posit that $m = K/2$. This is the minimal value of m that ensures that any pair of firms that are either directly or indirectly linked with each other end up with a non-zero share in each other's projects. It is the minimal number of iterations required for all entries of A_K to

be strictly positive, so that all firms in the component bear some reciprocal exposure. It is immediate to see that this exposure falls with the network distance among any two firms in the ring: the entries of A_K satisfy $a_{ij} > a_{lq}$ when $|i-j| < |l-q|$.[9] Finally, note that θ must be set at a level such that, as required in the general specification in the previous section, each firm retains a fraction α of claims on its own project; that is, all the main-diagonal entries of the matrix A_K have to be equal to α.

Complete Structures This corresponds to the situation where all firms in each component are directly linked among them – so, in the language of networks, all components are completely connected. In this case, the exchange of assets among firms within the component is captured by a square matrix A_K of the form:

$$A_K = \begin{pmatrix} \alpha & (1-\alpha)/K & \cdots & (1-\alpha)/K \\ (1-\alpha)/K & \alpha & \cdots & (1-\alpha)/K \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)/K & (1-\alpha)/K & \cdots & \alpha \end{pmatrix}, \tag{7}$$

where, as before, α is the fraction of its own assets a firm keeps for itself. Evidently, the pattern of exposures described by the above matrix would not change if the exchanges were iterated and hence we can set $m = 1$ with no loss of generality.

In the next sections we compare the performance with regard to shocks of alternative financial structures differing in the two above dimensions. As explained, optimality is identified with the minimization of the expected number of defaults when shocks hit the system. Our primary objective will then be to determine the optimal degree of segmentation (that is, the optimal size of the components) as well as the optimal network density (that is, whether it is preferable to have a ring or a completely connected structure), depending on the characteristics of the shock distribution.

Let $\varphi(A;\Phi(\cdot))$ denote the probability of bankruptcy of an arbitrary firm i when the financial network structure is described by the matrix A and the distribution of the b shocks is given by the cumulative distribution function $\Phi(\cdot)$ defined on $\mathbb{R}$. We have so:

$$\begin{gathered} \varphi(A;\Phi(\cdot)) = \\ q\pi_b \Pr\{\alpha(R-L_b) + (1-\alpha)R) < M\} + q\pi_b \textstyle\sum_{j\neq i}^{N} \Pr\{R - M < a_{ij}L_b\} \end{gathered} \tag{8}$$

where M satisfies (2). The first term on the right hand side is the probability of default when a b shock hits the firm under consideration, the second term is the probability of default of the same firm when a b shock hits any of the other $N-1$ firms. By the symmetry of A the probability of default is the same and given by the above expression for all i.

[9] It is immediate to see that the same properties hold for any other $m \geq K/2$. As $m \to \infty$, the limit matrix A_K has all off-diagonal entries with the same value, which is the same property exhibited by complete structures, as we see below.

As argued before,[10] the expected number of defaults in the system is $\varphi(A;\Phi(\cdot))\,N$. Hence the optimal financial structure is given by the value of K and the network density for which $\varphi(A;\Phi(\cdot))$ is minimal. It is worth highlighting that the first term on the second line of (8) is independent of the financial structure, as it does not depend on A. That is, the probability of default of a firm whose project is hit by a b shock is the same across all structures considered. These only differ in their ability to limit contagion – that is, in how effective they are in preventing large b shocks from inducing the default of firms connected to the one directly hit by the shock.

In the environment considered there is no cost of forming linkages, still we will see that the problem of identifying the optimal structure is a non trivial one and that this sometimes displays fewer connections than in a completely connected component. This is a key feature of our analysis that stands in marked contrast with the conclusions derived from the models studied in earlier work as, for instance, Allen and Gale (2001) or Allen, Babus, Carletti (2011).

The Continuum Approximation

The pattern of risk exposure induced by either the complete or the ring structures can be graphically depicted through a function that, for each firm i, specifies the fraction of the claims to the yield of the project of firm i that is held by any other firm j in the component. This pattern is given by a function of the corresponding network distances of other firms to i and is captured by the terms on the corresponding row of the matrix A_K. Given the assumption that $\alpha \geq 1/2$ the "exposure function" always reaches the highest value when $i = j$. In the particular case of the complete structure it takes a constant value for all other $j \neq i$, while for the ring it is a step function that is monotonically decreasing.

Because of the discreteness of the domain of this function – in particular, for the case of the ring structure – and the integer problems faced in the determination of the optimal degree of segmentation a formal analysis of the firms' probability of default for different network configurations becomes quite involved and tedious to carry out. To render the analysis more tractable, we study in what follows a continuum approximation of our model that abstracts from these considerations but still allows to capture the essential features of the problem. It involves approximating the discrete function $\varphi(A;\Phi(\cdot))$ by a continuous real function that embodies the same features as the original one.[11]

In the continuum approximation of the model, N is taken to be the measure of the firms in the system, while $K+1$ is the generic notation used to represent the measure of firms

[10] See footnote 4.

[11] This continuum formulation can be seen as representing a limit description of a context consisting of a large number of firms, each of them of small (relative) size. This interpretation, however, is not essential, so we prefer to view it simply as a continuous approximation of a context where the number of firms is not necessarily large.

belonging to a certain component. The returns on a firm's project are the same as in (1). To keep a formal parallelism with the discrete formulation, when a shock occurs it is taken to hit directly a unit mass of firms in one component. This mass of firms, therefore, plays the role of the single firm directly hit by a shock in the discrete context.

Segmentation is modelled as in the previous section, except that the size of a component, $K+1$, need not be an integer and can now be any real positive number lying between 1 and N. Differences in network density, on the other hand, are modelled as follows.

If the component is complete, the pattern of exposure to the returns on other firms' projects is as in the discrete model: the indirect exposure, to a shock that hits any other firm in the component, is constant and equal to $(1-\alpha)/K$. On the other hand, if the component has a ring structure, the pattern of risk exposure is now described by a continuous function $f(d;K)$, where $d \in [0, K/2]$ is the ring distance to the set of firms directly hit by the shock. This function has the following form:

$$\begin{aligned} f(d;K) &= \alpha - \tfrac{\alpha-H}{H} d && \text{for } 0 \leq d \leq H \\ &= \tfrac{HK}{K-2H} - \tfrac{2H}{K-2H} d && \text{for } H < d < K/2 \\ &= 0 && \text{for } d = K/2. \end{aligned} \tag{9}$$

for some suitably computed H. The function $f(\cdot)$ is defined for all $K \geq 1$, with the value of H being determined by the following adding-up constraint:

$$2\int_0^{K/2} f(x;K)\mathrm{d}x = (\alpha - H)\,H + 2H^2 + H\,(K/2 - H) = 1 - \alpha, \tag{10}$$

which reflects the requirement that firms externalize the fraction $(1-\alpha)$ of their risk. This condition leads to

$$H = \frac{2(1-\alpha)}{K+2\alpha}. \tag{11}$$

Recalling that $\alpha \geq 1/2$ and $K \geq 1$, it is immediate to verify that the function $f(d;K)$ exhibits the following properties that match those of the exposure function in the discrete set-up:

$$f(d;K) \text{ is positive and decreasing for all } K/2 > d > 0, \tag{12}$$

$$f(0) = \alpha, \tag{13}$$

$$f(K/2) = 0. \tag{14}$$

First, (12) says that every firm in a component is affected by a shock hitting any other firm in the component, but with an intensity that decreases with the distance to the source of the shock. In addition, (13) states that the level of exposure to a firm at minimal distance is the same as that to a direct shock, while (14) says that the exposure becomes vanishing small when the distance to the source of the shock is maximal in the component. Conditions (12), (13), and (14), together with the adding-up constraint (10) on the total fraction of the

risk externalized by firms, capture the essential features displayed by the exposure function in the discrete setup.

Note that $f(d;K)$ is a two-piece linear function with the kink at the bisectrix (i.e. at a distance $d = H$ such that $f(H) = H$). It is concave or convex depending on whether, respectively, K is smaller or larger than $2(1-\alpha)/\alpha$. An illustration of how it approximates the original function for the discrete setup is displayed in Figure 1.

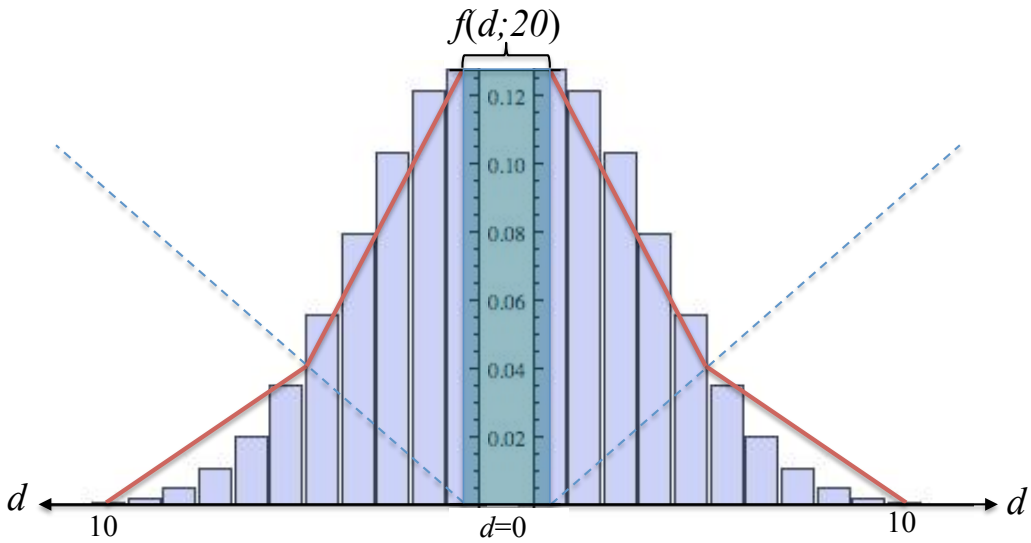

Figure 1: Continuum approximation of the exposure function. The step function describes the exposure in a ring with $K+1 = 21$ firms, as a function of the distance d to either side of the location occupied by any given firm. The exposure is the one induced by the matrix $A_{20} \equiv (B_{20})^{10}$, with B_{20} being the matrix of asset exchange specified in (5). In contrast, the solid line describes the exposure for the continuum approximation $f(d, 10)$.

Using the above specification of the exposure function for the ring and the complete structures, for any possible level of K, we are now in a position to determine the extent of default induced by any given shock of magnitude L for alternative financial structures. Note first that, as already argued for the discrete context, whether the unit mass of adjacent firms directly hit by a shock default or not is independent of the underlying financial network structure. For those firms will default if, and only if,

$$L > \frac{R-M}{\alpha}.$$

If the former inequality is violated and the firms hit by the shock do not default, no other firm in the corresponding component defaults either. This simply follows from the fact that $\alpha \geq (1-\alpha)/K$ since $\alpha \geq 1/2$, $K \geq 1$ and $f(d;K) \leq \alpha$ for any $d \geq 0$. But if those firms directly affected by the shock do default, what happens to all the others in the component naturally depends on the size $K+1$ of the component and on the pattern of the connections within it.

In the case where firms are connected through a *complete interaction pattern*, the uniformity of the exposure has the following immediate implication: *all* firms indirectly affected (i.e. not hit by the shock but lying in the component affected) will default if

$$L > \frac{K}{1-\alpha}(R-M) \tag{15}$$

whereas *none* of those firms will default otherwise. Thus, if we let $g_c(L;K)$ stand for the mass of firms that default when the shock hits some other firm in their component, that magnitude is given by the following step function:

$$g_c(L;K;M) = \begin{cases} 0 & \text{if } L \leq \frac{K}{1-\alpha}(R-M) \\ K & \text{if } L > \frac{K}{1-\alpha}(R-M) \end{cases} \tag{16}$$

In contrast, when the firms in the component are connected through a *ring interaction pattern* (as captured by the exposure pattern $f(\cdot)$), the conclusion is, in general, not so dichotomic. For, in this case, whether any particular firm in the component defaults or not depends on its ring distance d to those firms that have been directly affected. Such a firm defaults if, and only if,

$$L > \frac{1}{f(d;K)}(R-M),$$

which is to be contrasted with (15). Hence the threshold that marks the relevant "default range" is given by the distance $\hat{d}$ such that

$$f(\hat{d};K) = \frac{R-M}{L}, \tag{17}$$

so that a firm defaults if, and only if, its distance d from the set of firms directly hit by the shock is such that $d < \hat{d}$. Under a ring structure, therefore, the effect of different levels of the shock on the mass of firms defaulting is not discontinuous as in the complete structure. Rather, as the magnitude L of the shock increases, the mass of firms defaulting among those indirectly affected by it grows gradually (see Figure 2 for an illustration), as determined by the function $g_r(L;K,M) \equiv 2f^{-1}((R-M)/L;K)$ given by:

$$g_r(L;K,M) = \begin{cases} 0 & \text{for } L \leq \frac{R-M}{\alpha} \\ \frac{2\alpha H}{\alpha - H} - \frac{2H}{\alpha - H}\frac{R-M}{L} & \text{for } \frac{R-M}{\alpha} \leq L \leq \frac{R-M}{H} \\ K - \frac{K-2H}{H}\frac{R-M}{L} & \text{for } L \geq \frac{R-M}{H} \end{cases} \tag{18}$$

3 Optimal Financial Structures

In this section we identify the optimal network segmentation and network density. That is, we determine the financial structure that minimizes the expected mass of firms that default, in the continuum approximation of the model. To this end, we rely on the functions given in (18), and (16) that specify, respectively, the mass of firms defaulting in a component with

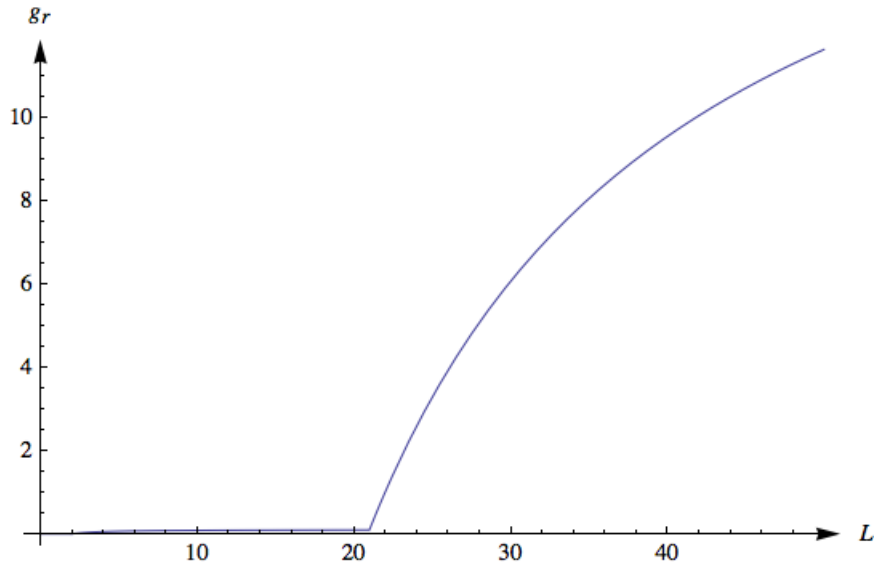

Figure 2: The function $g_r(L; 20, M)$, specifying the mass of firms that default when indirectly hit in a component of size $K + 1 = 21$ for any given magnitude L of the b shock, plotted for $\alpha = 1/2$ and the normalization $R - M = 1$, as posited below.

a complete or ring structure. The choice variables are not only the number C of different components in the system, but also the size K_i+1 of each component i, $i = 1, .., C$. We allow, therefore, for the possibility of asymmetric structures, with components of possibly different size. As explained in the previous section, the minimization of the expected mass of defaults is equivalent to the minimization of the probability that any given firm defaults. Hence, focusing on the latter, our optimization problem can be formulated as the minimization of the following magnitudes:

$$\varphi_\nu = qN\pi_b \left(\frac{1}{N}\Pi + \sum_{i=1}^{C} \frac{K_i+1}{N} \frac{K_i}{N} \frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M)}{K_i} \right) \tag{19}$$

where the subindex $\nu = c, r$ signifies that either a complete or a ring structure is considered, and Π is the probability that a firm defaults when directly hit by a b shock (which is the same for all structures since α is also the same across structures). The above expression reflects the fact that, when heterogeneous components are allowed, the ex-ante probability that any firm defaults is the average weighted probability that firms in each component default.

Consider first a fixed structure $\nu \in \{c, r\}$. Then, discarding irrelevant terms in (19), the optimal degree of segmentation is obtained as a solution of the following optimization problem:

$$\begin{array}{c} \min_{K_i,C} \sum_{i=1}^{C} \frac{K_i+1}{N} \mathbb{E}g_\nu(\widetilde{L}_b; K_i, M) \\ \text{s.t.} \sum_{i=1}^{C} \frac{K_i+1}{N} = 1 \end{array} \tag{20}$$

We will show that the solution is (essentially) symmetric for all cases under consideration,

with $K_i = K$ for all i. This implies that problem (20) can be reduced to minimizing $\mathbb{E}g_\nu(\widetilde{L}_b; K, M)$ with respect to K. Then, by comparing the optimum values of the solution of (20) for $\nu = r$ and $\nu = c$ we obtain the optimal network density.

Note that the value of $\mathbb{E}g_\nu(\widetilde{L}_b; K, M)$ depends not only on the distribution $\Phi(L_b)$ but also on the other parameters of the model, α, R, r, L_s. The specific values of these parameters have however little interesting bearing on the analysis and our primary focus is on the effects of the shock distribution on the optimal network structure. Hence in what follows we set[12] $\alpha = 1/2$, normalize $R - r$ to unity and set also $L_s = 1$. Furthermore, even though M is an endogenous variable (determined by (2)) and varies with the underlying network structure, for the purpose of determining the optimal structure we can make it equal to r (and hence $R - M = 1$) if $qN\pi_b$ is small.

To see this, let $M_F(qN\pi_b)$ denote the function specifying the value of M that satisfies the required arbitrage condition (2) under financial structure F, given by a pattern of interaction $\nu \in \{c, r\}$ and component sizes $K_1, ..., K_n$ all displaying the pattern ν, when the probability of the arrival of a b shock is $qN\pi_b$. Substituting equation (19) into (2), $M_F(qN\pi_b)$ is defined implicitly by the following equation:

$$M_F(qN\pi_b)\left[1 - qN\pi_b\left(\frac{1}{N}\Pi + \sum_{i=1}^{C}\frac{K_i+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_F(qN\pi_b))}{N}\right)\right] = r.$$

Since the function $\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_F(qN\pi_b))$ is bounded for each $\nu \in \{c, r\}$, as we see from (18), (16), $M_F(qN\pi_b)$ is continuous in $qN\pi_b$ at $qN\pi_b = 0$. Hence, if structure F_1 yields a higher expected mass of defaults when indirectly hit by a b shock than structure F_2, i.e.

$$\sum_{i=1}^{C^{F_1}}\frac{K_i^{F_1}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_{F_1}(0))}{N} > \sum_{i=1}^{C^{F_2}}\frac{K_i^{F_2}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_{F_2}(0))}{N},$$

then we also have, for $qN\pi_b$ sufficiently small,

$$\sum_{i=1}^{C^{F_1}}\frac{K_i^{F_1}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_{F_1}(qN\pi_b))}{N} > \sum_{i=1}^{C^{F_2}}\frac{K_i^{F_2}+1}{N}\frac{\mathbb{E}g_\nu(\widetilde{L}_b; K_i, M_{F_2}(qN\pi_b))}{N},$$

Therefore, in identifying the optimal structure, we can set $M = M_F(0) = r$ and thus have $R - M = 1$.

We organize our formal analysis in the rest of this section in three parts. First, in Subsection 3.1 we identify some clear-cut conditions regarding the probability distribution of the b shocks under which the optimal segmentation is one of the two polar extremes –

[12] A convenient consequence of this choice of α is that, just as for the discrete version of the model, the pattern of exposure is exactly the same for the complete and the ring structure if the component size is the smallest (i.e. $K = 1$). The difference between the two structures then grows wider the higher is K.

maximal or minimal – and the optimal degree of connectivity is complete. Then, in Subsection 3.2 we extend the analysis to more general specifications of the shock distribution, for which intermediate levels of segmentation are optimal. Finally, in Subsection 3.3 we identify scenarios where the optimal structure exhibits not only intermediate levels of segmentation but also low density of connections, as embodied by the ring structure.

3.1 Polarized segmentation

In order to get a clear understanding of the forces at work, we shall start by examining the case where the probability distribution of the b shocks is of the Pareto family with support $[1, \infty)$ and density $\gamma/L_b^{\gamma+1}$. By modulating the decay parameter γ, this formulation already allows the discussion of many questions of interest such as the contrast between fat and thin tails in the shock distribution (i.e. between scenarios where large shocks are relatively frequent or not). As mentioned above, our analysis will be carried out in two steps. Firstly, we shall characterize how γ affects the optimal degree of segmentation (as described by K) for the ring and for the complete structure. Secondly, we shall compare these two structures.

Let $D_r(K,\gamma) = \mathbb{E}_\gamma g_c(\widetilde{L}_b; K)$ denote the expected mass of firms in a ring of size $K+1$ who default when indirectly hit by a b shock with a Pareto distribution with parameter γ (that is, when the shock hits some other firm in the component). We have:[13]:

$$\begin{aligned} D_r(K,\gamma) &= \int_1^\infty g_r(L;K)\ \mathrm{d}\Phi(L;\gamma) \\ &= \int_{\frac{R-M}{H}}^\infty \left(K - \frac{K-2H}{H}\frac{1}{L}\right)\frac{\gamma}{L^{\gamma+1}}\mathrm{d}L + \int_{\frac{R-M}{\alpha}}^{\frac{R-M}{H}} \left(\frac{2\alpha H}{\alpha-H} - \frac{2H}{\alpha-H}\frac{1}{L}\right)\frac{\gamma}{L^{\gamma+1}}\mathrm{d}L \\ &= \gamma\left[K\frac{1}{\gamma(K+1)^\gamma} - [K(K+1)-2]\frac{1}{(\gamma+1)(K+1)^{\gamma+1}}\right] + \\ &2\gamma\left[-\frac{1}{K-1}\frac{1}{\gamma(K+1)^\gamma} + \frac{2}{K-1}\frac{1}{(\gamma+1)(K+1)^{\gamma+1}} + \frac{1}{K-1}\frac{1}{\gamma\, 2^\gamma} - \frac{2}{K-1}\frac{1}{(\gamma+1)2^{\gamma+1}}\right] \end{aligned} \tag{21}$$

We study next how the above expression behaves as K varies in its admissible range (recall that the minimum[14] admissible value of K is 1 and its maximal value is $N-1$). We establish in the following proposition that whenever $\gamma > 1$ (i.e. the distribution function of $\widetilde{L}_b$ does not have fat tails) $D_r(K,\gamma)$ attains a minimum at the highest value of K, i.e. $N-1$. On the other hand, when $\gamma < 1$ (i.e. the distribution has fat tails) the function attains a minimum at $K = 1$.

Lemma 1 *When the shock $\widetilde{L}_b$ has a Pareto distribution, the component size which minimizes the mass of firms defaulting is minimal ($K = 1$) if $\gamma > 1$, and maximal ($K = N-1$) if $\gamma < 1$.*

[13] Henceforth we shall use the parameter values $\alpha = 1/2$ and $R - M = 1$ specified at the end of the previous section and omit the argument M $(= R-1)$ in the functions throughout. Also, these values imply that $H = 1/(K+1)$.

[14] Strictly speaking, the expression in (21) holds for $K > 1$. For $K = 1$ we have $D_r(1,\gamma) = \gamma\left[K\frac{1}{\gamma(K+1)^\gamma} - [K(K+1)-2]\frac{1}{(\gamma+1)(K+1)^{\gamma+1}}\right]$. However, since we show in the proof of Lemma 1 that $D_r(K,\gamma)$ is continuous at $K = 1$, in what follows it suffices to work with (21).

On this basis, we can easily determine the optimal segmentation pattern in the system. To this end we must also take into account the constraint present in (20), $\sum_{i=1}^{C}(K_i+1)/N = 1$. Given the findings of the above lemma, this constraint only binds when N is odd and $\gamma < 1$, as in such case a structure with all components of the optimal (minimal) size 2 is not feasible. To find the optimal structure in this case we establish the concavity of $D_r(K,\gamma)$:

LEMMA 2 *When $\gamma < 1$ the function $D_r(K,\gamma)$ is strictly concave in K, for all $K > 1$.*

From Lemmas 1 and 2 it immediately follows that the optimal ring structure when $\gamma < 1$ and N is odd is "almost" symmetric, with $\frac{N}{2} - 1$ components of minimal size 2 and a residual component of size 3. In all other cases, the optimal ring structure is the one with all components of the same optimal size, determined in Lemma 1. We can then summarize our findings in the following statement:

PROPOSITION 1 *Suppose the shock $\widetilde{L}_b$ has a Pareto distribution. With $\gamma > 1$ (no fat tails) the optimal degree of segmentation for the ring structure is minimal, with a single component including all N firms. Otherwise, with $\gamma < 1$ segmentation is maximal, with all components of the minimal size 2 (except one, of size 3, when N is odd).*

The previous result shows that there is indeed a trade-off between risk sharing and contagion. On the one hand, when the distribution of the shocks exhibits fat tails (hence large shocks are relatively likely), the predominant consideration is to control contagion rather than achieve risk sharing. Hence the expected number of defaults is minimized by breaking the network into disjoint components of minimal size, which limits the extent to which a shock may spread its consequences far into the system. Instead, when the distribution has no fat tails, the most important consideration becomes risk-sharing, which is maximized by placing all firms in a single component.

Next, we turn to studying the analogous question for the case where the components are completely connected. In this case, the expected mass of firms who default in a completely connected component of size $K+1$ when indirectly hit by a b shock is:

$$D_c(K,\gamma) = \mathbb{E}_\gamma g_c(\widetilde{L}_b; K) = K \ \Pr\left(L \geq 2K\right) = K \ \left(\frac{1}{2K}\right)^{\gamma}. \tag{22}$$

Hence

$$\frac{\partial D_c}{\partial K} = -\left(\gamma - 1\right)\left(\frac{1}{2K}\right)^{\gamma} \gtreqless 0 \quad \Longleftrightarrow \quad \gamma \lesseqgtr 1,$$

which readily implies that the optimal component size is again minimal when $\gamma < 1$ (the shock distribution has fat tails), while it is maximal in the case where $\gamma > 1$. Hence the optimal segmentation structure is the same as for the ring, with all components of the optimal size when the constraint $\sum_{i=1}^{C}(K_i+1)/N = 1$ does not bind. In contrast, when $\gamma < 1$ and N odd, and hence the former constraint binds, the optimal structure is different from the one obtained for the ring in this situation. It is now exactly symmetric, with $\frac{N-1}{2}$

components, all of the same size (slightly larger than 2). This is because, as we show in the proof of the next proposition, $D_c(\cdot)$ is a convex function of K (remember that $D_r(\cdot)$ is a concave function of K). Hence we have the following:

PROPOSITION 2 *When the shock $\widetilde{L}_b$ has a Pareto distribution, the optimal degree of segmentation for the completely connected structure is minimal (one single component) if $\gamma > 1$, and maximal, with $N/2$ ($(N-1)/2$ if N is odd) identical components, if $\gamma < 1$.*

Finally, we need to compare the optimally segmented ring and complete network structures in order to identify which of the two is optimal when not only segmentation but also network density can be chosen. In view of Propositions 1 and 2, it is enough to compare the expected mass of firms defaulting under either maximal or minimum segmentation for the ring and the complete structures when, respectively, γ is lower or higher than unity. Note also that when K is at its minimal admissible value, 1, the pattern of exposure is the same for the two structures, $g_c(L_b; 1) = g_r(L_b; 1)$, hence a difference only arises when the optimal component size is greater than 1 for at least one structure. This leads to the following result.

PROPOSITION 3 *If the shock $\widetilde{L}_b$ has a Pareto distribution, for all values of $\gamma < 1$, when N is even the optimal structure is equivalently completely connected or a ring structure, while for all $\gamma > 1$ and for N large enough ($N > 1 + (1+\gamma)^{\frac{1}{\gamma-1}}$) the complete network strictly dominates the ring structure.*

Combining the results obtained in this subsection, we conclude that if shocks are Pareto distributed, the optimal network always displays maximal density and polarized (maximum or minimum) segmentation. In this case, therefore, the whole adjustment to the underlying risk conditions (i.e. to the different values of γ) is obtained only by varying the segmentation pattern. However, as our subsequent analysis will show, neither the optimality of the polarized segmentation pattern nor of the complete connectivity within components are features maintained for other, more complex, shock distributions.

3.2 Intermediate Degrees of Segmentation

Propositions 1 and 2 establish that, when the distribution of the shocks has a simple Pareto structure and thus it either has, or does not have, fat tails, the optimal degree of segmentation is always extreme, i.e. maximal or minimal. We show next that this is no longer true when the distribution of the shocks is more complex, as for instance when it is given by the mixture of two Pareto distributions.

PROPOSITION 4 *Suppose that the shock $\widetilde{L}_b$ is distributed as a mixture of a Pareto distribution with parameter $\gamma > 1$ and another Pareto distribution with parameter $\gamma' < 1$, with respective weights p and $1-p$. Then, there exist p_0, p_1, $0 < p_0 < p_1 < 1$, such that, whenever $p \in (p_0, p_1)$,*

the optimal pattern of segmentation for the completely connected structure is symmetric with components of intermediate size $K^* + 1$, $1 < K^* < N - 1$.

Abusing slightly previous notation, denote by $D_c(K, \gamma, \gamma', p) = p\mathbb{E}_\gamma g_c(\widetilde{L}_b; K) + (1 - p)\mathbb{E}_{\gamma'} g_c(\widetilde{L}_b; K)$ the expected mass of firms who default in a complete component of size $K+1$ when indirectly hit by a b shock hitting a firm in the component, and the distribution of this shock is as in the statement of the above proposition. We show in the proof of this result that the function $D_c(K, \gamma, \gamma', p)$ attains a minimum at an intermediate value $\hat{K} \in (1, N-1)$. A symmetric structure with all components of size $\hat{K}$ is however generically not feasible now (i.e. violates the condition $\sum_{i=1}^{C}(K_i + 1)/N = 1$). We then show that the optimal structure is still symmetric, with all components of size K^*, smaller or equal to $\hat{K}$.

A numerical analysis of the problem shows that a similar conclusion applies to the case where the components display a ring structure. That is, an intermediate level of segmentation continues to being optimal when the shock distribution involves a mixture of Pareto distributions with both fat and thin tails. However, since an analytic result in this case is hard to obtain, we illustrate matters through the following example.

Example 1 *Set $\gamma = 2$, $\gamma' = 0.5$ and $p = 0.95$. For these values we find that the value of K which minimizes $D_c(K, \gamma, \gamma', p)$ is $\hat{K}^c = 5.65$, and at this value the expected mass of defaults (when a b shock hits some other firms in the component) is 0.13. The value of K that minimizes the corresponding expression for the ring structure, $D_r(K, \gamma, \gamma', p)$, is higher, $\hat{K}^r = 8.02$, and the expected mass of defaults in this case is also higher, equal to 0.145. The fact that $\hat{K}^r > \hat{K}^c$ can be intuitively understood as a reflection of the fact that, when arranged optimally, components with a ring structure compensate for their lower density of connections with a larger size. See Figure 3 for a graphical depiction of these two functions.*

To find the optimal financial structure for the whole system we also need to specify the value of N. Let $N = 10$. In this case it is clear that a symmetric structure with equal components of size $\hat{K}^c + 1$ is not *feasible when the components are completely connected, and the same applies to components of size $\hat{K}^r + 1$ when they are rings. We find that both for the complete and the ring structures the optimal configuration of the system is given by two equal-sized components of size $K^* + 1 = 5$. Moreover, in line with the conclusions stated above for the values of the mass of expected defaults at $\hat{K}^r$ and $\hat{K}^c$, we find that the optimal complete structure still dominates the optimal ring structure: the former yields an expected mass of defaults equal to 0.26 in contrast with 0.32 induced by the latter (see Figure 4). This conclusion is robust to alternative specifications of the parameter values of the environment.*

3.3 Sparse Connections

Let us consider now the case where the probability distribution of the b shocks is not smooth because it has some atoms. More precisely, let $\Phi(L_b)$ be the mixture of a Pareto

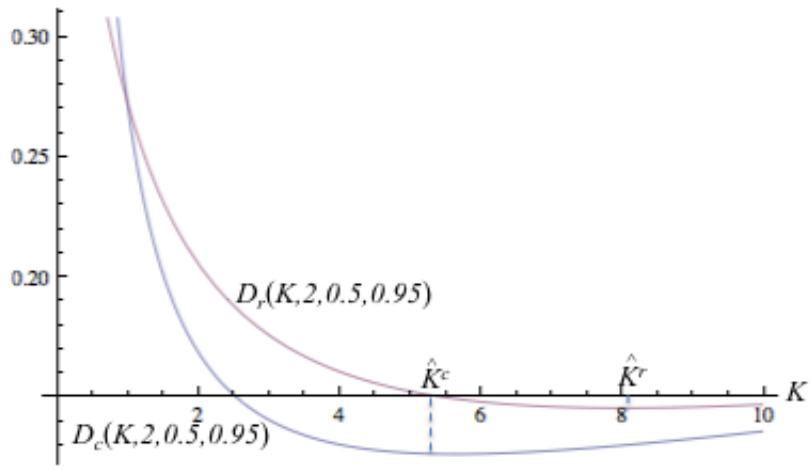

Figure 3: Expected mass of firms in a component of size K indirectly affected by a b shock who default when this shock hits their component, as given by the function $D_v(K,\cdot)$ of the component's size K, for both complete and ring structures ($v = c, r$). The parameter values are $\gamma = 2$, $\gamma' = 0.5$, $p = 0.95$.

distribution with $\gamma > 1$ and a Dirac distribution putting all probability mass on a shock of magnitude $\bar{L} > 2(N-1)$. On the one hand, note that the Pareto distribution considered has no fat tails. With such a distribution, as we saw in Section 3.1, minimal segmentation ($K = N-1$) is optimal and, in addition, the completely connected structure dominates the ring structure. On the other hand, the shock $\bar{L}$ selected by the Dirac distribution has the following property. If a shock of that magnitude occurs and firms are arranged in a single component, all firms default when they are completely connected while some survive if arranged in a ring. Combining the previous considerations, we show below that there is an open region of parameter values for which the second effect prevails over the first one and hence the optimal financial structure is a ring. Under such conditions, therefore, sparse connections are optimal.

Proposition 5 *Let $\widetilde{L}_b$ be distributed as a mixture of a Pareto distribution with parameter $2 > \gamma > 1$ and a Dirac distribution with all mass concentrated on $\bar{L} = 2(N-1)+1$, with weights respectively given by p and $1-p$. Then, for all values of N such that*

$$N > 1 + \left(\frac{1}{4^{\gamma-1}} - \frac{1}{5^{\gamma}} + \frac{\frac{1}{2^{\gamma-1}} \frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}} \right)^{\frac{1}{2-\gamma}} \tag{23}$$

and provided that

$$\frac{(1-p)}{p} < (\gamma - 1) \left(\frac{1}{2(N-1)} \right)^{\gamma}, \tag{24}$$

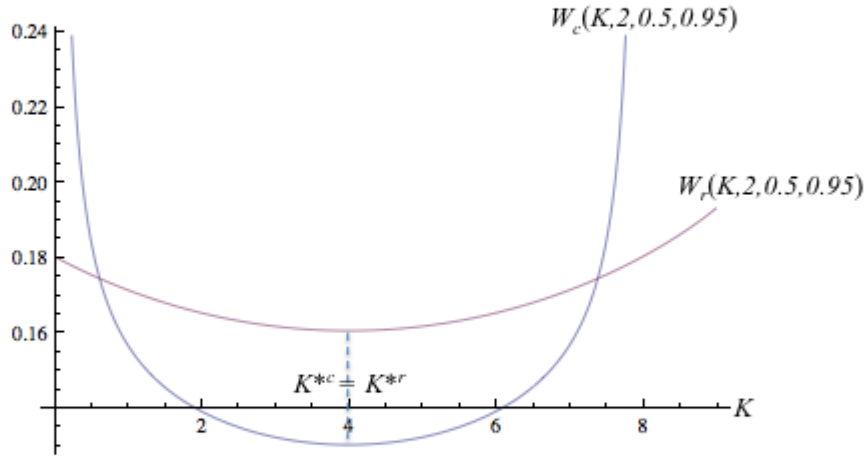

Figure 4: Expected mass of firms in a system with N firms indirectly hit by a b shock who default when this shock hits a component in the system, as given by the function $W_v(K,\cdot) \equiv \frac{K+1}{N} D_v(K,\cdot) + \frac{N-K-1}{N} D_v(N-K-2,\cdot)$, for the complete and ring structures ($v = c, r$). The system consists of $N = 10$ firms divided into two components of size K and $N - K$. The parameter values are as in Figure 3, i.e. $\gamma = 2$, $\gamma' = 0.5$, $p = 0.95$.

there exists an open set of values of p such that the optimal financial structure is a single ring component.

The role of inequality (24) is to guarantee that the weight p on the Pareto distribution is sufficiently high that the optimal segmentation structure is determined by it and hence a *single* component is optimal for the completely connected structures. But then, given that there is also a non negligible probability that a large shock arrives that cannot be fully absorbed, some attempt at "controlling the induced damage" may be in order. And this is indeed what the ring achieves – a suitable compromise between the extent of *risk sharing* allowed by extensive *indirect* connectivity (i.e. minimal segmentation) and the limits to wide *risk contagion* resulting from sparse *direct* connections.

4 Stability and optimality

We now examine the relationship between the optimal pattern of linkages derived in the previous section and the individual incentives to form those linkages. We explore, in other words, whether social welfare is aligned with the maximization of individual payoffs. To model the strategic considerations involved in the creation of networks structures, we will model a network-formation game. Since we have only explored the payoffs accruing to structures formed by complete components and rings, we need to restrict our analysis to such kinds of structures. For simplicity we concentrate on completely connected component

structures in what follows. The network-formation game is then assumed to be conducted as follows:

- Firms independently submit their proposals concerning the components to which they want to belong. Formally, a strategy of each firm i is a subset $S_i \subset N$, with the property that $i \in S_i$.
- A particular proposal S_i is accepted only when all the firms $j \in S_i$ make the same proposal, i.e. $S_j = S_i$. Formally, given a profile of strategies $\mathbf{S} \equiv (S_i)_{i \in N}$ a component S is established if and only if for all the firms $i \in S$, $S_i = S$. If the proposal S_i of firm i is rejected, that is if for some $j \in S_i$, $S_j \neq S_i$, firm i stays isolated.

Given the above network-formation rules, a specific network $\Gamma(\mathbf{S})$ is induced by each strategy profile $\mathbf{S}$. As maintained throughout, the payoff of each firm i is decreasing in $\varphi_i(\Gamma(\mathbf{S}))$, its own default probability resulting from the network induced by $\mathbf{S}$. Formally, we can w.l.o.g. identify its payoff with $-\varphi_i(\Gamma(\mathbf{S}))$, the opposite of that probability.

In such a network-formation game, an undesirable feature of the standard concept of Nash Equilibrium is that it leads to a vast multiplicity of equilibrium networks, a consequence of the fact that the formation of any link induces a coordination problem between the two agents involved. (As an extreme illustration, note that the empty network can always be supported by a Nash equilibrium where every agent proposes nobody to link with.) To address this issue, it is common in the literature to consider a strengthening of the Nash equilibrium notion that reduces miscoordination by allowing sets of agents to deviate jointly (see e.g. Goyal and Vega Redondo (2007) or Calvó-Armengol and Ikiliç (2009)). In the context of our model, we shall capture this idea by means of the concept we label *Coalition-Proof Equilibrium* (CPE), where any group (i.e. coalition) of agents can coordinate their deviations:

Definition 1 *A strategy profile* $\mathbf{S} \equiv (S_i)_{i \in N}$ *of the network-formation game defines a* Coalition-Proof equilibrium *(CPE) if there is* no subset *of firms* W *and a strategy profile for these firms,* $(S'_j)_{j \in W}$*, such that*[15]

$$\forall i \in W, \quad \varphi_i \left[\Gamma \left((S'_j)_{j \in W}, (S_k)_{k \in N \setminus W} \right) \right] < \varphi_i[\Gamma(\mathbf{S})].$$

The CPE notion precludes joint profitable deviations by any arbitrary set of firms, so it is obviously a refinement of the standard notion of Nash Equilibrium. A requirement typically imposed on coalition-based notions of equilibrium is that the coalitional deviations considered should be robust, in the sense of being themselves immune to a subcoalition profitably deviating from it. This requirement has no bite for our analysis since we can show

[15] In line with what was postulated in Section 2, we shall continue to assume that, after any change in connections has been implemented, the firms involved in the change continue to distribute the fraction $1-\alpha$ of their own assets among all their neighbors (old and new) in a uniform manner

that all the profitable deviations that need to be allowed are robust in the aforementioned sense.[16]

Consider first the case where the optimal completely connected structure consists of identical components of the size that minimizes the mass of defaults in a component – thus, in this case, the feasibility constraint $\sum_{i=1}^{C}(K_i+1)/N = 1$ does not bind (as for instance in Proposition 2). Then it is immediate to verify that the optimal structure is also a CPE. In contrast, we show below (Proposition 6) that this is no longer necessarily true when the feasibility constraint binds.

To make the point, we analyze the CPE for the class of distributions considered in Section 3.2 and show that, in that context, there is typically a conflict between social optimality and individual incentives. To understand the conflict, first recall from Proposition 4 that there is an open set of the parameter space for which, under the aforementioned class of shock distributions, the optimal completely connected structure involves a symmetric segmentation of the whole population into several components of equal size, K^*+1. Moreover, K^*+1 is typically smaller than the individually optimal component size $\hat{K}+1$ that minimizes default *within* a given component.

This gap produces a conflict between individual and social incentives, which is reflected by the inefficiency of CPE configurations. In a CPE the outcome is asymmetric: individual incentives (supported by coalitional deviations) give rise to all but one component being of the individually optimal size $\hat{K}+1$ and one other component of a size smaller than K^*+1. But this, to repeat, cannot be socially optimal because standard convexity considerations favor a more balanced configuration where the sizes of all components are equal.

Proposition 6 *Consider the same environment as in Proposition 4. Then, for $p \in (p_0, p_1)$, the socially optimal completely connected structure cannot be supported at a CPE of the network formation game. Among the completely connected structures, the only CPE configuration is asymmetric with all but one component displaying the size $\hat{K}$ that minimizes $D_c(K, \gamma, \gamma', p)$ and one component of a lower size.*

The basis for the above result is two-fold. First, it is clear that no component prevailing at a CPE can be larger than $\hat{K}+1$, the size that is individually optimal. For, if it were, an obvious coalitional deviation would be available. Moreover, if there is any component of a size lower than $\hat{K}+1$, there can be no more than one. For otherwise a profitable deviation towards a larger component would be feasible for a suitable coalition. In this context, the source of the conflict between social and individual optimality lies in the externality that is imposed by the fact that the adding-up constraint must hold in the aggregate across all components. Therefore, as firms in any component strive for the size that is individually

[16] See Remark 2 in the Appendix.

(or component-wise) optimal, this will generally force other firms to remain in a component that is too small to share risk efficiently.

As a simple illustration of the problem, refer back to Example 1, where the efficient configuration involves two completely connected components of common size equal to $K^* + 1 = 5$, while the individually optimal size for each component is $\hat{K}^c + 1 = 6.65$, the value that minimizes $D_c(K, 2, 0.5, 0.95)$. In contrast the only CPE configuration is asymmetric with two components, one displaying a size of 6.65 and the other a smaller "residual size" of 3.35.

The existence of a conflict between efficiency and strategic stability is of course hardly novel nor surprising in the field of social networks (see e.g. Jackson and Wolinsky (1996) for an early instance of it). For, typically, the creation or destruction of any link between two agents impose externalities on others that are not internalized by the two agents involved in the linking decision. In the context of risk-sharing, this tension has been studied in a recent paper by Bramoullé and Kranton (2007) – hereafter labelled BK – and it is interesting to understand its differences with our approach. We close, therefore, this section with a brief comparison of the two models.

BK consider an environment consisting of a finite number of agents affected by i.i.d. income shocks. Linkages generate risk sharing in a way similar to that of our model, except in two important respects: (a) risk-sharing is complete (i.e. uniform) across all members in a component; (b) there is no risk of contagion, so the size of optimal components is just limited by the fact that links are assumed to be costly. Focusing on the notion of strategic stability (which is weaker than ours),[17] BK are also interested in comparing efficient and equilibrium configurations. They find that whenever equilibrium structures exist (not always), there are at most two asymmetric components, with sizes smaller than the optimal one.

The contrast between BK's conclusions and ours derives from the nature of the externality in the two cases. In BK, given that the cost of any new link is borne only by the two agents involved, the equilibrium induces an underinvestment in link formation (which is a "public good"). Instead, in our case there are no linking costs, so the nature of the externality that is not internalized is quite different. It has to do with the fact that, when firms deviate to reach a larger component of individually optimal size, they do not internalize that the firms that are left behind will be forced to components that are inefficiently small. Thus, in the end, it is the need to meet an overall feasibility constraint imposed by the finite measure of the population that typically generates inefficiencies.

[17] Strategic stability allows only for coalitions of at most two players and rules out as well the simultaneous creation and destruction of links. See Jackson and Wolinsky (1996) for details.

5 Asymmetric environments

So far we have concentrated the discussion on a situation where all firms in the system are identical. Although this allows us to obtain analytical results and gather intuition, it is important to extend our analysis to situations where firms are significantly different in size or in their shock distribution, as often happens in the real world. To this end, we shall simplify the analysis and focus our attention in the next subsection on the case where structures are completely connected. In the following subsection we also allow for some asymmetry in network structures. Under these circumstances, there is no real gain in using the continuum approximation of the patterns of firms' exposure and thus we revert to the original specification, where firms are conceived as discrete entities (which can be of different types and sizes, as we explain below).

5.1 Shock and size asymmetry

To begin with, consider the case where all firms have equal resources R in the absence of shocks but they differ in the distribution of the shocks that may hit their project's returns. More precisely, suppose that the N firms are partitioned into subsets $N_1, ...N_n$, so that $N = \cup_{l=1}^{n} N_l$, and for every firm $j \in N_l$ the big shock follows a distribution F_{N_l} with common expected value for all $l \in \{1, ..., n\}$.

For each $l = 1, .., n$ we denote by $D(K, \Phi_{N_l})$ the expected number of firms who default in a (complete) component of size $K + 1$ when indirectly hit by a b shock hitting a firm j belonging to subset N_l . Note that, since all firms are of equal size, each firm has the same capacity of absorbing shocks. This is what allows us to have the function $D(\cdot)$ depend only on the size of the component (as determined by K) and the distribution Φ_{N_l} of the shock hitting the firm, and not also of the types of other firms in the component. In particular, each firm retains a fraction α of its own project and acquires the same exposure $(1 - \alpha)/K$ to shocks hitting any other firm in the component[18]. We have so:

$$D(K, \Phi_{N_l}) = K \left[1 - \Phi_{N_l} \left(\frac{K}{1 - \alpha} \right) \right]. \tag{25}$$

Let $K^*_{N_l}$ be the minimizer in K of this function.

We are interested in identifying the optimal segmentation structure, which now requires determining not only the number of firms that should lie in each component but also the composition of each component in terms of the different types of firms included in it. One important concern will be to understand whether firm matching within components should be assortative or dissortative – that is, whether components should be more or less homogeneous. In this context, therefore, a structure must specify a set of components $\mathcal{C} \equiv \{C_i\}_{i=1,...,m}$ and, for each component i, the corresponding type distribution

[18] This is a consequence of the fact that, even though the risks are different, the exchange of assets is still one for one for every pair of firms.

$\{N_l^i\}_{i=1,\ldots,n}$, where $C_i = \cup_{l=1}^n N_l^i$ and each N_l^i stands for the subset of firms in N_l that belong to component i.

Proposition 7 *Suppose that, for every l, the cardinality $|N_l|$ of the set of firms of type l is a multiple of $K_{N_l}^* + 1$. Then, the optimal (completely connected) structure has all firms belonging to homogeneous components, where firms are all of the same type, and $K_{N_l}^* + 1$ is the common size of every component consisting of firms of any given type l.*

The intuition for this result is as follows. For each l, conditional on the b shock hitting a firm of type l, the optimal arrangement is to have a homogeneous component with $K_{N_l}^* + 1$ firms all of type l, as $K_{N_l}^*$ is the value that minimizes the function $D(K, \Phi_{N_l})$ given in (25). If instead the system included some heterogeneous component with firms of type j, k, its size could not be set to equal *simultaneously* the (generally different) optimal sizes for the cases of shocks hitting type j and type k firms. Such a configuration could then not be optimal if a segmentation in optimal homogeneous components is feasible. Of course, a key feature required for this conclusion to hold is that, as already noticed, the expected number of indirect defaults in a component only depends on the type l of the firm directly hit by the shock and the size of the component. That is, the indirect effect of the shock does *not* depend on the distribution of types among the firms in the component not directly hit by the shock.

Next, we turn our attention to the case where firms are heterogeneous in size. In light of the previous result, which establishes the optimality of extreme assortativity by distributional types, we now concentrate on environments where all firms are of the same type as far as the distribution of the shock hitting them is concerned. For simplicity, let us consider a situation with just two possible sizes. On the one hand, there are firms of unit size, identical to the ones we have been considering so far. On the other hand, there are firms of size $\beta > 1$, such larger size having the following two implications. First, the return on the projects of these larger firms when no shock affects them is βR, so it is scaled up by β compared to those of the smaller firms. (Naturally, we also assume that the same factor β applies to the value of their liabilities.) Second, the larger firms are supposed to face a probability of being directly hit by a shock that is β times larger (that is, equal to βq) Hence, in this case, we make an assumption that is polar to the case considered before (and thus leads to a complementary analysis): size affects the probability of arrival of the shocks, *not* their distribution. In a sense, therefore, one can view a large firm as a full merge of β small firms of the sort we have been considering so far.

Of course, in order to maintain equal values in trade, a firm of size β must exchange assets with a firm of size 1 in a proportion $1/\beta$ to 1. Let us denote by N_1 the overall set of firms of size 1, and by N_β the overall set of firms of size β, while N_1^i and N_β^i stand for the corresponding subsets of small and big firms within a component C_i. The size of this component is thengiven by $K_i + 1 = \left|N_1^i\right| + \beta \left|N_\beta^i\right|$ where, as before, $|\cdot|$ stands for the

cardinality of the set in question. Then the expected number of defaults in the component if the firm hit is small (i.e. of size 1) is:[19]

$$\begin{aligned} D(K_i,1) &= \left(\left|N_1^i\right|-1\right)\Pr\left\{\frac{(1-\alpha)L}{K_i}>1\right\}+\beta\left|N_\beta^i\right|\Pr\left\{\frac{\beta(1-\alpha)L}{K_i}>\beta\right\} \\ &= K_i\Pr\left\{\frac{(1-\alpha)L}{K_i}>1\right\}, \end{aligned} \tag{26}$$

while if the firm hit is large (of size β) the expected number of defaults is

$$\begin{aligned} D(K_i,\beta) &= \left|N_1^i\right|\Pr\left\{\frac{(1-\alpha)L}{K_i+1-\beta}>1\right\}+\beta\left(\left|N_\beta^i\right|-1\right)\Pr\left\{\frac{\beta(1-\alpha)L}{K_i+1-\beta}>\beta\right\} \\ &= (K_i+1-\beta)\Pr\left\{\frac{(1-\alpha)L}{K_i+1-\beta}>1\right\}. \end{aligned}$$

For each $s\in\{1,\beta\}$, define K_s^* as the minimizer in K of $D(K,s)$. The total expected number of defaults in a system divided into m components when a b shock hits a firm is then:

$$\begin{aligned} & \sum_{i=1}^m \frac{K_i+1}{N}\left(\frac{\left|N_1^i\right|}{K_i+1}D(K_i,1)+\frac{\left|N_\beta^i\right|}{K_i+1}D(K_i,\beta)\right). \\ = & \sum_{i=1}^m\left(\frac{\left|N_1^i\right|}{N}D(K_i,1)+\frac{\left|N_\beta^i\right|}{N}D(K_i,\beta)\right) \\ \geq & \sum_{i=1}^m\left(\frac{\left|N_1^i\right|}{N}D(K_1^*,1)+\frac{\left|N_\beta^i\right|}{N}D\left(K_\beta^*,\beta\right)\right) \\ = & D(K_1^*,1)\frac{|N_1|}{N}+D\left(K_\beta^*,\beta\right)\frac{|N_\beta|}{N} \end{aligned}$$

This establishes the following result.

Proposition 8 *Suppose that $|N_1|$ and $|N_\beta|$ are a multiple, respectively, of K_1^*+1 and K_β^*+1. Then the optimal (completely connected) structure is such that all firms are in components with all firms of the same size (either small or large) and the common size of every component consisting of small (resp. large) firms is K_1^*+1 (K_β^*+1).*

It is worth noting that the same result would hold if firm size, 1 or β, instead of scaling up the probability were to scale the size of the b shock hitting firms – that is, if the probability that a firm of size 1 is hit by a shock of magnitude L were the same as the probability that a firm of size β is hit by a shock of size βL. Then, it can be easily checked that,

[19] To derive the expression below note that, as a result of the fact that a small firm exchanges equity at a rate $1/\beta$ with the big firm, each small firm ends up having a share $(1-\alpha)/(K_i+1-\beta)$ of the big firm, while the big firm has a share $(1-\alpha)\beta/K_i$ of each small firm.

even though the counterpart of expressions (26) and (27) would be different, they would still retain the property of being independent of the composition between small and large firms in the component. And, as explained before, this is the key feature that yields the above conclusion, that the optimal configuration involves the segmentation of the population into homogeneous components, with all firms of equal size. The intuition is similar to the one discussed when motivating Proposition 7: allowing for heterogeneous components would mean failing to take advantage of the generally one-to-one correspondence between component size and type – be it associated to size or shock distribution – of the firm hit by the shock, that assures optimality.

5.2 Asymmetry in structure

Once we allow for different sizes of firms, we may also examine new types of structures in which large and small firms may play asymmetric roles. The key question we now study is whether it is optimal that firms of different size have a different position in the network, and hence a different role in the pattern of risk sharing trades. To this end, our analysis will focus on comparing two structures: (a) one *symmetric*, where components are completely connected and each component involves only firms of identical size; (b) another one asymmetric, where components have a *"star" structure*, consisting of a large firm that acts as a central hub and is directly connected to various small firms.

More precisely, we consider the case where there are only two large firms and 2β small ones. Hence in structure (a) we have two completely connected components of the same total size: one with the two large firms; the other with the 2β small firms. In contrast, in structure (b) we have two identical star components, each consisting of β small firms that are solely connected to a large firm (that is, the large firm acts as a hub and there are β spokes of unit size). Hence in this case each component is heterogeneous and not fully connected.

The first step is to specify the pattern of exchange in the presence of firms of different sizes and hence to determine the induced risk exposure in each of the two cases. In structure (a), the situation is analogous to that of the completely connected structures considered in Subsection 2.2. The exposure pattern can then be described by a matrix A_K of the form specified in (7) for a component of size $K+1$, with $K = 1$ and $K = 2\beta - 1$ for the complete components consisting of large and small firms, respectively.

On the other hand, in a star structure (b) the pattern of mutual exposure obtained in

each component after the direct exchanges of assets is:[20]

$$\widetilde{B} = \begin{pmatrix} \theta & (1-\theta)/\beta & (1-\theta)/\beta & \cdots & (1-\theta)/\beta \\ (1-\theta) & \theta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\theta) & 0 & 0 & \cdots & \theta \end{pmatrix}$$

The entries of the matrix $\widetilde{B}$ reflect the fact the exchange of assets among firms of different size is no longer one for one: the large firm (indexed by $j = 1$) must offer only a share $(1-\theta)/\beta$ of its assets for a larger share $(1-\theta)$ in the assets of small firms (indexed by $j = 2, 3, ..., \beta + 1$). Maintaining the feature that trades are iterated a minimal number of times so that each firm has a non-zero exposure to any other firm in the component, we need here one round of securitization so that the pattern of exposure among firms after both direct and indirect trades is $\widetilde{A} = \widetilde{B}^2$, or

$$\widetilde{A} = \begin{pmatrix} \theta^2 + (1-\theta)^2 & 2\theta(1-\theta)/\beta & (1-\theta)/\beta & \cdots & (1-\theta)/\beta \\ 2\theta(1-\theta) & \theta^2 + (1-\theta)^2/\beta & (1-\theta)^2/\beta & \cdots & (1-\theta)^2/\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\theta(1-\theta) & (1-\theta)^2/\beta & (1-\theta)^2/\beta & \cdots & \theta^2 + (1-\theta)^2/\beta \end{pmatrix}.$$

From the above expression for $\widetilde{A}$ we see that, for each value of θ, the fraction of claims held by each small firm on its own project:

$$\alpha = \theta^2 + (1-\theta)^2/\beta, \tag{27}$$

is smaller than the analogous fraction held by the large firm:

$$\alpha' = \theta^2 + (1-\theta)^2. \tag{28}$$

Naturally, the difference $\alpha' - \alpha > 0$ grows larger with β. This is simply a reflection of the fact that the higher is the size asymmetry between large and small firms the lower the share of its own project that the large firm needs to trade for any given share on the project of a small firm. As in Subsection 2.2, the value of θ will be set so that each firm maximizes the extent of risk sharing consistent with *every* firm in the system (large or small) retaining at least a share of 1/2 on the yields of its own project. From (27) and (28), it is clear that this happens when $\alpha = 1/2$ and $\alpha' = 1/2 + (1-\theta)\frac{\beta-1}{\beta}$.

As in Section 2.1, any firm i defaults when a shock L hits the project of firm j in the same component (possibly $i = j$) and the firm's exposure to the shock, $\tilde{a}_{ij}L$, exceeds the firm's net returns (here $\tilde{a}_{ij}$ denotes the ij entry of $\widetilde{A}$). This now amounts to a different condition if the firm under consideration is big or small, since each firm has a different "return buffer": if i is small, default occurs when $\tilde{a}_{ij}L > 1$; instead, if it is large the condition is $\tilde{a}_{ij}L > \beta$.

The next result compares the performance of structures (a) and (b), for different values L of the magnitude of the b shock hitting a randomly selected firm:

[20] We dispense here for notational simplicity with any reference to the component size ($K = \beta + 1$).

Proposition 9 *Assume $\beta > 2$. Then, if*

$$\frac{1}{1-\alpha'} < L \leq 2\beta, \tag{29}$$

expected defaults are lower in structure (a) with homogeneous components than in structure (b) with star components, while if

$$2(2\beta - 1) < L \leq \max\left\{\frac{\beta - 1}{\alpha' - 1/2}, \frac{\beta^2}{1-\alpha'}\right\} \tag{30}$$

the opposite conclusion holds.

First, the superiority of the symmetric (completely connected) structure in the range given by (29) is easy to explain. In the star structure a large firm (the hub) engages in risk sharing trades with a set of smaller firms (the spokes). This ends up limiting the possibilities of risk sharing when a shock hits a large firm because, in order to match the lower value of the assets of the smaller firms, the large firm is forced to hold more of its own assets than in the symmetric structure. As a consequence, when a shock of an intermediate size as in (29) hits a large firm, it triggers the default of the small firms in the star structure as well as, possibly, that of the large firm. Instead such a shock hitting the big firm would be absorbed with no defaults in a symmetric structure, due to the enhanced risk sharing possibilities it would enjoy when connected to a firm of the same size.

On the other hand, it is precisely this limitation to the risk sharing possibilities in a star structure that protects the hub as well as the other spokes against the shocks hitting a small firm. As a consequence, the star structure performs better than the symmetric structure for shocks of larger size, as in (30). In a symmetric component, these shocks are so large that they trigger the default of all the firms linked to a firm directly hit by the shock. In contrast, in a star component none of these firms defaults when the shock hits one of the spoke firms – only the small firm hit defaults. In this case, therefore, the large firm acts a sort of buffer, preventing a sizeable fraction of the shock to spread and cause further defaults in the component.

6 Conclusion

We have proposed a stylized model to study the problem that arises when firms need to share resources to weather shocks that can threaten their survival, but then are exposed to the risk coming from those same connections that help them in the time of need. Depending on the characteristics of the shock distribution, a wide variety of different configurations can be optimal. For example, maximal segmentation in small groups is optimal if big shocks are likely, while very large groups are optimal when most shocks are of moderate magnitude. There are also conditions, however, when an intermediate group size is optimal or when groups should be large but display some internal "detachment" (i.e. sparse connectivity).

The former consideration pertains to social optimality, i.e. to the minimization of the default rate within the whole system. We have also explored whether such social optimum is aligned with individual optimality. And we have seen that, in general, there is a conflict between strategic incentives and social welfare. This tension arises from the fact that firms have always an incentive to form connected components of the size that minimizes the default probability of their members, thus ignoring the negative externality this behavior imposes on other firms. Finally, we have found that when heterogeneity among firms is allowed but network components are taken to be fully and symmetrically connected, optimality is achieved under perfect assortativity. But if the network structure can display some asymmetries (e.g. a "large central" agent may act as a hub), it is optimal to take advantage of this possibility as a *firebreak* under certain conditions.

The present analysis represents a first step in studying the welfare implications of alternative risk-sharing network structures, as well as their (in)consistency with individual incentives. By restricting to suitable families of network structures, we have arrived at clear-cut conclusions as to what kind of segmentation, sparseness, or asymmetries in connections attain optimality. The basic insights obtained from this analysis could inform the regulation of financial systems, which are indeed subject to some of the key mechanisms contemplated by our theoretical framework. Should one separate commercial and investment banking? Should the financial systems of different regions be insulated from each other, or should overall integration be pursued? Is it advisable to allow for some large central institutions to have a buffer role in mitigating the shocks affecting smaller ones? And, in the end, how effectively can we trust any prescriptions along these lines to be indeed impl emented by the economic agents themselves (i.e. to be compatible with their own individual incentives)? These are some of the questions that will naturally and crucially arise in the design and regulation of financial systems in the real world.

Admittedly, however, our approach is too stylized to be *directly* useful for policy advise. But we hope that it helps to highlight some of the main considerations that should underlie a systematic analysis of those policy issues. Of course, a proper discussion of the such risk-sharing and contagion phenomena in the real world would have to account for many other important dimensions of the problem. One of these is the wide and intricate heterogeneity that abounds everywhere. For the sake of tractability, our analysis restricts to regular or otherwise schematic structures that are far from what we typically observe in real-world contexts. And, a priori, it seems likely that some of the insights and recommendations applicable to those contexts should depend on such rich heterogeneity. Another need will be to integrate the risk-management decisions studied here with other considerations (e.g. inter-agent cooperation, or the exploitation of synergies) that also underlie e conomic connections in the real world. All of this will demand a richer theoretical framework and a more powerful methodology, the development of which can hopefully build in a fruitful manner upon the present effort.

Appendix

Proof of Lemma 1: Rearranging terms in (21) and simplifying we get:

$$D_r(K,\gamma) = \left(K\left(\frac{1}{\gamma+1}\right) - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma} + \frac{1}{K-1}\frac{1}{(\gamma+1)}\left(\frac{1}{2}\right)^{\gamma-1}, \quad (31)$$

and hence

$$\begin{aligned}\tfrac{\partial D_r}{\partial K}(K,\gamma) &= -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\ &+ \left(K\left(\tfrac{-\gamma}{\gamma+1}\right)\tfrac{1}{K+1} + \tfrac{2}{K-1}\tfrac{1}{K+1}\tfrac{\gamma}{\gamma+1} + \left(\tfrac{1}{\gamma+1}\right) + \tfrac{2}{(K-1)^2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{K+1}\right)^{\gamma}.\end{aligned} \quad (32)$$

Now note that the inequality $\partial D_r(K,\gamma)/\partial K > 0$ is equivalent to:

$$\frac{2(K-1)}{K+1}\gamma + (K-1)^2 + 2 > (K+1)\left(\frac{K+1}{2}\right)^{\gamma-1} + \gamma K\frac{(K-1)^2}{K+1},$$

or

$$(K-1)^2\left(1-\frac{\gamma K}{K+1}\right) + 2\left(1+\gamma\frac{K-1}{K+1}\right) > (K+1)^{\gamma}\frac{1}{2^{\gamma-1}},$$

which can be rewritten as

$$(K-1)^2 + 2 + \gamma(K-1)(2-K) > (K+1)^{\gamma}\frac{1}{2^{\gamma-1}}. \quad (33)$$

So, using the identities

$$(K-1)^2 + 2 + (K-1)(2-K) = K+1$$

and

$$(K+1)^{\gamma}\frac{1}{2^{\gamma-1}} = 2\left(\frac{K+1}{2}\right)^{\gamma}$$

we can equivalently write (33) as follows:

$$\frac{K+1}{2} - \left(\frac{K+1}{2}\right)^{\gamma} - \frac{1}{2}(1-\gamma)(K-1)(2-K) > 0.$$

Denote by $\Xi(K,\gamma)$ the term on the left hand side of the previous inequality, conceived as a function of K and γ. Then, to complete the proof, we establish the following property:

$$\forall K > 1, \quad \Xi(K,\gamma) \gtrless 0 \Leftrightarrow \gamma \lessgtr 1. \quad (34)$$

To show this property, note first that $\Xi(K,1) = 0$ for all K, so that $\frac{\partial D_r}{\partial K}(K,\gamma) = 0$ for $\gamma = 1$ and all K. On the other hand,

$$\begin{aligned}\frac{\partial \Xi}{\partial \gamma}(K,\gamma) &= -\left(\frac{K+1}{2}\right)^{\gamma}\ln\frac{K+1}{2} + \frac{1}{2}(K-1)(2-K) \\ &\leq -\ln\frac{K+1}{2} + \frac{1}{2}(K-1)(2-K),\end{aligned}$$

the inequality being strict for all $K > 1$. It is then easy to verify that the terms on the two sides of the above inequality are equal to 0 when $K = 1$ and the term on the right hand side is negative[21] for all $K > 1$, establishing (34) and hence also $\partial D_r(K,\gamma)/\partial K \gtrless 0 \Leftrightarrow \gamma \lesseqgtr 1$.

We conclude, as stated in the proposition, that the minimum of $D_r(K,\gamma)$ is attained at the maximum admissible value of K (i.e. $N-1$) when $\gamma > 1$, while it is attained at the lowest value of K (i.e. $K = 1$)[22] when $\gamma < 1$. This completes the proof of the proposition. ■

Proof of Lemma 2 Note first that the expression of $\partial D_r(K,\gamma)/\partial K$ obtained in (31) can be conveniently rewritten as follows:

$$\begin{aligned}
&\tfrac{\partial D_r}{\partial K}(K,\gamma) = -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\
&+\left(\tfrac{-K^2+K+2}{K-1}\tfrac{1}{K+1}\tfrac{\gamma}{\gamma+1}+\tfrac{1}{\gamma+1}+\tfrac{2}{(K-1)^2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{K+1}\right)^{\gamma} \\
&= -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\
&+\tfrac{1}{\gamma+1}\tfrac{K^2-2K+3}{(K-1)^2}\left(\tfrac{1}{K+1}\right)^{\gamma}-\tfrac{\gamma}{K+1}\left(K^2-K-2\right)\tfrac{1}{K-1}\left(\tfrac{1}{K+1}\right)^{\gamma}
\end{aligned}$$

Differentiating then again with respect to K yields:

$$\begin{aligned}
\frac{\partial^2 D_r(1/2,K,\gamma)}{\partial K^2} &= 2\frac{\left(\frac{1}{2}\right)^{\gamma-1}}{(\gamma+1)(K-1)^3}-\frac{1}{\gamma+1}\frac{\left(\frac{1}{K+1}\right)^{\gamma}}{(K-1)^3(K+1)}\times \\
&\quad \left(K^3\left(-\gamma^2+\gamma\right)+2K^2\left(2\gamma^2-\gamma\right)+5K\left(-\gamma^2+\gamma\right)+4K+2(\gamma-1)^2+2\right) \\
&= \frac{1}{2^{\gamma}(\gamma+1)(K-1)^3}\left(4-\frac{2^{\gamma}}{(K+1)^{\gamma+1}}\left((K-1)^2K\left(\gamma-\gamma^2\right)\right.\right. \\
&\quad \left.\left.+2\gamma^2(K-1)^2+4\gamma(K-1)+4(K+1)\right)\right)
\end{aligned}$$

Hence

$$\frac{\partial^2 D_r(1/2,K,\gamma)}{\partial K^2} < 0$$

if and only if

$$G(K) \equiv \frac{\left(\frac{K+1}{2}\right)^{\gamma+1}}{\frac{(K-1)^2}{8}\left(K\left(\gamma-\gamma^2\right)+2\gamma^2\right)+\gamma\frac{(K-1)}{2}+\frac{(K+1)}{2}} < 1 \tag{35}$$

[21] We have in fact

$$\frac{d\left(-\ln\frac{K+1}{2}+\frac{1}{2}(K-1)(2-K)\right)}{dK} = \frac{K+1-2K^2}{K+1} < \frac{2K(1-K)}{K+1} < 0$$

[22] To complete the argument we verify the claimed continuity property of $D_r(K,\gamma)$, as in (31), at $K = 1$:

$$\begin{aligned}
&\lim_{K\to 1} D_r(K,\gamma) \\
&= \left(\frac{1}{2}\right)^{\gamma}\frac{1}{\gamma+1}-\lim_{K\to 1}\frac{1}{K-1}\frac{2}{\gamma+1}\left[\left(\frac{1}{K+1}\right)^{\gamma}-\left(\frac{1}{2}\right)^{\gamma}\right] \\
&= \left(\frac{1}{2}\right)^{\gamma}\left(\frac{1}{\gamma+1}\right)-\frac{-\gamma 2^{\gamma}}{(\gamma+1)4^{\gamma}} = \left(\frac{1}{2}\right)^{\gamma} = D_r(1,\gamma)
\end{aligned}$$

First, we observe that $G(1) = 1$. Thus, to establish (35), it is enough to show that G is decreasing for all $K > 1$. Letting $x \equiv K - 1$ for notational simplicity, $\frac{dG(K)}{dK} < 0$ if, and only if,

$$\frac{\frac{d}{dx}\left(\left(\frac{x}{2}+1\right)^{\gamma+1}\right)}{\left(\frac{x}{2}+1\right)^{\gamma+1}} < \frac{\frac{d}{dx}\left(\gamma x\left(\frac{x}{8}\left(x\left(1-\gamma\right)+1+\gamma\right)+\frac{1}{2}\right)+\frac{x}{2}+1\right)}{\left(\gamma x\left(\frac{x}{8}\left(x\left(1-\gamma\right)+1+\gamma\right)+\frac{1}{2}\right)+\frac{x}{2}+1\right)},$$

or:

$$\frac{\frac{\gamma+1}{2}\left(\frac{x}{2}+1\right)^{\gamma}}{\left(\frac{x}{2}+1\right)^{\gamma+1}} = \frac{\gamma+1}{x+2} < \frac{\frac{1}{2}\gamma+\frac{1}{4}x\gamma+\frac{1}{4}x\gamma^2+\frac{3}{8}x^2\gamma-\frac{3}{8}x^2\gamma^2+\frac{1}{2}}{\frac{x}{2}+\frac{1}{2}x\gamma+\frac{1}{8}x^2\gamma+\frac{1}{8}x^3\gamma+\frac{1}{8}x^2\gamma^2-\frac{1}{8}x^3\gamma^2+\frac{1}{2}}$$

The above inequality is equivalent to the following one:

$$\left(\gamma+\frac{1}{2}x\gamma+\frac{1}{2}x\gamma^2+\frac{3}{4}x^2\gamma-\frac{3}{4}x^2\gamma^2+1\right)(x+2) > (\gamma+1)\left(x+\gamma x+\frac{1}{4}x^2\gamma+\frac{1}{4}x^3\gamma+\frac{1}{4}x^2\gamma^2-\frac{1}{4}x^3\gamma^2+1\right),$$

or

$$\begin{array}{c}\gamma x+\frac{1}{2}x^2\gamma+\frac{1}{2}x^2\gamma^2+\frac{3}{4}x^3\gamma-\frac{3}{4}x^3\gamma^2+x \\ +2\gamma+x\gamma+x\gamma^2+\frac{3}{2}x^2\gamma-\frac{3}{2}x^2\gamma^2+2\end{array} > \begin{array}{c}x+\gamma x+\frac{1}{4}x^2\gamma+\frac{1}{4}x^3\gamma+\frac{1}{4}x^2\gamma^2-\frac{1}{4}x^3\gamma^2+1+ \\ \gamma x+\gamma^2 x+\frac{1}{4}x^2\gamma^2+\frac{1}{4}x^3\gamma^2+\frac{1}{4}x^2\gamma^3-\frac{1}{4}x^3\gamma^3+\gamma\end{array},$$

or

$$\frac{7}{4}x^2\gamma+\frac{1}{2}x^3\gamma+\gamma+1+\frac{1}{4}x^3\gamma^3 > \frac{1}{4}x^2\gamma^3+\frac{3}{4}x^3\gamma^2+\frac{3}{2}x^2\gamma^2.$$

That is

$$\frac{x^2\gamma}{2}\left(\frac{7}{2}-\frac{\gamma^2}{2}-3\gamma\right)+\frac{1}{4}x^3\gamma\left(2+\gamma^2-3\gamma\right)+\gamma+1 > 0.$$

the above inequality being always true if $\gamma < 1$, which completes the proof. ■

Proof of Proposition 3: From (22) and (31) we get:

$$\begin{aligned} & D_c(K,\gamma) - D_r(K,\gamma) \qquad (36) \\ = \; & \left(\frac{1}{2}\right)^{\gamma}\left(\frac{1}{K}\right)^{\gamma-1} - K\left(\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma} + \frac{2}{K-1}\frac{1}{\gamma+1}\left(\left(\frac{1}{K+1}\right)^{\gamma}-\left(\frac{1}{2}\right)^{\gamma}\right)\end{aligned}$$

As shown in Propositions 1 and 2, when $\gamma < 1$ and N is even, the optimal structure both for the ring and the completely connected structures has all components of size $K + 1 = 2$. As we noticed, when $K = 1$ the pattern of exposure is identical for the ring and the completely connected structure, hence the value of the above expression equals zero in that case, as can be verified.[23]

Consider now the case $\gamma > 1$, for which $K = N - 1$ (i.e. minimal segmentation) is optimal for both structures. Evaluating (36) at this value of K we find:

[23] Strictly speaking, we can show that that its limit for $K \to 1$ equals zero.

$$D_c(N-1,\gamma) - D_r(N-1,\gamma) =$$
$$= \left[\left(\frac{1}{N}\right)^{\gamma} - \left(\frac{1}{2}\right)^{\gamma}\right]\left[\frac{2}{N-2}\frac{1}{\gamma+1} - \frac{N-1}{1+\gamma}\right] + \left(\frac{1}{2}\right)^{\gamma}\left[\left(\frac{1}{N}\right)^{\gamma-1} - \frac{N-1}{1+\gamma}\right]$$

Since $2 \leq (N-1)(N-2)$ for $N \geq 3$, we have that for all[24] $N > 1 + (1+\gamma)^{\frac{1}{\gamma}}$ the desired conclusion follows:

$$D_c(K,\gamma) - D_r(K,\gamma) < 0.$$

This completes the proof. ■

Proof of Proposition 4: From (22), we can write:

$$D_c(K,\gamma,\gamma',p) = p\,K\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,K\left(\frac{1}{2K}\right)^{\gamma'}$$

Hence

$$\frac{\partial D_c}{\partial K}(K,\gamma,\gamma',p) = -p\,(\gamma-1)\left(\frac{1}{2K}\right)^{\gamma} - (1-p)\,(\gamma'-1)\left(\frac{1}{2K}\right)^{\gamma'}. \tag{37}$$

and $\frac{\partial D_c}{\partial K} > 0$ is equivalent to

$$(1-p)\,(1-\gamma')\left(\frac{1}{2K}\right)^{\gamma'} > p\,(\gamma-1)\left(\frac{1}{2K}\right)^{\gamma},$$

or, since $\gamma > 1$ and $\gamma' < 1$,

$$K > \frac{1}{2}\left(\frac{p\,(\gamma-1)}{(1-p)\,(1-\gamma')}\right)^{\frac{1}{\gamma-\gamma'}}.$$

This implies that $D_c(K,\gamma,\gamma',p)$ is minimized at the point

$$\hat{K}(p) = \frac{1}{2}\left(\frac{p\,(\gamma-1)}{(1-p)\,(1-\gamma')}\right)^{\frac{1}{\gamma-\gamma'}}$$

provided this point is admissible, i.e. $\hat{K}(p) \in [1, N-1]$.

Compute next the second derivative of $D_c(\cdot)$:

$$\begin{aligned}\frac{\partial^2 D_c}{\partial K^2}(K,\gamma,\gamma',p) &= p\,(\gamma-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,(\gamma'-1)\,\frac{\gamma'}{K}\left(\frac{1}{2K}\right)^{\gamma'} \\ &\geq p\,(\gamma-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,(\gamma'-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma'} \\ &= -\frac{\gamma}{K}\frac{\partial D_c}{\partial K}(K,\gamma,\gamma',p)\end{aligned}$$

Thus $\frac{\partial^2 D_c}{\partial K^2}(1/2,K,\gamma,\gamma',p) > 0$ for all feasible $K < \hat{K}(p)$, i.e. the function $D_c(\cdot)$ is convex in this range.

[24] Note that $(1+\gamma) < (N-1)^{\gamma-1}(N-1) < (N)^{\gamma-1}(N-1)$

The optimal degree of segmentation for the completely connected structure is obtained as a solution of problem 20. Denote by $(K_i^*)_{i=1}^C$ a vector of component sizes that solves this optimization problem. We will show that there exists some appropriate range $[p_0, p_1]$ such that if $p \in [p_0, p_1]$, the optimal component sizes are such that $K_i^* = K_j^* = K^*$ for all $i, j = 1, 2, \ldots, C$ and some common K^* with $2 \leq K^* \leq N - 2$.

Choose p_0 such that $\hat{K}(p_0) = \frac{N}{2} - 1$. Such a choice is feasible and unique since by A.3 $N > 4$, $\hat{K}(\cdot)$ is increasing in p, $\hat{K}(0) = 0$, and $\hat{K}(p) \to \infty$ as $p \to 1$. Next we show that, for all $p \geq p_0$, whenever $C \geq 2$,the vector $(K_i^*)_{i=1}^C$ solving problem 20 satisfies:

$$\forall i, j = 1, 2, \ldots, C, \quad K_i^* = K_j^* \leq \hat{K}(p) \tag{38}$$

Let K_i^* and K_j^* stand for any two component sizes that are part of the solution to the optimization problem. First note that, since $\hat{K}(p) \geq N/2 - 1$, if $K_i^* > \hat{K}(p)$ then we must have that $K_j^* < \hat{K}(p)$. But such asymmetric arrangement cannot be part of a solution to problem 20 because $D_c(\cdot, \gamma, \gamma', p)$ is increasing at K_i^* and decreasing at K_j^*. Hence a sufficiently small increase of K_j and a decrease of K_i, which keeps $K_i + K_j$ unchanged, is feasible and allows to decrease the expected mass of defaults. The only possibility, therefore, is that $K_j^* \leq \hat{K}(p)$ and $K_j^* \leq \hat{K}(p)$.

To complete the argument and establish (38), suppose that at an optimum we have $K_i^* \neq K_j^*$ for at least two components i, j. Since, as shown in the previous paragraph, neither K_i^* nor K_j^* can exceed $\hat{K}(p)$, both K_i^*, K_j^* lie in the convex part of the function $D_c(\cdot, \gamma, \gamma', p)$. It follows, therefore, that if we replace these two (dissimilar) components with two components of equal size $\frac{1}{2}(K_i^* + K_j^*)$, feasibility is still satisfied and the overall expected mass of defaults is reduced, contradicting that the two heterogenous components of size K_i^*, K_j^* belongs to an optimum configuration.

We have thus shown that, when $p \geq p_0$, if at the optimum we have $C \geq 2$, the unique optimal configuration involves a uniform segmentation in components of common size $K^*(p) \leq \hat{K}(p)$. It remains then to show that at the optimum we indeed have $C \geq 2$. At $p = p_0$ the optimum exhibits two components, $C = 2$, since the optimal component size $\hat{K}(p_0) = N/2 - 1$ is feasible. Since $\hat{K}(p)$ is increasing and continuous in p and $D_c(K, \gamma, \gamma', p)$ is continuous in K, by continuity there exists some p_1, with $p_0 < p_1 < 1$, such that for all $p \in (p_0, p_1)$ the expected mass of defaults in a structure with two components, both of size $N/2 - 1$, is still smaller than that in a single component of size N. That is, at the optimum $C \geq 2$.

Since $N/2 - 1 > 1$, this completes the proof that the optimal component size $K^* + 1$ is "intermediate," i.e. satisfies $1 < K^* < N - 1$. ■

Proof of Proposition 5: For the probability distribution of the b shock stated in the claim, the expected mass of firms not directly hit by a b shock who default in a completely connected component of size K when a b shock hits the component is:

$$D_c(K, \gamma, p) = (1 - p)K + pK \left(\frac{1}{2K}\right)^{\gamma}. \tag{39}$$

Differentiating the above expression with respect to K yields:

$$\frac{\partial D_c(K,\gamma,p)}{\partial K} = (1-p) - (\gamma-1)\,p\,\left(\frac{1}{2K}\right)^{\gamma},$$

which is negative for all K as long as (24) is satisfied. This establishes that the optimal degree of segmentation for the complete structure is minimal, that is obtains at $K = N-1$.

Next, using (31) and (18), noting that $\bar{L} > \frac{1}{H} = K+1$ for all K, we obtain the following expression for the expected mass of defaults in the case of the ring structure:

$$\begin{aligned} D_r(K,\gamma,p) &= (1-p)\left(K - \left(K - \frac{2}{K+1}\right)\frac{K+1}{\bar{L}}\right) \\ &+ p\left[\left(\frac{K}{\gamma+1} - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}\right] + p\left[\frac{1}{K-1}\frac{1}{2^{\gamma-1}}\frac{1}{\gamma+1}\right] \end{aligned}$$

It suffices then to show that the expected mass of defaults is smaller for the ring than for the completely connected structure when $K = N-1$: $D_c(N-1,\gamma,p) > D_r(N-1,\gamma,p)$ or, substituting the above expressions:

$$\begin{aligned} (1-p)\,(N-1) + p\,(N-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma} &> (1-p)\left(N-1-\tfrac{N^2-N-2}{2N-1}\right) + \\ p\left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} &+ p\left[\tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1}\right], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &\left(\tfrac{1-p}{p}\right)\tfrac{N^2-N-2}{2N-1} > \\ &\left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + \tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1} - (N-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma}. \end{aligned}$$

Using (24) the above inequality holds for an open interval of values of p if

$$\begin{aligned} &(\gamma-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma}\tfrac{N^2-N-2}{2N-1} > \\ &\left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + \tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1} - (N-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma} \end{aligned}$$

or

$$\begin{aligned} &(\gamma-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma} 2\left(\tfrac{(N-1)^2+N-3}{2(2(N-1)+1)}1\right) + (N-1)\ \left(\tfrac{1}{2(N-1)}\right)^{\gamma} \\ &-2\left(\tfrac{(N-1)}{2}\left(\tfrac{1}{\gamma+1}\right) - \tfrac{1}{\gamma+1}\left(\tfrac{1}{N-2}\right)\right)\left(\tfrac{1}{N}\right)^{\gamma} - \left(\tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{(\gamma+1)}\right) > 0 \end{aligned}$$

Noticing that by A.3 and (23) we have $N \geq 5$ and this in turn implies

$$\frac{(N-1)^2+N-3}{4\,(N-1)+2} \geq \frac{N-1}{4},$$

a sufficient condition for the above inequality to hold is that:

$$\begin{aligned} &(\gamma-1)\ \left(\frac{1}{2\,(N-1)}\right)^{\gamma} 2\left(\frac{N-1}{4}\right) + \frac{2}{\gamma+1}\frac{1}{N-2}\left(\frac{1}{N}\right)^{\gamma} \\ &+(N-1)\ \left(\frac{1}{2\,(N-1)}\right)^{\gamma} - \left(\frac{N-1}{\gamma+1}\right)\left(\frac{1}{N-1}\right)^{\gamma} - \left(\frac{1}{N-2}\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) \\ = \ &\frac{N-2}{(N-1)^{\gamma-1}}\left((\gamma-1)\ \frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma}} - \frac{1}{\gamma+1}\right) + \frac{2}{\gamma+1}\left(\frac{1}{N}\right)^{\gamma} - \left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) > 0. \end{aligned}$$

Since $\gamma \in (1,2)$, this inequality is in turn satisfied if the following hold:

$$\left[\frac{N-2}{(N-1)^{\gamma-1}} + \left(\frac{1}{N}\right)^{\gamma}\right]\left(\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}\right) - \left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) > 0$$

or

$$(N-1)^{2-\gamma} - \left(\frac{1}{(N-1)^{\gamma-1}} - \frac{1}{N^{\gamma}}\right) > \frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}}$$

which is implied by the inequality

$$(N-1)^{2-\gamma} - \left(\frac{1}{4^{\gamma-1}} - \frac{1}{5^{\gamma}}\right) > \frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}}$$

that is in turn equivalent to (23). This completes the proof of the proposition. ■

Proof of Proposition 6: Let $\mathbf{S} \equiv (S_i)_{i\in N}$ be a CPE of the network-formation game with completely connected components. Denote by C^S the number and by $\{K_j+1\}_{j=1}^{C^S}$ the sizes of the different components of the CPE network $\Gamma(\mathbf{S})$. Recall that $\hat{K}$ denotes the unique value that minimizes $D_c(K,\gamma,\gamma',p)$ (c.f. the proof of Proposition 4).

The proof has two main steps. We show first that for each component $j = 1,..,C^S$ we have $K_j \leq \hat{K}$. That is, all components in the CPE have size smaller or equal than the individually optimal one. Suppose not: $K_j > \hat{K}$ for some j. Choose then a subset $\mathcal{D}$ of the firms belonging to component j whose measure satisfies $K_j - \hat{K} \geq |\mathcal{D}| > 0$. Since the function $D_c(K,\gamma,\gamma',p)$ is increasing in K for $K > \hat{K}$, $D_c(K_q - |\mathcal{D}|,\gamma,\gamma',p) < D_c(K_q,\gamma,\gamma',p)$. This implies that all the firms who are in component j except those in $\mathcal{D}$ could enjoy a lower default probability by proposing a component with all the firms in j except those with the firms in $\mathcal{D}$. This establishes the desired conclusion.

Next, we show that at a CPE we have $K_{\widetilde{h}} < \hat{K}$ for some $\widetilde{h} \in \{1,2,...,C^S\}$ and $K_j = \hat{K}$ for all $j \neq \widetilde{h}$. We proceed again by contradiction. Suppose that there are two components of $\Gamma(\mathbf{S})$, j and q, such that $0 < K_j < K_q < \hat{K}$. Pick then a subset of firms belonging to the first component: $\mathcal{D}_j \subset K_j$, of measure $|\mathcal{D}_j| \leq \hat{K} - K_q$. Consider the following joint deviation for the firms in $\mathcal{D}_j$ as well as for all those in component q from their strategies in $\mathbf{S}$. Each of the firms in $\mathcal{D}_j$ together with the firms in component q propose a component consisting of all of them (and only them). Then, we have:

$$D_c(K_q + |\mathcal{D}_j|,\gamma,\gamma',p) < D_c(K_q,\gamma,\gamma',p) < D_c(K_j,\gamma,\gamma',p)$$

Hence all firms involved benefit from the deviation, which contradicts the fact that at a CPE we have $0 < K_j < K_q < \hat{K}$. This contradiction establishes the above claim and completes so the proof of the Proposition. ■

Remark 2 *Note that the deviations contemplated in the proof of Proposition 6 are "internally consistent" in the following sense. Given that any set of firms of the required composition is set*

to deviate, this deviation is itself in the interest of any subset of this set that might reconsider the situation. This requirement (which is commonly demanded in the game-theoretic literature for coalition-based notions of equilibrium), is clearly satisfied in our case since, by refusing to follow suit with the deviation, any firm in this subset can only lose, because of the monotonicity of the function $D_c(.)$.

Proof of Proposition 7: If a component i is hit a by a b shock, the expected number of defaults among firms not directly hit by it is:

$$\sum_{l=1}^{n} \frac{\left|N_l^i\right|}{K_i+1} D\left(K_i, \Phi_{N_l}\right),$$

where K_i+1 is as usual the size of component i, in this case given by $\sum_{l=1}^{n}\left|N_l^i\right|$. Then the total expected number of defaults in the system when a b shock hits a firm is

$$\begin{aligned} & \sum_{i=1}^{I} \frac{K_i+1}{N} \sum_{l=1}^{n} \frac{\left|N_l^i\right|}{K_i+1} D\left(K_i, \Phi_{N_l}\right). \\ = & \sum_{i=1}^{I} \sum_{l=1}^{n} \frac{\left|N_l^i\right|}{N} D\left(K_i, \Phi_{N_l}\right). \end{aligned}$$

Exchanging the summation indices, we get

$$\begin{aligned} \sum_{l=1}^{n} \sum_{i=1}^{I} \frac{\left|N_l^i\right|}{N} D\left(K_i, \Phi_{N_l}\right) & \geq \sum_{l=1}^{n} \sum_{i=1}^{I} \frac{\left|N_l^i\right|}{N} D\left(K_{N_l}^*, \Phi_{N_l}\right) \\ & = \sum_{l=1}^{n} D\left(K_{N_l}^*, \Phi_{N_l}\right) \sum_{G_i=1}^{I} \frac{\left|N_l^i\right|}{N} = \sum_{l=1}^{n} \frac{\left|N_l\right|}{N} D\left(K_{N_l}^*, \Phi_{N_l}\right). \end{aligned}$$

Since $|N_l|$ is a multiple of $K_{N_l}^*+1$ the lower bound

$$\sum_{l=1}^{n} \frac{\left|N_l\right|}{N} D\left(K_{N_l}^*, \Phi_{N_l}\right)$$

is also the expected number of defaults if every firm is part of a group of optimal size $K_{N_l}^*+1$ with all firms of the same type. ■

Proof of Proposition 8: See the main text. ■

Proof of Proposition 9: To prove the result, we compute the expected number of bankruptcies associated to the two structures (star and symmetric) for all possible levels of the b shock.

Noting that $\alpha' - \alpha = (1-\theta)^2(\beta-1)/\beta$, the exposure matrix for the star structures can be conveniently rewritten as follows:

$$\widetilde{A} = \begin{pmatrix} \alpha' & (1-\alpha')/\beta & (1-\alpha')/\beta & \cdots & (1-\alpha')/\beta \\ 1-\alpha' & \alpha & (\alpha'-\alpha)/(\beta-1) & \cdots & (\alpha'-\alpha)/(\beta-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\alpha' & (\alpha'-\alpha)/(\beta-1) & (\alpha'-\alpha)/(\beta-1) & \cdots & \alpha \end{pmatrix},$$

Recall that $\alpha = 1/2$ while α' is determined, together with θ, by (27) and (28). Its properties are characterized below:

Lemma 3 *For all $\beta > 2$, the solution of (27) and (28) is unique and given by continuous, monotonically increasing functions $\theta(\beta)$ and $\alpha'(\beta)$, such that*

$$5/9 \leq \alpha'(\beta) < 2 - \sqrt[2]{2} \tag{40}$$

Proof of Lemma 3:

It can be easily verified that for all $\beta > 2$ there is only one admissible (i.e., lying between 0 and 1) solution of (27), given by

$$\frac{2 + \sqrt{2\beta^2 - 2\beta}}{2\beta + 2}.$$

This expression defines the function $\theta(\beta)$, which is increasing if and only if the following inequality is satisfied:

$$\frac{(4\beta - 2)}{2\sqrt{2\beta^2 - 2\beta}}(2\beta + 2) > \left(4 + 2\sqrt{2\beta^2 - 2\beta}\right),$$

which is equivalent to

$$2\beta^2 + \beta - 1 > 2\sqrt{2\beta^2 - 2\beta} + 2\beta^2 - 2\beta$$

or

$$9\beta^2 - 6\beta + 1 > 8\beta^2 - 8\beta$$

always satisfied for $\beta > 2$. The minimal value of θ in this range is then $\theta(2) = 2/3$, while the maximum is $\lim_{\beta\to\infty} \theta(\beta) = 1/\sqrt[2]{2}$.

Also, $\alpha'(\beta)$ is also increasing in β

$$\frac{d\alpha'}{d\beta} = 2(2\theta - 1)\frac{d\theta}{d\beta}.$$

Hence its minimum value is $\alpha'(\beta) = 5/9$ and its maximum is $2 - \sqrt[2]{2}$. $\square$

Denote by $G_{star}(L)$ and $G_{sym}(L)$ the functions that specify the total size of all[25] the firms who default, resp. for the star and the symmetric structure, as a function of the magnitude L of the b shock.

[25]Note that, in contrast to the functions $g_\nu(L; K, M)$, $\nu = c, r$, introduced in Section 2.2, the functions $G_{star}(\cdot)$, $G_{sym}(\cdot)$ describe the number of *all firms* who default, that is including the firm that is directly hit by the shock L. Since the degree of risk externalization, as we noticed, may now differ across different structures, so does the probability that a firm directly hit by a shock defaults.

Let us begin by determining the total size of firms defaulting if the shock hits a *small firm* and the structure of the component to which it belongs is a star. For any given value of L, it is given by the following function[26]:

$$G^s_{star}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} = 2 \\ 1 & \text{for } 2 < L \leq \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ \beta \text{ if } \frac{\beta-1}{\alpha'-1/2} \leq \frac{\beta^2}{1-\alpha'} & \text{for } \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} < L \leq \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ 1+\beta \text{ otherwise} & \\ 2\beta & \text{for } L > \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \end{cases}$$

Instead, if the structure is still a star but the shock hits a *large firm* (i.e. the hub), the expected number of defaults is given by the following function:

$$G^\ell_{star}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{1-\alpha'} \\ \beta & \text{for } \frac{1}{1-\alpha'} < L \leq \frac{\beta}{\alpha'} \\ 2\beta & \text{for } L > \frac{1}{1-\alpha'} \end{cases},$$

since the upper and lower bounds on α' established in the previous Lemma imply that $\beta/\alpha' > 1/(1-\alpha')$. *Ex ante*, a shock hitting a component of the network has the same probability of striking a large firm or a small firm. Hence, $G_{star}(L) = \left(G^s_{star}(L) + G^\ell_{star}(L)\right)/2$, or:

$$G_{star}(L) = \begin{cases} 0 & \text{for } L \leq 2 \\ \frac{1}{2} & \text{for } 2 < L \leq \frac{1}{1-\alpha'} \\ \frac{1}{2} + \frac{1}{2}\beta & \text{for } \frac{1}{1-\alpha'} < L \leq \frac{\beta}{\alpha'} \\ \frac{1}{2} + \frac{1}{2}2\beta & \text{for } \frac{\beta}{\alpha'} < L \leq \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ \frac{\beta}{2} + \beta \text{ if } \frac{\beta-1}{\alpha'-1/2} \leq \frac{\beta^2}{1-\alpha'} & \text{for } \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} < L \leq \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ \frac{1+\beta}{2} + \beta \text{ otherwise} & \\ 2\beta & \text{for } L > \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \end{cases}, \tag{41}$$

since again it can be verified, given the previous lemma, that

$$2 < \frac{1}{1-\alpha'} < \frac{\beta}{\alpha'} < \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\}.$$

Consider next the case where the structure is symmetric, with two completely connected components, the first one with the two large firms, the second one with the 2β small firms. In this case, since there is no asymmetry within each component, every firm retains a fraction exactly equal to $\alpha = 1/2$ of claims to the returns of its own project. The off diagonal terms of the exposure matrix are then equal to $1/2$ for the component with the two large firms

[26] Recall that a small firm i defaults when a shock hits a firm j if and only if $\tilde{a}_{ij}L > 1$, while a large firm k defaults if and only if $\tilde{a}_{kj}L > \beta$.

and $1/[2(2\beta-1)]$ for the second one, with 2β firms. Since the shock reaches with equal probability each of the two components, the expected number of defaults is given by the following function:

$$G_{sym}(L) = \begin{cases} 0 & \text{for } L \leq 2 \\ \frac{1}{2} & \text{for } 2 < L \leq 2\beta \\ \frac{1}{2} + \frac{1}{2}2\beta & \text{for } 2\beta < L \leq 2(2\beta-1) \\ 2\beta & \text{for } L > 2(2\beta-1) \end{cases} \tag{42}$$

since it can be easily verified that $2 < 2\beta < 2(2\beta-1)$.

A straightforward comparison of the functions $G_{star}(L)$ and $G_{sym}(L)$ given in (41) and (42), noting that $\frac{\beta}{\alpha'} < 2\beta$ and $\frac{\beta}{\alpha'} < 2(2\beta-1) < \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\}$, yields then the claim in the proposition. ■

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