# Balancing the Power to Appoint Officers* 

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#### Abstract

Rules of k names are frequently used methods to appoint individuals to office. They are two-stage procedures where a first set of agents, the proposers, select k individuals from an initial list of candidates, and then another agent, the chooser, appoints one among those k in the list. In practice, the list of k names is often arrived at by letting each of the proposers screen the proposed candidates by voting for v of them and then choose those k with the highest support. We then speak of v -rules of k names. Our main purpose in this paper is to study how different choices of the parametrs v and k affect the balance of power between the proposers and the choosers. From a positive point of view, we analyze a strategic game where the proposers interact to determine what list of candidates to submit. From a normative point of view, we study the performance of different rules in expected terms, under the sustained hypothesis that agents' preferences are unknown at the design period, but realized at the time of voting. The choice of v and k is then analyzed from the perspectives of efficiency, fairness and compromise.


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## 1 Introduction

Appointing people to office is one of the main ways how the powerful exert their influence in society. But the ability of any authority to appoint officers is often limited by the existence of other "de iure" or "de facto" powers.

In many historical circumstances, different groups have fought and competed for the ability to appoint people to influential positions. The history of the Roman Church is full of instances where the secular rulers and the clergy have struggled to decide who had the possibility of appointing new bishops. In many European countries, University Rectors have been appointed sometimes by the Government, sometimes by the University community itself, sometimes by combinations of inputs from both. Even the President of the United States has to submit his proposals for cabinet members, for supreme court judges and for many other appointments to the approval of the legislators. In other types of societies, or for other types of appointments, the power to choose one's candidate for a post may be almost unlimited temporarily, but is likely to be challenged at same point.

In this paper we study a class of methods that allow several agents to share the power to appoint. These methods are widely used in the present, and were also used in the past. We call them rules of $k$ names, and they work as follows. The set of deciders is divided into two groups: the proposers and the chooser. Proposers consider the set of all candidates to a position and screen $k$ of them. Then, the chooser picks the appointee out of these $k$ names. Indeed, rules of $k$ names can vary, depending on the composition of the sets of proposers, on the value of $k$, and also on the rules that the different participants adopt when deciding how to choose a list of candidates, or one candidate among many.

Here we focus on a specific family of procedures used to determine the list with $k$ names that are, in fact, also adopted in many practical cases. This family of screening rules has the following form: each proposer submits a list of $v$ candidates (for $v \leq k$ ), and then the $k$ most voted candidates get into the list. Though one can think of more general methods to select the $k$ names, the ones we consider are simple and frequently used. We call these procedures the $v$-rules of $k$ names.

The main purpose in this paper is to study how different choices of the parameters $v$ and $k$ affect the balance of power between the proposers and the chooser. From a positive point of view, we analyze a strategic game where the proposers interact to determine what list of candidates to submit. From a normative point of view, we study the performance of
different rules in expected terms, under the sustained hypothesis that agents' preferences are unknown at the design period, but realized at the time of voting. The choice of $v$ and $k$ is then analyzed from the perspectives of efficiency, fairness and compromise.

Notice that a great variety of methods that are used in practice do differ on the values of both $k$, the size of the list, and $v$, the number of candidates that each proposer can vote for. There are cases where in order to participate in the choice of $k$ candidates, each voter is allowed to submit $k$ names. The rule used to elect Irish bishops or prosecutor-general in most of Brazilian states are of this sort, with $k=v=3$. Yet, in most cases we know, each proposer is asked to submit a vote for $v$ candidates, with $v$ less than $k$. This is the case, for example, when choosing public university rectors in Brazil ( $k=3, v=1$ ), members of the Chile's Courts of Justice $(k=3, v=2)$ or Chile's Supreme Court ( $k=5, v=3$ ). The case where $k=v$ is very special as it allows a simple majority to impose the whole list. As we shall see, for a given society the chooser may prefer a large $v$, or a small $v$, depending on the distribution of preferences among the proposers and the extent to which their own preferences are aligned with those of the majority of proposers. Hence, the actual normative choice of one pair $(k, v)$ will not only depend on the planner's objectives, but also on her expectations regarding the possible preference profiles.

Our first task is to understand the intricacies of the decision making process that will take place, under any given rule and for every specific society. We shall assume, given $k, v$ and a tie breaking rule, that proposers engage in a normal form game and play strong Nash equilibrium strategies. The proposers' strategies are the possible lists of $v$ candidates that they will support, and the outcome function is given by the chooser's best alternative among those with higher support (after tie breaking). ${ }^{1}$

In view of the complexities of the analysis, our positive results will take two complementary routes. One is to proceed with the study of the generalcase, providing conditions for the existence and eventual uniqueness of strong equilibria in our games. One of the things we learn from this analysis is that the impact on the distribution of power among agents of different choices of $k$ and $v$ is not one-directional, as the strategic interactions between agents may affect the final outcome of the rule in very rich ways. A second direction we take, regarding positive results, consists in identifying a family of situations ( represented by what we call the polarized proposers' model) under which the existence and

[^1]uniqueness of equilibria is guaranteed, and for which one can get much more conclusive comparative static results.

In order to plunge the reader directly into our problem, consider the following example:
There are five candidates $\{a, b, c, d, e\}$ and eleven proposers. Each proposer is allowed to vote for one candidate $(v=1)$ and a list will be formed with the names of the three most voted candidates ( $k=3$ ), with ties being broken according to the order $b \succ a \succ e \succ d \succ c$. The type (preferences) and the number of agents are given in the following table.

## Preference Profile

| 1 type 1 proposer | 7 type 2 proposers | 3 type 3 proposers | Chooser |
| :---: | :---: | :---: | :---: |
| $b$ | $a$ | $c$ | $b$ |
| $a$ | $c$ | $a$ | $c$ |
| $e$ | $d$ | $e$ | $a$ |
| $d$ | $b$ | $d$ | $e$ |
| $c$ | $e$ | $b$ | $d$ |

We shall argue, in what follows, $c$ can be the outcome induced from the strong Nash equilibrium play of the proposers when the chooser always picks his best alternative in the list.

Consider the following strategy profile that sustains $c$ as a strong Nash equilibrium outcome: the seven type 2 proposers cast four votes for $a$ and three votes for $e$. The only one type 1 proposer casts a vote for $b$, while the three type 3 proposers cast three votes for $c$. Thus, the selected list is $\{a, c, e\}$ and $c$ is the winning candidate.

The argument behind this equilibrium is quite clear. Type 3's go ahead in support of $c$, and then the type 2's have to prevent $b$ from becoming the outcome by "wasting" their remaining votes in support of $e$.

But there is another, maybe more interesting equilibrium. Notice that any coalition with at least three proposers can impose at least one candidate in the list, and that the chooser and the three proposers of type 3 prefer $c$ to $a$. In spite of this, candidate $a$ can also be sustained as a strong Nash equilibrium outcome! To verify it, consider the following strategy profile: the seven type 2 proposers cast three votes for $a$, two votes for $e$, one vote for $b$ and one vote for $d$. Type 1 proposer casts a vote for $b$, while the three type 3 proposers cast two votes for $d$ and one for $e$. So, $a, d$ and $e$ will have three votes each, while $b$ only two. Thus, the selected list is $\{a, d, e\}$ and $a$ is the winning candidate.

The reader can check that no coalition of voters can profitably deviate.
Now, here is a intuition for this equilibrium, where the two proposers of type 2 cleverly distribute their votes in order to prevent the type 3's from being able to select $c$, even if they all vote for it. Voters of type 2 ensure that candidate $a$, their favorite, is among the proposed ones, by casting three votes in its favor. They also give enough support to candidate $b$ so that, along with the vote of type $1, b$ is still not chosen, but would be as soon as candidates with two votes enter the list. Then, since $b$ has two votes, proposers of type 3 cannot vote for their favorite, $c$, because if they all spent their votes on $c$, which would make $c$ eligible, then some alternative with two votes would come in, and in this case it would be $b$, which they hate but is the chooser's best. Given that they cannot get $c$, they then concentrate, in alliance with type 2 people, in getting $e$ and $d$ into the list, both above their worse alternative $b$, in order to at least get their second alternative.

Thus, the presence of the type 1 proposer voting for $b$ leads types 2 and 3 into a sort of race: if one of them uses the most rewarding strategy in one of the two equilibria, the other must concede. If both used their most rewarding strategies, then $b$, that they both hate, would come out!

In this example, we can observe several types of strategic behavior on the side of agents. This example becomes complex, in spite of the small number of agents and alternatives, because there are no restrictions on the distribution of voters' preferences.

In a large part of the paper we shall concentrate on cases where, thanks to some control on the characteristics of voters, one may characterize, either fully or partially, the set of strong Nash equilibria. This will allow us to establish a number of comparative static results for the better determined cases. And we will also provide a number of precise reasons and examples, in addition to the one we just presented, that help to understand the intricacies of a fully general analysis.

Our second purpose is normative. We want to evaluate the performance of different rules of $k$ names from an ex ante point of view, by computing the utility that different participants in the decision process may expect. This would allow us to eventually arbitrate among different proposals for specific $v$-rules of $k$ names, on the basis of their "expected" performance and of the "expected" satisfaction they can provide to different parts of society. For that purpose, we need to make different modelling decisions. One is on how to measure the utility of individuals, and how to calculate the eventual ex-
pected utility. In the absence of additional information, we consider that agents have utility functions whose argument is the ranking of alternatives, and treat them as Von Neumann-Morgenstern utility functions over lotteries.

Under different distributional assumptions on the preferences of the proposers and of the chooser, within the polarized proposer's model, we provide explicit formulas to compute the expected ranking, for these different agents, of the strategic outcome of each $v$-rule of $k$ names. This allows us to discuss the ex ante efficiency of different rules, their ability to distribute the power among proposers and choosers, or the expected outcome of bargaining processes eliding to the choice of a rule.

Our preceding papers on the subject Barberà and Coelho (2006) and Barberà and Coelho (2010) provided an initial analysis of rules of $k$ names and of possible ways to screen candidates. Our main results in Barberà and Coelho (2010) did focus on majoritarian rules. Here we extend the analysis to the much wider class of $v$-rules of $k$ name, of which only the case $v=k$ is majoritarian. In addition, we provide a fresh start toward the normative evaluation of these rules.

We now comment on some related papers. Unfortunately there does not seem to be a body of literature specifically devoted to study appointment rules with their checks and balances. Of course there exist many voting rules that can be adapted to this specific purpose, but we feel that it may be useful to focus on those that are especially fit for appointment. To mention some related work, Holzman and Moulin (2011) and Alon et al. (2011) concentrate on what they call nomination rules, leading to the choice of a fixed number of candidates where the candidates are also the voters. Even if different from our analysis, these papers show how being specific on the nature of the choice to be made can help in focusing on new axioms and new questions. We would also like to mention some sequential methods where different agents play different roles, as voters or vetoers, like Mueller's voting by veto (see Mueller,1978), Moulin's successive elimination procedures (Moulin, 1982) or Stevens, Brams and Merril's final-offer arbitrage procedures (Brams and Merrill (1986) and Stevens (1966)). All of them are multi-stage procedures that also demand a game theoretic and a normative analysis, though in fact they are all different from each other and of $v$-rules of $k$ names. What we can certainly say is that $v$-rules of $k$ names are among to the most widely used methods in that general vein.

As for the normative analysis, the papers closest to ours are those that study the
design of egalitarian and utilitarian voting schemes. See for instance, Rae's (1969), Curtis (1972), Badger (1972), Coelho (2004) and Barberà and Jackson (2004). However, this literature focuses on the case where a society faces dichotomous choices. As for our game theoretical analysis, the more related literature is the one characterizating the set of strong Nash equilibrium outcomes of voting games. See Barberà and Coelho (2010), Ertemel, Kutlu and Sanver (2010), Sertel and Sanver (2004), Polborn and Messner (2007), Moulin (1982) and Gardner (1977).

The paper is organized as follows. In the next section (Section 2), we formally describe the $v$-rule of $k$ names, the game induced by this rule (Constrained Chooser Game) and the equilibrium concept use (Strong Nash Equilibrium). In Section 3, we study how the set of Strong Nash equilibrium outcomes responds to changes in the parameters $k$ and $v$ defining the rule. In Section 4, we undertake a normative analysis. We describe the agents' expected utility as a function of $(k, v)$, in order to characterize the optimal values of these rules' parameters, according to standard normative criteria and under the sustained hypothesis that agents' preferences are unknown at the design period, but realized and known by all agents at the time of voting.

## 2 The setup

In this section we formally define rules of $k$ names and the games they induce. We observe that, in addition to other structural features, like the number of proposers, the number of candidates and the size $k$ of proposed candidates, a full specification of a rule of $k$ names also requires to define the screening rules by which the proposers decide what names go into the list. In principle, this method could remain unspecified, or be rather complicated. But in actual practice simple and well specified screening rules are usually set. Basically, proposers are allowed to vote for a number $v$ of candidates, and then the $k$ most voted ones are selected (with a tie break if needed). These votes will typically be cast as the result of strategic calculations that may involve the cooperative coordination among players.

Denote by $\mathbf{A}=\{1, \ldots, a\}$ the finite set of candidates, and by $\mathbf{A}_{k} \equiv\{B \subseteq \mathbf{A} \mid \# B=k\}$ the set of all possible subsets of $\mathbf{A}$ with cardinality $k$ where $\# B$ stands for the cardinality of $B$. Denote by $\mathbf{N}=\{1, \ldots, n\}$ the finite set of proposers, who will select a set $B$ from $\mathbf{A}_{k}$
from which an individual that does not belong to $\mathbf{N}$, the chooser, selects one candidate for office.

Let $W$ be the set of all strict orders (transitive ${ }^{2}$, asymmetric ${ }^{3}$, irreflexive $^{4}$ and complete ${ }^{5}$ ) on A. Each member $i \in \mathbf{N} \cup\{$ chooser $\}$ has a strict preference $\succ_{i} \in W$. For any nonempty subset $B$ of $\mathbf{A}, B \subseteq \mathbf{A} \backslash \emptyset$, we denote by $\alpha\left(B, \succ_{i}\right) \equiv\left\{x \in B \mid x \succ_{i} y\right.$ for all $y \in B \backslash\{x\}\}$ the preferred candidate in $B$ according to preference $\succ_{i}$.

Definition 1 Let $M^{\mathbf{N}} \equiv M_{1} \times \ldots \times M_{n}$ with $M_{i}=M_{j}=M$ for all $i, j \in \mathbf{N}$ where $M$ is the space of actions of a proposer in $\mathbf{N}$. Given $k \in\{1,2, \ldots, a\}$, a screening rule for $k$ names is a function $S_{k}: M^{\mathbf{N}} \longrightarrow \mathbf{A}_{k}$ associating to each action profile $m_{\mathbf{N}} \equiv\left\{m_{i}\right\}_{i \in \mathbf{N}} \in$ $M^{\mathbf{N}}$ the $k$-element set $S_{k}\left(m_{\mathbf{N}}\right)$.

In words, a screening rule for $k$ names is a voting procedure that selects $k$ alternatives from a given set, based on the actions of the proposers. In general, these actions may consist of single votes, sequential votes, the submission of preference of rankings, the filling of ballots, etc...For example, if the actions in $M^{\mathbf{N}}$ are casting single votes for some candidates, then $M \equiv \mathbf{A}$. If the actions in $M^{\mathbf{N}}$ are submissions of strict preference relations among candidates, then $M \equiv W$.

We concentrate on what we call $v$-screening rules. We have already observed that it is usual in practice to specify the number of votes that each proposer can cast for different candidates, and then use plurality count to determine those that will be selected. Our next definitions refer to this particular and important subclass of screening rules, and to the rules of $k$ names that use them.

Definition $2 A v$-screening rule for $k$ names can be described as follows: each proposer votes for $v$ candidates, and the list is formed with the names of the $k$ most voted candidates, with a tie breaking rule when needed. The parameters $v$ and $k$ satisfy $v \leq k<a$ and $v . n \geq k$. The tie breaking criterion is a strict ordering of alternatives. This ordering can either be fixed for all profiles, or coincide with the preferences of some

[^2]agent at each given profile ${ }^{67}$
Definition 3 A v-rule of $k$ names is described as follows: given a set of candidates for office, a committee of size $n$ chooses $k$ members from this set by using a v-votes screening rule for $k$ names. Then a single individual from outside the committee selects one of the listed names for office. ${ }^{8}$

Having defined our rules, we now want to consider the type of strategic interactions that may arise among the proposers, as a function of their preferences and those of the chooser. We model these interactions as a normal form game with complete information, and concentrate our analysis on the study of its strong Nash equilibria.

Definition 4 (Barberà and Coelho, 2010) Given $k \in\{1,2, \ldots, a\}$, a screening rule for $k$ names $S_{k}: M^{\mathbf{N}} \longrightarrow \mathbf{A}_{k}$ and a preference profile $\succ \equiv\left\{\succ_{i}\right\}_{i \in \mathbf{N} \cup\{\text { chooser }\}} \in W^{\mathbf{N}+1}$, the Constrained Chooser Game can be described as follows: It is a simultaneous game with complete information where each player $i \in \mathbf{N}$ chooses a strategy $m_{i} \in M_{i}$. Given $m_{\mathbf{N}} \equiv\left\{m_{i}\right\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}, S_{k}\left(m_{\mathbf{N}}\right)$ is the chosen list with $k$ names and the winning candidate is $\alpha\left(S_{k}\left(m_{\mathbf{N}}\right), \succ_{\text {chooser }}\right)$.

In the Constrained Chooser Game, the chooser's strategy set is restricted to a single element. In that sense, we could say that he is not an active player. Specifically, we take it that the chooser will simply select that candidate that is best for him among those that he will be presented with. Thus, the chooser's preferences will condition the outcome function, and therefore will have an impact on the equilibrium play of the proposers. But we exclude the possibility that he announces a choice rule that is not in accordance to his preferences, which are known in each game.

We choose to analyze the set of strong Nash equilibria of this game. This is consistent with the idea that proposers have complete information about their preferences and those

[^3]of the chooser, and that they must find ways to cooperate among themselves, in order to come up with a favorable list.

Definition 5 Given $k \in\{1,2, \ldots, a\}$, a screening rule for $k$ names $S_{k}: M^{\mathbf{N}} \longrightarrow \mathbf{A}_{k}$ and a preference profile $\succ \equiv\left\{\succ_{i}\right\}_{i \in \mathbf{N} \cup\{\text { chooser }\}} \in W^{\mathbf{N}+1}$, a joint strategy $m_{\mathbf{N}} \equiv\left\{m_{i}\right\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}$ is a pure strong Nash equilibrium of the Constrained Chooser Game if and only if, given any coalition $C \subseteq N$, there is no $m_{\mathbf{N}}^{\prime} \equiv\left\{m_{i}^{\prime}\right\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}$ with $m_{j}^{\prime}=m_{j}$ for every $j \in \mathbf{N} \backslash C$ such that $\alpha\left(S_{k}\left(m_{\mathbf{N}}^{\prime}\right), \succ_{\text {chooser }}\right) \succ_{i} \alpha\left(S_{k}\left(m_{\mathbf{N}}\right), \succ_{\text {chooser }}\right)$ for each $i \in C$.

## 3 Cooperation and conflict under $v$-rules of $k$ names.

In this section we discuss how different $v$-rules for $k$ names will tilt the decision power to the benefit of the chooser or (reciprocally) of the proposers. To do so, we study the equilibria of the games induced by our rules, for different values of $v$ and $k$, and the impact of these two defining parameters on the interests of the chooser. Notice that under $v$-rules of $k$ names, the messages that agents are required to send consist of subsets of candidates with cardinality $v$.

We begin in Subsection 3.1, by studying conditions that any strong Nash equilibria outcome must satisfy. These conditions are helpful to locate equilibria and provide a first step toward their characterization, when they exist! However, we also show that they are not sufficient for either existence or uniquess, and provide examples indicating the variety of problems that make it hard to fill the gap of a full characterization. We also show that if a candidate is at same time the proposers's strong Condorcet winner and the chooser's best candidate, then it is the unique equilibrium outcome for any $v$-rule of $k$ names. This proves that, when there is little conflict between the proposers and the chooser's objectives, then any choice of $v$ and $k$ will do. But it is also highlights the fact that, otherwise, choosing $v$ and $k$ has consequences on the balance of power between the interested parties, We finish this subsection by giving a counterintuitive example where the chooser is better off under a small $k$ and a large $v$.

In Subsection 3.2, we study a model of simple societies (the polarized proposers model) for which we are able to prove the existence and uniqueness of strong Nash equilibrium outcomes, and to characterize them. A special case covered by this model is the homo-
geneous proposer's case ${ }^{9}$, where all proposers are in agreement. We show that in the polarized case the chooser always prefer rules with a lower $k$ and a smaller $v$. We conclude that it is possible to sign the impact of our parameters upon the chooser in polarized societies, but that the general case requires a delicate case by case analysis.

### 3.1 Necessary conditions for equilibrium outcomes

Clearly, any specific screening rule will endow each subgroup of proposers with some power to determine what candidates to include in the proposal. Understanding how this power is distributed is a prerequisite to discuss, later on, the strategic interaction among voters, depending on their diverse interests. The definitions we provide now will be useful for this purpose. ${ }^{10}$

Definition 6 Given a v-screening rule for $k$ names $S_{k}: M^{\mathbf{N}} \longrightarrow \mathbf{A}_{k}$ and $X \subseteq A$ such that $\# X \leq k$, let $q_{k}^{v}(X)$ be the minimum $\widehat{q}$ such that for any coalition $C \subseteq N$ of voters with $\# C \geq \widehat{q}$ implies that there exists $m_{C} \in M^{C}$ such that for every profile of the complementary coalition $m_{\mathbf{N} \backslash C} \in M^{\mathbf{N} \backslash C}$ we have $X \subseteq S_{k}\left(m_{C}, m_{\mathbf{N} \backslash C}\right)$.

In words $q_{k}^{v}(X)$ is the minimum $\widehat{q}$ such that any coalition, with size higher or equal to $\widehat{q}$, can impose the choice of $X$, under $v$-screening rule.

Remark 1 The values of $q$ evolve monotonically with those of $k$ and $v$. For any $A$ and $v<v^{\prime}<k<k^{\prime}<\# A$, we have that:

1. $q_{k}^{v}(X) \geq q_{k}^{v^{\prime}}(X)$ for any $X \in A_{k}$;
2. $q_{k}^{v}\left(X^{\prime}\right) \geq q_{k}^{v}(X)$ for any $X \in A_{k}$ and $X^{\prime} \in\left\{Y \in A_{k^{\prime}} \mid X \subset Y\right\}$;
3. $q_{k}^{v^{\prime}}(\{x\}) \geq q_{k}^{v}(\{x\})$ for any $x \in A$;
4. $q_{k}^{v}(\{x\}) \geq q_{k^{\prime}}^{v}(\{x\})$ for any $x \in A$.
[^4]These $q$ values may differ depending on the set or on the candidate, due to the tie breaking rule. We can find common bounds for all of them, and compute explicit formulas for their values.

Definition 7 For any $v$-screening rule for $k$ names, let $q_{1}^{v} \equiv \operatorname{Max}_{y \in A}\left\{q_{k}^{v}(\{y\})\right\}$ and $q_{k}^{v} \equiv \operatorname{Max}_{Y \in A_{k}}\left\{q_{k}^{v}(Y)\right\} .{ }^{11}$

Definition 8 A candidate is a chooser's $\ell$-top candidate if and only if he is among the $\ell$ best ranked candidates according to the chooser's preference.

Once endowed with the preceding definitions, we can state Proposition 1, that provides necessary conditions for a candidate to be a strong Nash equilibrium outcome of the Constrained Chooser Game.

Proposition 1 Consider any $v$-rule of $k$ names. If candidate $x$ is a strong Nash equilibrium outcome of the Constrained Chooser Game, then it satisfies the following four conditions

C1: It is among the chooser's ( $a-k+1$ )-top candidates.
C2: If $y \neq x$ is among chooser's (a-k+1)-top candidates then $\#\left\{i \in N \mid y \succ_{i} x\right\}<q_{k}^{v}(Y)$ for any $Y \in A_{k}$ such that $y$ is the chooser's best candidate in $Y$.

C3: If $y$ is the chooser's best candidate then $\#\left\{i \in N \mid y \succ_{i} x\right\}<q_{k}^{v}(\{y\})$.
C4: If $y$ is the chooser's best candidate and also ranked above than $x$ by the tie breaking criterion then $\#\left\{i \in N \mid x \succ_{i} y\right\} \geq q_{k}^{v} .{ }^{12}$

It is not always easy to identify those candidates that may be elected at a strong Nash equilibrium of the game. But knowing the necessary conditions alone is already of great help. We illustrate this point though an example.

Example 1 Let $\mathbf{A}=\{a, b, c, d, e\}$ and let $\mathbf{N}=\{1,2,3\}$. Suppose that each proposer votes for one candidate and the three most voted candidates form the list, with a tie breaking rule when needed: $b \succ a \succ e \succ d \succ c$. The preferences of the chooser and the committee

[^5]members are as follows:

## Preference Profile

Proposer 1 Proposer 2 Proposer 3 Chooser

| $e$ | $e$ | $e$ | $a$ |
| :---: | :---: | :---: | :---: |
| $d$ | $d$ | $d$ | $b$ |
| $c$ | $c$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | $d$ |
| $b$ | $b$ | $c$ | $e$ |

Notice that, we have that $q_{k}^{v}(\{x\})=1$ for any $x \in A$ and $q_{k}^{v}(X)=3$ for any $X \in A_{k}$.
The first step in describing the equilibrium outcomes is to identify those candidates that satisfy the three necessary conditions established in Proposition 1.
Inspecting the preference profile above, we have that:

1. Condition 1: $\{a, b, c\}$.
2. Condition 2: $\{a, b, c, d, e\}$.
3. Condition 3:\{a,d,e\}.
4. Condition 4: $\{a, b, d, e\}$.

So, only candidate a that satisfies all four conditions. Now we have to check whether there is a strategy profile that sustains candidate a as a strong Nash equilibrium candidate. The following strategy profile sustains $a$ as a strong Nash equilibrium outcome: Proposer 1 votes for a, Proposer 1 votes for $d$ and Proposer 3 votes for $b$.

The table below presents the set of strong Nash equilibrium for different values of $v$. Notice that, in this example, the chooser is weakly worse off as $v$ increases.
Set of strong Nash equilibrium outcomes

$$
\begin{array}{lll}
k=3 & v=1 & \{a\} \\
k=3 & v=2 & \{a\} \\
k=3 & v=3 & \{c\}
\end{array}
$$

In the preceding example, the choice of candidates satisfying the necessary conditions could be in fact be sustained with an appropriate set of strong equilibrium strategies. But this need not be the case. In fact, there may be candidates that satisfy the necessary conditions and yet cannot be the outcome of any equilibrium. Worse of that, equilibria may not exist even if some candidates meet the necessary conditions, as shown by our next example.

Example 2 Let $\mathbf{A}=\{a, b, c, d\}$ and let $\mathbf{N}=\{1,2,3\}$. Suppose $k=2$ and $v=1$, with the following tie breaking rule when needed: $a \succ c \succ b \succ d$. The preferences of the chooser and the committee members are as follows:

## Preference Profile

| Proposer 1 | Proposer 2 | Proposer 3 | Chooser |
| :---: | :---: | :---: | :---: |
| $c$ | $b$ | $b$ | $a$ |
| $b$ | $c$ | $a$ | $c$ |
| $d$ | $d$ | $c$ | $b$ |
| $a$ | $a$ | $d$ | $d$ |

We have that $q_{k}^{v}(\{a\})=q_{k}^{v}(\{c\})=1, q_{k}^{v}(\{x\})=2$ for any $x \in A \backslash\{a, c\}$ and $q_{k}^{v}(X)=2$ for any $X \in A_{k}$ such that $a \in X$ and $q_{k}^{v}(Y)=3$ for any $Y \in A_{k}$ such that $a \notin Y$. Inspecting the preference profile above, we have that:

1. Condition 1: $\{a, b, c\}$.
2. Condition 2: $\{b, c\}$.
3. Condition 3: $\{a, b\}$.
4. Condition 4: $\{a, b\}$.

So, only candidate b satisfies all four necessary conditions stated in Proposition 1.
However, $b$ is not an equilibrium outcome, since Proposer 1 always have incentive in preventing the election of $b$ by casting a vote for $c$.

Notice also that the proposers' preference profile satisfies single peakedness. So, this example teaches us that even this strong property cannot guarantee the existence of an equilibrium. If we had considered a 2-votes screening rules for two names, candidate $b$ would be the unique strong Nash equilibrium outcome of the game.

Before engaging in any other discussion, let us notice that when there is sufficient agreement among agents regarding candidates, then all rules will yield the same outcome. This result emphasizes the fact that the choice of specific rules makes a difference when there is disagreement.

Definition 9 A candidate $x$ is the proposers' strong Condorcet winner if and only if $\#\left\{i \epsilon N \mid x \succ_{i} y\right\}>\frac{n+1}{2}$ for every $y \in A \backslash\{x\}$.

Proposition 2 If a candidate is the proposers' strong Condorcet winner and also the chooser's best candidate then it is the unique Strong Nash equilibrium outcome for any $v$-rule of $k$ names.

Proposition 2 provides a simple sufficient condition for the existence and uniqueness of strong Nash equilibrium in our game for any pair of $(v, k)$. we know other sufficient conditions and they appear in Appendix C, Section C1.

In the spirit of the result above, notice that in case of disagreement, the choice of $v$ and $k$ will have an impact on the balance between the satisfaction of the chooser and that of the proposers. In the next subsection we will show that, in many cases, the chooser will prefer a larger to a smaller $k$, and a smaller rather than a larger $v$. But the following example shows that, without any further restrictions, these preferences can be reversed: the chooser may be happier as $k$ decreases and as $v$ increases.

Example 3 Let $A=\{a, b, c, d\}$, and let $N=\{1,2,3\}$. Each proposer votes for one candidate and the list has the names of the two most voted candidates, with a tie breaking rule when needed: $c \succ d \succ b \succ a$.

## Preference Profile

| Proposer 1 | Proposer 2 | Proposer 3 | Chooser |
| :---: | :---: | :---: | :---: |
| $b$ | $a$ | $a$ | $b$ |
| $d$ | $c$ | $c$ | $a$ |
| $c$ | $d$ | $d$ | $c$ |
| $a$ | $b$ | $b$ | $d$ |

We have that $q_{k}^{v}(\{c\})=q_{k}^{v}(\{d\})=1, q_{k}^{v}(\{x\})=2$ for any $x \in A \backslash\{c, d\}$ and $q_{k}^{v}(\{c, d\})=2$ and $q_{k}^{v}(Y)=3$ for any $Y \in A_{k} \backslash\{\{c, d\}\}$. Inspecting the preference profile above, we have that:

1. Condition 1: $\{a, b, c\}$.
2. Condition 2: $\{a, c\}$.
3. Condition 3: $\{a, b, c, d\}$
4. Condition $4:\{b, c, d\}$

Thus, only c satisfies all four conditions. In fact, $c$ is the unique strong Nash equilibrium outcome under $v=1$ and $k=2$.

Here there is a intuition for this result: notice that candidate a cannot be a strong equilibrium outcome of the Constrained Chooser Game, because as long as proposer 1 votes for b, proposers 2 and 3 cannot get a to be the outcome, even if they can force a to be in the list. Short of that, proposers 2 and 3 coordinate their actions so that one of them votes for $c$ and the other for $d$. If 1 persists in voting for $b$, this creates a tie between the three candidates that is solved in favor of $c$ and $d$, out of which the chooser selects $c$. If 1 votes for c instead, the same outcome ensues. And all other actions by any combination for agents would lead some of them to outcomes that would be worse than c for some of them. Hence, $c$ is the unique strong Nash equilibrium of the Constrained Chooser Game under our proposed rule.
The cases $(v, k)=(1,1)$ and $(v, k)=(2,2)$ are majoritarian, and lead to the election of $a$.

The table below presents the set of strong Nash equilibrium for different values of $v$ and $k$. Notice that, in this example, the chooser is weakly worse off as $v$ increases.
Set of strong Nash equilibrium outcomes

$$
\begin{array}{lll}
k=1 & v=1 & \{a\} \\
k=2 & v=1 & \{c\} \\
k=2 & v=2 & \{a\}
\end{array}
$$

Notice that, with $v$ fixed at 1 , the chooser is better off when $k=1$ than when $k=2$. This is quite surprising, since $k=1$ means that the chooser has no power at all! On the other hand, for $k=2$, the chooser prefers the higher value $v=2$ to that of $v=1$. Again, this not completely intuitive, as one may think that a smaller $v$ takes away power from the proposers. What the example shows is that simple intuitions on the issues may be misleading unless we provide a careful strategic analysis.

## 4 The polarized proposers model

We now present a specification of possible societies for which, as we shall see, the equilibria outcomes always exist, is unique and can be easily characterize. We call them polarized societies, and they are described as follows:

1. (Assumption 1). Proposers are partitioned into $G_{1}$ and $G_{2}=N \backslash G_{1}$, with sizes

$$
\# G_{1}=m>\# G_{2}=n-m .
$$

2. (Assumption 2). All proposers in $G_{1}$ share the same preferences over the set of candidates.
3. (Assumption 3). All proposers in $G_{2}$ share the same preferences over the set of candidates and it is the reverse of the preferences of the proposers in $G_{1}$.
4. (Assumption 4). The tie breaking rule coincides with at least one of the agent's preferences over the set of candidates. ${ }^{13}$

Proposition 3 Consider the Polarized Proposers Model and any v-rule of $k$ names. A strong Nash equilibrium outcome of the Constrained Chooser Game always exists and it is unique. In addition:

1) Suppose that the tie breaking criterion coincides with the majoritarian group's preferences over the set of candidates.
If $m \geq q_{k}^{v}>n-m$ then the strong Nash equilibrium outcome is the best candidate of the majoritarian group out of chooser's $(a-k+1)$-top alternatives;

If $q_{k}^{v}>m \geq q_{1}^{v}>n-m$ then the strong Nash equilibrium outcome is the chooser's best candidate out of the majoritarian group's $k$-top candidates;

If $q_{k}^{v}>m>n-m \geq q_{1}^{v}$ then the equilibrium outcome is the chooser's best candidate.
2) Suppose that the tie breaking criterion coincides with the chooser's preferences over the set of candidates or with the minoritarian group's preferences over the set of candidates. If $m \geq q_{k}^{v}$ then the strong Nash equilibrium outcome is the best candidate of the majoritarian group out of chooser's $(a-k+1)$-top candidates; If $q_{k}^{v}>m$ then the strong Nash equilibrium outcome is the chooser's best candidate.

A simple and interesting case of polarized societies arises when all proposers share the same preferences, i.e., $G_{1}=N$ and $G_{2}=\emptyset$. We call it homogeneous proposers model.

Corollary 1 Consider the homogeneous proposers model $(m=n)$. The strong Nash equilibrium outcome is the proposers's best candidate out of chooser's $(a-k+1)$-top candidates.

[^6]The following three corollaries follow from Proposition 3 and Remark 1.

Corollary 2 Consider the Polarized Proposers Model. The chooser cannot be worse off under $v^{\prime}$-rule for $k$ names than under $\widetilde{v}-$ rule for $k$ names whenever $\widetilde{v}>v^{\prime}$.

Corollary 3 Consider the Polarized Proposers Model and $v$-rule of $k$ names. The chooser cannot be worse off under a more polarized set of proposers (small m) than under a less polarized set of proposers (big m).

Corollary 4 Consider the Polarized Proposers Model. The chooser cannot be worse off under $v$-rule for $k$ names than under $v$ - rule for $\widetilde{k}$ names whenever $k>\widetilde{k}$.

## 5 An ex-ante analysis of different rules: egalitarianism, efficiency and bargaining.

We now turn to the normative analysis of our rules. We suggest a general framework within which one can make ex ante calculations of the consequences of adopting any specific collective decision rule. Then we apply our general methods to specific instances, as an indication of how to proceed. These instances are chosen in such a way that we can provide exact calculations of expected values and perform comparisons among different rules of $k$ names, on the basis of our previous analysis. But the interested reader will be able to adapt the same analytical principles to any situation and any set of rules of interest.

The main ingredients of the analysis are the following. First, we'll need a probability distribution over social preference profiles.This is a way to specify the uncertainty faced by a planner who must choose one among different rules, regarding the possible worlds on which that rule will need to be used, and with what frequency. Then, we will have to make some assumption about how much information the agents will be endowed with, once the uncertainty is resolved and they are about to make a decision. This will allow us to determine the expected behavior of voters at each realization of the world. On that basis, we'll be able to compute the expected utility of the different voters. Finally, we'll need an objective function indicating the normative criterion that the planner wishes to achieve when choosing among different rules.

In order to indicate how to make this general approach operational, we provide an analysis of some specials cases, based on the following assumptions.

1. Regarding the possible worlds, we concentrate on preference profiles where all of the agents' utilities are identified with the inverse of the ranking of the alternatives they get. That is, we assume all agents to be risk neutral and to measure the performance of any rule by the ranking of the alternative they will obtain at each realization of the world. Under this assumption, we study two different cases. The first one arises when there is only one proposer and one chooser, whose preferences are independently randomly drawn from a uniform distribution over all possible rankings of candidates. The second case assumes that all probability is concentrated on polarized worlds, where the preferences of the majoritarian segment of the proposers and those of the chooser are again independently drawn from a uniform distribution on the universal domain.
2. Regarding the information of agents at the time of choice, we assume that they are all fully informed about the preference profile that has materialized. Then, we use our previous analysis and make the expected utility calculations that result from the agents' playing a strong Nash equilibrium.
3. Regarding the objectives of the planner, in the one-proposer case we show that three possible criteria would converge into the same one. These are utilitarianism (minimizing the sum of expected rankings), egalitarianism (minimizing the difference between the expected ranking of the proposer and that of the chooser) and Nash bargaining (minimizing the product of the expected rankings for both agents). In general, and given that the chooser remains as a single agents as the number of proposers grows large, utilitarianism and Nash bargaining become less attractive, and we concentrate on the type of rules that tend to equalize the expected ranking of the outcome for the chooser with that of the average proposer.

Sections 5.1 and 5.2 will develop the two cases we just outlined. Before getting there, let us just remark that each of our preceding modeling decisions could be altered without changing the essence of our analysis. Regarding the specification of possible worlds, it is not hard to extend it to cases where the preferences of agents are still based on the ranking of the outcomes but exhibit different degrees of risk aversion. As for the informational assumptions, one could also study easily the polar case where, once a profile is realized, each agent is only informed about her own preference, but remains ignorant
about those of the rest. In that case, it becomes natural to assume that agents will behave sincerely, rather than strategically, and the computations carry over in a similar manner ${ }^{14}$ Finally, it is clear that one could resort to alternative evaluation criteria, like weighted utilitarianism, or any sort of distributional criterion regarding power, other than egalitarianism. But again, our main message here is that the use of some method over another may be discussed in expected utility terms and that, in our case, it is even possible to get of feeling of the trade-offs involved in the choice of any pair $(v, k)$ over any other $\left(v^{\prime}, k^{\prime}\right)$ through explicit nimerical computations.

### 5.1 The case of one proposer

In this subsection, we consider the case where there is only one proposer. The next proposition characterizes the strong Nash equilibrium outcome for this special case of the Polarized proposer model ( $m=n=1$ ).

Proposition 4 Consider any v-rule of $k$ names and only one proposer. The Strong Nash equilibrium outcome of the rule of $k$ names is the proposer' best candidate out of chooser' $s(a-k+1)$-top candidates.

Here is the intuition for Proposition 4. The proposer knows that once he selects a list with $k$ names, the winning candidate is the chooser's best candidate out of this list. Thus, he knows that, in practice, his set of alternatives is restricted to the chooser' s ( $a-k+1$ )-top alternatives, since only for candidates in this set it is possible to form a list with $k$ names where a given candidate is the chooser's best candidate in the list. The proposer must select his best candidate out of chooser's $(a-k+1)$-top candidates and submit a list where this candidate is the chooser's best candidate in the list, since Proposition 4 is a direct corollary of Proposition 3, its proof is omitted.

Let us assume that the proposer and the chooser's preferences are the result of independent random draws from a uniform distribution over the domain of strict preferences. Given that the agents's preferences are random variables, the ranking of the equilibrium outcome according to the preferences of one of the agents is also a random variable. Denote by $R_{c}$ and $R_{p}$ the random variables that represent the ranking of the equilibrium outcome according to the chooser and the proposer's preference relation and by $r_{c}$ and $r_{p}$

[^7]the realized values that $R_{c}$ and $R_{p}$ may take. $r_{i}=1+\#\left\{y \in N \mid y \succ_{i} x\right\}$ if $x$ is the equilibrium outcome. Notice that if the equilibrium outcome is the agent i's best candidate then $r_{i}=1$ and if it is the agent i's worst candidate then $r_{i}=a$ (where $a$ denotes the number of candidates). Let us assume that the agents' Bernoulli utility functions assign to each candidate the negative values of its ranking, i.e., $u_{c}\left(r_{c}\right)=-r_{c}$ and $u_{p}\left(r_{p}\right)=-r_{p}$.

The random variable $R_{c}$ has the same distribution than a discrete random variable uniformly distributed over $\{1,2, \ldots, a-k+1\}$.
$R_{p}$ has the same distribution as the smallest element of a random sample with size $s=a-k+1$ drawn without replacement from a uniformly distributed population $D=$ $\{1,2, \ldots, a\}$. Thus, following standard results of order statistics literature, we have:

$$
\begin{align*}
& E\left(u_{p}\left(R_{p}\right) \mid k, a\right)=-\frac{a+1}{a-k+2}  \tag{1}\\
& E\left(u_{c}\left(R_{c}\right) \mid k, a\right)=-\frac{a-k+2}{2} \tag{2}
\end{align*}
$$

From equations 1 and 2 above, notice that the proposer's expected utility is strictly decreasing with $k$, while the chooser's expected utility is strictly increasing with $k$. Thus, when $k=1$ the chooser's expected utility reaches its minimum and $E\left(u_{c}\left(R_{c}\right) \mid k=1, a\right)=$ $-\frac{a+1}{2}$, while proposer's reaches its maximum, $E\left(u_{p}\left(R_{p}\right) \mid k=1, a\right)=-1$.

Definition 10 A $k \in\{1, \ldots, a\}$ is an egalitarian solution if $\left|E\left(u_{p}\left(R_{p}\right) \mid k, a\right)-E\left(u_{c}\left(R_{c}\right) \mid k, a\right)\right| \leq$ $\left|E\left(u_{p}\left(R_{p}\right) \mid k^{\prime}, a\right)-E\left(u_{c}\left(R_{c}\right) \mid k^{\prime}, a\right)\right|$ for every $k^{\prime} \in\{1, \ldots, a\}$. We denote by $S_{e}$ the set of all values of $k$ that are egalitarian solutions.

Definition $11 A k \in\{1, \ldots, a\}$ is a utilitarian solution if $E\left(u_{p}\left(R_{p}\right) \mid k, a\right)+E\left(u_{c}\left(R_{c}\right) \mid k, a\right) \geq$ $E\left(u_{p}\left(R_{p}\right) \mid k^{\prime}, a\right)+E\left(u_{c}\left(R_{c}\right) \mid k^{\prime}, a\right)$ for every $k^{\prime} \in\{1, \ldots, a\}$. We denote by $S_{u}$ the set of all values of $k$ that are utilitarian solutions.

Definition $12 A k \in\{1, \ldots, a\}$ is a Nash bargaining solution with disagreement point $(d, d)$ where $d \leq-\frac{a+1}{2}$, if $\left(E\left(u_{p}\left(r_{p}\right) \mid a, k\right)-d\right)\left(E\left(u_{c}\left(r_{c}\right) \mid a, k\right)-d\right) \geq\left(E\left(u_{p}\left(R_{p}\right) \mid a, k\right)-\right.$ $d)\left(E\left(u_{c}\left(R_{p}\right) \mid a, k\right)-d\right)$ for every $k^{\prime} \in\{1, \ldots, a\}$. We denote by $S_{n}$ the set of all values of $k$ that are Nash bargaining solutions.

Proposition 5 The egalitarian, utilitarian and Nash bargaining solutions for $k$ coincide in the one proposer case. Moreover, if $x=a+\frac{5}{2}-\sqrt{2 a+\frac{9}{4}}$ is not an integer $S_{u}=S_{n}=$ $S_{e}=\{\lfloor x\rfloor\}$. And if $x$ is an integer then $S_{u}=S_{n}=S_{e}=\{x-1, x\}$.

The intuition is simple: the combination of expected utilities for the proposer and for the chooser that we get as $k$ changes constitute a symmetric set. Since the egalitarian and the utilitarian solution satisfy Nash's axiom of symmetry, and our bargaining problem is symmetric, they both coincide with Nash's solution in this nice case.

Corollary 5 Let $y=a-\sqrt{2 a+2}+2$. If $y$ is an integer number then $S_{u}=S_{n}=S_{e}=\{y\}$ and $E\left(u_{p}\left(r_{p}\right) \mid a, k=y\right)=\left(E\left(u_{c}\left(r_{c}\right) \mid a, k=y\right) .{ }^{15}\right.$

Corollary 6 Any $k$ that is an egalitarian, utilitarian or Nash bargaining solution must be greater or equal than $\frac{a+1}{2}$.

Corollary 7 Consider any $k$ that is an egalitarian, utilitarian or Nash bargaining solution. At any realization of the preference profiles, the chooser's payoff of the strong Nash equilibrium cannot be lower than $-\frac{a+1}{2}$, i.e., $u_{c}\left(r_{c}\right) \geq-\frac{a+1}{2}$. Moreover, there exists some realizations of the preference profiles where the proposer's payoff is lower than $-\frac{a+1}{2}$, i.e., $u_{p}\left(r_{p}\right)<-\frac{a+1}{2}$.

The first part of the corollary above follows from Condition 1 of Proposition 1 that states that any strong Nash equilibrium outcome must be among the chooser's $(a-k+1)$ top candidates, from Corollary 5. To show the second part of the corollary, consider $a=7$. By Proposition 5, we have the egalitarian $k$ solution is $\{5\}$. Now consider the case where the chooser's preferences are the reverse of those held by the proposer. Then $u_{c}\left(r_{c}\right)=-3 \geq-\frac{a+1}{2}$ and $u_{p}\left(r_{p}\right)=-5<-\frac{a+1}{2}$.

Example 4 Consider $a=5$. Applying Proposition 6, we have that $S_{u}=S_{n}=S_{e}=\{3,4\}$ since $a+\frac{5}{2}-\sqrt{\left(2 a+\frac{9}{4}\right)}=4$. Consider now $a=7$ then $S_{u}=S_{n}=S_{e}=\{5\}$ since $a+\frac{5}{2}-\sqrt{\left(2 a+\frac{9}{4}\right)}=5.47$. Notice also that $a-\sqrt{2 a+2}+2=5$ when $a=7$. Thus, by Corollary 4, we have that $E\left(u_{p}\left(r_{p}\right) \mid a=7, k=5\right)=\left(E\left(u_{c}\left(r_{c}\right) \mid a=7, k=5\right)\right.$.

[^8]Corollaries 6 and 7 may be a bit disturbing, because in real life we observe the use of small values of $k$. But he is due to the specificity of the one proposer case, where, as shown by Proposition 7 in the next subsection, the proposer gets a large advantage, that can only be compensated by a larger $k$. In the next subsection, we will see that these $k$ values become smaller as diversity among proposers increases.

### 5.2 The case of several proposers

In this section we study the impact of $v$ and $k$ the expected value of $v$ and $k$ on the expected value of the ranking of equilibrium outcomes for the chooser and for the average proposer in the polarized proposers model.

The preference of the majoritarian group of proposers and that of the chooser are drawn independently from a uniform distribution, and this generates the distribution over polarized profiles. Again, we assume that the utilities of agents assign each candidate the negative of its rank.

We denote by $R_{p} \equiv \frac{m}{n} R_{G_{1}}+\frac{n-m}{n} R_{G_{2}}$ and $u_{p}\left(r_{p}\right) \equiv \frac{m}{n} u_{G_{1}}\left(r_{G_{1}}\right)+\frac{n-m}{n} u_{G_{2}}\left(r_{G_{2}}\right)$ the average utility of an outcome for the proposers, given $r_{G_{1}}$ and $r_{G_{2}}$.

We now have our two parameters in operation, and finding egalitarian solutions will entail the simultaneous choice of values for $v$ and $k$.

Definition 13 Consider the Polarized Proposers Model. A pair $(k, v)$, such that $k \in$ $\{1, \ldots, a\}$ and $v \in\{1, \ldots, k\}$, is an egalitarian solution if $\left|E\left(u_{p}\left(R_{p}\right) \mid a, k, v\right)-E\left(u_{c}\left(R_{c}\right) \mid a, k, v\right)\right| \leq$ $\left|E\left(u_{p}\left(R_{p}\right) \mid a, k^{\prime}, v^{\prime}\right)-E\left(u_{c}\left(R_{c}\right) \mid a, k^{\prime}, v^{\prime}\right)\right|$ for every $k^{\prime} \in\{1, \ldots, a\}$ and $v^{\prime} \in\left\{1, \ldots, k^{\prime}\right\}$. We denote by $S_{e}$ the set of all values of $(k, v)$ that are egalitarian solutions.

Our next two propositions may be a bit tedious, but we still include them in order to show that one may compute exact values for expected utilities in our model, and use them to determine the egalitarian values $(v, k)$.

Proposition 6 Consider the Polarized Proposers Model with the majoritarian group's preferences as the tie breaking criterion. Suppose that the (common) preference of majoritarian proposers are randomly drawn from a uniform distribution over the set all rankings of candidates, and so is the preference of the chooser. Remember that the preferences of the minority of proposers will then be automatically specified to be the opposite of that of
the majority. Then, for any $v$-rule of $k$ names, the agents' expected utilities are given by the following expressions:

1. If $m \geq q_{k}^{v}>n-m$ :

$$
\begin{aligned}
& E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=-\frac{m}{n} \frac{(a+1)}{(a-k+2)}-\frac{n-m}{n} \frac{(a+1)(a-k+1)}{(a-k+2)} \\
& E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)=-\frac{(a-k+2)}{2}
\end{aligned}
$$

2. If $q_{k}^{v}>m \geq q_{1}^{v}>n-m:$

$$
\begin{aligned}
& E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=-\frac{m}{n} \frac{(k+1)}{2}-\frac{n-m}{n} \frac{(2 a-k+1)}{2} \\
& E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)=-\frac{(a+1)}{(k+1)}
\end{aligned}
$$

3. If $q_{k}^{v}>m>n-m \geq q_{1}^{v}$ :

$$
\begin{aligned}
& E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=-\frac{(a+1)}{2} \\
& E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)=-1
\end{aligned}
$$

Notice that the different cases in Proposition 6 arise because, in view of the size of the majorities, and the power assigned by the choice of $v$ and $k$ to the majority and the minority, equilibria will be differently characterized as shown by Proposition 6.

The following proposition gives a partial characterization of the egalitarian solution under the Polarized Proposers Model.

Remark 2 Notice that, ceteris paribus, the egalitarian value of $k$ for polarized societies is non increasing in $m$. This is due to Corollary 3.

Proposition 7 Consider the Polarized Proposers Model with the majoritarian group's preferences as the tie breaking criterion. Suppose that the (common) preference of majoritarian proposers are randomly drawn from a uniform distribution over the set all rankings of candidates, and so is the preference of the chooser. Remember that the preferences of the minority of proposers will then be automatically specified to be the opposite of that of the majority. Then, for any $v$-rule of $k$ names, we have that:
$S_{e} \subseteq S_{1} \cup S_{2}$.
where
$S_{1}=\left\{(k, v) \in\left\{\left\lfloor\tau_{1}\right\rfloor,\left\lceil\tau_{1}\right\rceil\right\} \times\left\{1, \ldots,\left\lceil\tau_{1}\right\rceil\right\} \mid m \geq q_{k}^{v}\right\}$
$S_{2}=\left\{(k, v) \in\left\{\left\lfloor\tau_{2}\right\rfloor,\left\lceil\tau_{2}\right\rceil\right\} \times\left\{1, \ldots,\left\lceil\tau_{2}\right\rceil\right\} \mid q_{k}^{v}>m \geq q_{1}^{v}>n-m\right\}$
$\tau_{1}=\frac{m}{n}\left(\frac{n}{m}+(a+1)-\sqrt{\frac{n}{m}\left(2-\frac{n}{m}\right)+(2 a+1)+a^{2}\left(\frac{n}{m}-1\right)^{2}}\right) ;$
$\tau_{2}=\frac{\frac{m}{n}}{2 \frac{m}{n}-1}\left((a-1)-a \frac{n}{m}+\sqrt{\frac{n}{m}\left(2-\frac{n}{m}\right)+(2 a+1)+a^{2}\left(\frac{n}{m}-1\right)^{2}}\right) .{ }^{16}$
These values arise from minimizing the expressions resulting from comparing the expected values for the chooser and the average proposer, as expressed in Proposition 6. For each inequality in Proposition 6, we obtain first the values of $k$ that minimize differences between the agents' expected utilities, then for each value of $k$ we find the values of $v$ 's that would be compatible with its corresponding inequality. In fact, we can ignore inequality 3 since it is dominated by the other inequalities (since if inequality 3 holds, the chooser would have all the power).

Armed with these explicit calculations, one can proceed to analyze specific cases. Without going into much detail, the following example provides the solution to the egalitarian power distribution choice of rules for a specific case. ${ }^{17}$

Example 5 Let $a=10, n=7, m=5$ and suppose that the agents's utilities are $u(r)=-r$. Applying Proposition 7, we have that $\tau_{1}=4.4633, \tau_{2}=1.919$. Hence, $\left.S_{1}=\{(4,4),(4,3),(5,5),(5,4)\}, S_{2}=\{(5,3)(1,1),(2,1))\right\}$ and the egalitarian solution is $S_{e}=\{(2,1)\}$. Now, consider $m=6$. Again, applying Proposition 7, we have that $S_{e}=$ $\{(6,6),(6,5),(6,4),(6,3),(6,2)\}$. Notice that as the size of the majority increases from 5 to 6 , the egalitarian $k$ increases from 2 to 6 . In the homogeneous proposers' case $(m=n)$, the set of egalitarian solutions would be $S_{e}=\{(7,7),(7,6),(7,5),(7,4),(7,3),(7,2),(7,1)\}$.

These examples suggest that, while the choice of $k$ is rather stringent, the values of $v$ that allow societies to reach an egalitarian distribution of power, given $k$, are not so precisely determined. This lack of full uniqueness is not surprising, nor bothersome, we believe, given that we are working in a setting with integer parameters. However, it suggests yet a final possible exercise, that we briefly discuss now.

[^9]Indeed, there are cases whether the constitutional detail provided by a planner stops short of fully specifying both parameters $k$ and $v$. In particular, if $k$ is exogenously fixed, one can still inquire what could be a power equalizing choice for $v$.

Here is an interesting example where this kind of problem arose in practice.
According to the Brazilian Constitution, one-third of the members of the Superior Court of Justice shall be chosen in equal parts among lawyers and members of the Public Prosecution. When there is position vacant assigned to be occupied by a lawyer, the constitution states the National Lawyer Association must propose six candidates to the court (so, $a=6$ ). Upon receiving the set of candidates, the court shall organize a list of three names and send it to the President of the Republic, who selects one of the listed names.

However, the constitution does not determine what screening rules should be used to screen the six initial names and then the three out of them. If we consider the list of six candidates as given, and concentrate on the choice of screening rule to determine the three to be sent to the President, it turns out that the Superior Court will be facing the question we just mentioned in the abstract. Given $k=3$, what $v$ should be used by the court to select the three names.?

It turns out that, in fact, a specific screening rule was decided upon and is now established in the bylaws of Superior Court.

On what grounds was this rule chosen? We cannot tell. But, can we rationalize the proposed rules through our analysis?

Since there are 33 ministers in the Superior Court of Justice, when there is a position vacant, the number of proposers of the three names is 32 . Therefore, we have the following parameters $n=32, a=6$ and $k=3$.

If we suppose that the assumptions of the Polarized Proposers Model hold and that $m=24$, the size of the majority group, what would be the value of $v$ that minimizes the absolute difference between the president's expected utility and the average of the ministers' expected utilities would be equal to three: that is, $v=k=3$, which is a majoritarian method. In fact, $v=k=3$ would be a egalitarian solution for any size of the majority group $(m)$. The rule actually chosen by the ministers was in fact a cumbersome sequential method, which however boils down, in terms of the implied equilibrium, to using the majoritarian $v$ rule with $v=k=3$, precisely the value that would correspond
to our calculations. Of course, we are not claiming that this was the reasoning underlying the choice of the screening rule that appears in the bylaws. But the example at least shows what kind of reasoning they could have adopted, and the use of our approach in selecting not only first best solutions, but also to perform second best analysis.

## 6 Concluding Remarks

Rules that contemplate several stages of choice are widely used. Some people are in charge of screening, then others choose among those candidates that were not screened out. We have concentrated in the case with only one chooser, because it is actually used in many cases, and also for simplicity, but hope to keep deepening our understanding of the advantages of each of the many forms in which societies divide their decision tasks.

In fact, as mentioned at the end of our introduction, the very idea to divide the tasks may arise from very diverse reasons. The one we have concentrated upon is to divide the decision power. This is in line with Arrowian tradition, where the interests of agents are taken as given, and the rules are methods to mitigate conflicts. But there is at least a second fundamental reason to subdivide decisions, this one based on common values, more in line of Condorcet Jury Theorem. This reason is to assign each agent to the partial decision that she is better informed about. When candidates can be judged on a multidimensional scale, different decision-makers in a team may contribute to a final choice by screening out candidates based on the dimension that they are better fit to judge. In this context, rules of $k$ names can be seen as methods to make proper use of expert advise.

Even within our present framework, we are aware that our normative analysis can be enriched by endowing agents with more complex preferences, considering a wider range of distributions over preference profiles, relaxing the full information assumption and/or considering alternative equilibrium concepts under maybe different specifications of the game they interact within.

Finally, let us re-emphasize that, even if widely used, $v$-rules of $k$ names are only one class among many others through people are eventually appointed. Given the power that comes attached with the possibility to appoint people to offices, we hope that these, along with other rules, can be systematically scrutinized and compared. We would like to think
of our work as part of this potential stream of work.

## References

[1] Alon, N., Fisher, F., Procaccia, A. and Tennenholtz, M. (2011) Sum of us: strategyproof selection from the selectors, in: Proc. 13th Conference on Theoretical Aspects of Rationality and Knowledge, pp. 101-110.
[2] Barberà, S., Sonnenschein, H. and Zhou, L. (1991) Voting by committees. Econometrica 59:595-609.
[3] Barberà, S., Bossert, W., Pattanaik, P. (2004) Ranking sets of objects. In: Barberà, S., Hammond, P. and Seidl, C.(Eds.) Handbook of Utility Theory, Volume II Extensions. Kluwer Academic Publishers 893-977
[4] Badger, W.W. (1972) Political individualism, positional preferences and optimal decision-rules. In: Niemi, R.G., Weisberg, H.F. (eds.) Probability Methods for Collective Decision Making. Merril Publishing, Columbus, Ohio.
[5] Barberà, S., Jackson, M. (2004) Choosing how to choose: Self-stable majority rules and constitutions. Quarterly Journal of Economics, 119 (3).
[6] Barberà, S., Coelho, D. (2010) On the rule of $k$ names. Games and Economic Behavior, 2010, vol. 70, p. 44-61.
[7] Barberà, S., Coelho, D. (2008) How to choose a non-controversial list with $k$ names. Social Choice and Welfare 31:79-96.
[8] Bernheim, D., Peleg, B. and Whinston, M (1987), Coalition-Proof Nash Equilibria I. Concepts. 42:1-12.
[9] Brams, S. J., and Merrill, S. (1986) Binding versus final-offer arbitration: A combination is best. Management Science 32:1346-1355.
[10] Coelho, D. (2005) Maximin choice of voting rules for committees. Econ. Gov. (2005) 6: 159-175
[11] Curtis, R. (1972) Decision rules collective values in constitutional choice. In: Niemi, R.G., Weisberg, H.F. (eds.) Probability Methods for Collective Decision Making. Merril Publishing, Columbus, Ohio.
[12] Gardner, R. (1977) The Borda Game. Public Choice, vol 30 (1):43-50.
[13] Gehrlein W. (1985) The Condorcet criterion and committee selection. Mathematical Social Science 10:199-209
[14] Ertemel, S., Kutlu, L. and Sanver, R. (2010). Voting Games of Resolute Social Choice Correspondence. Draft
[15] Kaymak, B. and Sanver, M. R. (2003) Sets of alternatives as Condorcet winners. Social Choice and Welfare 20:477-494.
[16] Moulin, H.(1982) "Voting with Proportional Veto Power," Econometrica, Econometric Society, vol. 50(1), pages 145-62, January.
[17] Moulin, H. (1988) Axioms of Cooperative Decision Making. Cambridge University Press, Cambridge.
[18] Holzman, J. and Moulin, H. (2011) Impartial nomination for a prize, mimeo.
[19] Mueller, D.: "Voting by Veto," Journal of Public Economics, 10(1978), 57-75.
[20] Peleg, B. (1984) Game Theoretic Analysis of Voting in Committees. Econometric Society Monographs in Pure Theory, Cambridge University Press, Cambridge.
[21] Polborn, M. and Messner, M. (2007) Strong and Coalition - Proof Political Equilibria under Plurality and Runoff Rule. International Journal of Game Theory, 35:287-314.
[22] Rae, D. (1969) Decision rules and individual values in constitutional choice. American Political Science Review 63: 40-56.
[23] Ratliff T (2003) Some startling inconsistencies when electing committees. Social Choice Welfare. 21:433-454
[24] Sertel, M. R., and Sanver M. R. (2004) Strong equilibrium outcomes of voting games are the generalized Condorcet winners. Social Choice and Welfare 22: 331-347.
[25] Stevens, C.M. (1966) Is compulsory arbitration compatible with bargaining. Industrial Relations 5:38-52.

## Appendix A

Proof of Proposition 1. Suppose that candidate $x$ is the outcome of a strong Nash equilibrium of the Constrained Chooser Game. In any strong Nash equilibrium where $x$ is the outcome, the screened set is such that $x$ is the best candidate in this set according to the chooser's preferences. This implies that $x$ is a chooser's $(a-k+1)$-top candidate. To prove that Condition 2 is necessary take any candidate $y \neq x$ among those that are chooser's $(a-k+1)$-top candidates and let $Y$ be any list with $k$ names where $y$ is the chooser's best candidate in $Y$. Notice that $y$ cannot be considered better than $x$ by any coalition with at least $q_{k}^{v}(Y)$ candidates. Otherwise, this coalition could impose $Y$, preventing $x$ from being elected. So, if $y$ is a chooser's $(a-k+1)$-top candidate, then $\#\left\{i \in N \mid y \succ_{i} x\right\}<q_{k}^{v}(Y)$ for any $Y \in A_{k}$ such that $y$ is the chooser's best candidate in $Y$.

Now, to justify Condition 3, suppose, by contradiction, that it is not true that $\#\{i \in$ $\left.N \mid y \succ_{i} x\right\} \geq q_{k}^{v}(y)$. Let $C_{1} \equiv\left\{i \in N \mid y \succ_{i} x\right\}$, so $\# C_{1} \geq q_{k}^{v}(y)$. Then, the coalition of proposers in $C_{1}$ would be able to impose the inclusion of $y$ in the list (since $\# C_{1} \geq q_{k}^{v}(y)$ ), and the chooser would select it instead of $x$. Hence, if $y$ the chooser's best candidate, we have that $\#\left\{i \in N \mid y \succ_{i} x\right\}<q_{k}^{v}(y)$.
Finally, consider Condition 4. Let $y$ be the chooser's best candidate, and assume that it is ranked above $x$ by the tie breaking criterion. Suppose, by contradiction, that it is not true that $\#\left\{i \in N \mid x \succ_{i} y\right\} \geq q_{k}^{v}$. Hence, at any strategy profile that includes $x$ in the selected list, the coalition $C_{1} \equiv\left\{i \in N \mid y \succ_{i} x\right\}$ can find a profitable deviation to include $y$, becomes the winning candidate. Therefore, $x$ cannot be a strong Nash equilibrium outcome.

Proof of Proposition 2. Consider any $v$-rule of $k$ names and suppose that $x$ is the proposers‘ strong Condorcet winner, as well as the chooser's best candidate. First let us show that there is a strategy profile that sustains $x$ as a strong Nash equilibrium outcome. Consider the strategy profile, where all proposers vote for $x$. Notice that $x$ will be in the chosen list.

Then, candidate $x$ will be elected since he will be in the list and he is the chooser's top
candidate. The only way to change this result is to avoid the inclusion of $x$ in the chosen list. But any coalitions with size smaller than $\frac{n+1}{2}$ cannot avoid the inclusion of $x$ in the chosen list, because the complementary coalition has size higher than $\frac{n+1}{2}$ and all the proposers are voting for $x$. Notice that no coalition with size higher or equal to $\frac{n+1}{2}$ will have any incentive to deviate, since there is no $y \in A \backslash\{x\}$ that is considered better than $x$ by all proposers in the coalition (recall that $x$ is the Proposers' strong Condorcet winner). Therefore, this strategy profile is a strong Nash equilibrium of the Constrained Chooser Game.
Now let us show that $x$ is the unique strong Nash equilibrium outcome. Suppose, by contradiction, that there is a strategy profile that sustains $y \in A \backslash\{x\}$ as a strong Nash equilibrium outcome. By Condition 3 of Proposition $1, \#\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(\{x\})$. Notice that $q_{k}^{v}(\{x\}) \leq \frac{n+1}{2}$. Hence, $\#\left\{i \in N \mid x \succ_{i} y\right\}<\frac{n+1}{2}$. It is a contradiction since $x$ is the proposers' Condorcet winner.

## Proof of Proposition 3.

1) Suppose that the tie breaking criterion coincides with the majoritarian group's preferences over the set of candidates.
1.1) Consider $m \geq q_{k}^{v}$.

Let $x$ be the best alternative of the majoritarian group out of the chooser's $(a-k+1)$-top candidates. Since $m \geq q_{k}^{v}$, and by definition of $q_{k}^{v}$, there is a strategy profile that can be adopted by the majoritarian group that leads to the election of $x$, and the minoritarian group is unable to change it. Notice also that the majoritarian group will not have any incentive in changing this outcome. Therefore, there exists a strategy profile that sustains $x$ as a strong Nash equilibrium outcome.
Now let us show that $x$ is the unique strong Nash equilibrium outcome. Suppose, by contradiction that there is another strong Nash equilibrium outcome $y \neq x$. By Condition 2 of Proposition 1, we have that $\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(X)$ where $x$ is the chooser's best alternative in $X$. This is a contradiction since $\left\{i \in N \mid x \succ_{i} y\right\}>m>q_{k}^{v}>q_{k}^{v}(X)$.
1.2) Consider $q_{k}^{v}>m \geq q_{1}^{v}>n-m$.

Let $x$ be the chooser's best alternative out of the majoritarian group's $k$-top candidates. Let $X$ be the set of $k$-top candidates for the majoritarian group's. We first show that there exists a strategy profile that sustains $x$ as an equilibrium outcome. Notice that $q_{k}^{v}>m \geq q_{1}^{v}>n-m$ implies that $m \geq q_{k}^{v}(X)$ and $n-m \geq q_{k}^{v}(\{x\})$. Consider the fol-
lowing strategy profile: the majoritarian group adopts a strategy profile that can allows it to impose the list $X$ and the minoritarian group adopts a strategy profile that allows it to impose $x$ in the list. In order to change the outcome, one of the groups could try to block the inclusion of $x$, but neither of them alone can do it. Notice also that only the majoritarian group would be able to include another candidate better than $x$ in the list sent to the chooser. But this candidate would be worse than $x$ for the majoritarian group. Therefore, there exists no coalition of proposers that has an incentive to deviate. Thus, we have proved that there exists a strategy profile that sustains $x$ as an equilibrium outcome.
Now we shall prove that $x$ is the unique strong Nash equilibrium outcome. By contradiction, suppose that there is another strong Nash equilibrium outcome $y \neq x$. By Condition 1 of Proposition 1, $x$ is among the chooser's $(a-k+1)$-top candidates. By Condition 2 of Proposition 1, we have that $\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(X)$. This is a contradiction since $\left\{i \in N \mid x \succ_{i} y\right\}>m \geq q_{k}^{v}(X)$.
1.3) Consider $q_{k}^{v}>m>n-m \geq q_{1}^{v}$.

Let $x$ be the chooser's best candidate. First let us show that there exists a strategy profile that sustains $x$ as an equilibrium outcome. Notice that $n-m \geq q_{1}^{v}$ implies that $m>n-m>q_{v}^{v}(\{x\})$. Consider the following strategy profile: every proposer casts a vote for $x$. Thus, $x$ will be in the selected list and it will be elected. No group can take $x$ out from the selected list by a unilateral deviation, since both have size larger than $q_{v}^{v}(\{x\})$. Since both group has the reverse preference profile of the other, they do not have incentive to jointly deviate from this strategy profile. Therefore, this strategy profile sustains $x$ as an strong Nash equilibrium outcome.
Now let us prove that $x$ is the unique strong Nash equilibrium outcome. By contradiction, suppose that there is another strong Nash equilibrium outcome $y \neq x$. By Condition 3 of Proposition 1, we have that $\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(\{x\})$. This is a contradiction since $m>n-m \geq q_{1}^{v}(\{x\})$.
2) Suppose that the tie breaking criterion coincides with the chooser's preferences over the set of candidates.
2.1) Consider $m \geq q_{k}^{v}$.

Let $x$ be the best alternative of the majoritarian group out of the chooser's $(a-k+1)$-top
candidates. Since $m \geq q_{k}^{v}$, and by definition of $q_{k}^{v}$, there is a strategy profile that can be adopted by the majoritarian group that leads to the election of $x$, and the minoritarian group is unable to change it. Notice also that the majoritarian group will not have any incentive in changing this outcome. Therefore, there exists a strategy profile that sustains $x$ as a strong Nash equilibrium outcome.
Now let us show that $x$ is the unique strong Nash equilibrium outcome. Suppose, by contradiction that there is another strong Nash equilibrium outcome $y \neq x$. By Condition 2 of Proposition 1, we have that $\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(X)$ where $x$ is the chooser's best alternative in $X$. This is a contradiction since $\left\{i \in N \mid x \succ_{i} y\right\}>m>q_{k}^{v}>q_{k}^{v}(X)$. 2.2) Consider $q_{k}^{v}>m$.

Let $x$ be the chooser's best candidate. Notice that $q_{k}^{v}>m$ implies that $n-m \geq$ $q_{k}^{v}(\{x\})$.Suppose the following strategy profile: every proposer cast a vote for $x$. Thus, $x$ will be in the selected list and it will be elected. No group can take $x$ from the selected list by a unilateral deviation, since both has size larger than $q_{v}^{v}(\{x\})$. Since both group have the reverse preference profile than the others, they do not have incentive to joint deviate from this strategy profile. Therefore, this strategy profile sustains $x$ as strong Nash equilibrium outcome.
Now let us prove that $x$ is the unique strong Nash equilibrium outcome. By contradiction, suppose that there exists another strong Nash equilibrium outcome $y \neq x$. By Condition 3 of Proposition 1, we have that $\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(\{x\})$. This is a contradiction since $m>n-m \geq q_{1}^{v}(\{x\})$.
3) Suppose that the tie breaking criterion coincides with the minoritarian group's preferences over the set of candidates.
3.1) Consider $m \geq q_{k}^{v}$.

Let $x$ be the best alternative of the majoritarian group out of the chooser's $(a-k+1)$-top candidates. Since $m \geq q_{k}^{v}$, and by definition of $q_{k}^{v}$, there is a strategy profile that can be adopted by the majoritarian group that leads to the election of $x$, and the minoritarian group is unable to change it. Notice also that the majoritarian group will not have any incentive in changing this outcome. Therefore, there exists a strategy profile that sustains $x$ as a strong Nash equilibrium outcome.
Now let us show that $x$ is the unique strong Nash equilibrium outcome. Suppose, by con-
tradiction that there is another strong Nash equilibrium outcome $y \neq x$. By Condition 2 of Proposition 1, we have that $\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(X)$ where $x$ is the chooser's best alternative in $X$. This is a contradiction since $\left\{i \in N \mid x \succ_{i} y\right\}>m>q_{k}^{v}>q_{k}^{v}(X)$.

## 3.2) Consider $q_{k}^{v}>m$.

Let $x$ be the chooser's best candidate. Consider the following strategy profile: every proposer casts a vote for $x$. Thus, $x$ will be in the selected list and it will be elected. Notice that the minoritarian group cannot take $x$ out from the selected list by a unilateral deviation since $m \geq q_{v}^{v}(\{x\})$. If some $y$ is better than $x$ for the majoritarian group, then it will be ranked below than $x$ by the tie breaking criterion. Thus, the majoritarian group cannot deviate in any way that simultaneously includes $y$ and excludes $x$ from the selected list. Since both groups have the reverse preferences, they do not have incentive to jointly deviate from this strategy profile. Therefore, this strategy profile sustains $x$ as an strong Nash equilibrium outcome. Now let us prove that $x$ is the unique strong Nash equilibrium outcome. By contradiction, suppose that there is $y \neq x$ that is also a strong Nash equilibrium outcome. Suppose that the minoritarian group of proposers prefers $y$ to $x$. By Condition 3 of Proposition 1, we have that $\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(x)$ which implies that $m<q_{k}^{v}(x)$. This is a contradiction, since $m \geq q_{1}^{v}(\{x\})$. Suppose that the majoritarian group of proposers prefers $y$ to $x$. Thus $x$ is ranked above than $y$ according by the tie breaking criterion. By Condition 4 of Proposition 1, we have that $\left\{i \in N \mid y \succ_{i} x\right\} \geq q_{k}^{v}$, which implies that $m \geq q_{k}^{v}$. This is a contradiction since $q_{k}^{v}>m$. Therefore, $y$ cannot be strong Nash equilibrium outcome.

Proof of Proposition 5. Proposition 5 is a direct consequence of lemmas 1-4.
Lemma 1 Let $a-\sqrt{2 a+2}+2$ be an integer number. If $k=a+2-\sqrt{2 a+2}$ then $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)=E\left(u_{c}\left(r_{c}\right) \mid a, k\right)$.

Proof of Lemma 1. First notice that for every $k$ we have that:
$E\left(u_{p}\left(r_{p}\right) \mid a, k\right) E\left(u_{c}\left(r_{c}\right) \mid a, k\right)=\frac{a+1}{2}$
Take any $k^{*} \in[1, a]$ such that $E\left(u_{p}\left(r_{p}\right) \mid a, k^{*}\right)=E\left(u_{c}\left(r_{c}\right) \mid a, k^{*}\right)$. Thus,
$E\left(u_{c}\left(r_{c}\right) \mid a, k^{*}\right)^{2}=\frac{a+1}{2}$
$E\left(u_{c}\left(r_{c}\right) \mid a, k^{*}\right)=\frac{a-k^{*}+2}{2}=\sqrt[2]{\frac{a+1}{2}}$
Therefore, $k^{*}=a+2-\sqrt[2]{2 a+2}$.

Lemma $2 A k \in\{1, \ldots a\}$ maximizes $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)$ if and only if it minimizes $\left.\mid E\left(u_{p}\left(r_{p}\right) \mid a, k\right)-E\left(u_{c}\left(r_{c}\right) \mid a, k\right)\right) \mid$.

Proof of Lemma 2. First notice that for every $k$ we have that:
$E\left(u_{p}\left(r_{p}\right) \mid a, k\right) E\left(u_{c}\left(r_{c}\right) \mid a, k\right)=\frac{a+1}{2}$
The equality above implies that
$\left(E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)\right)^{2}=E\left(u_{p}\left(r_{p}\right) \mid a, k\right)^{2}+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)^{2}+(a+1)$
The expression above implies that, given that $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)<0$, a $k \in$ $\{1, \ldots a\}$ maximizes $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)$ if and only if it minimizes $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)^{2}+$ $E\left(u_{c}\left(r_{c}\right) \mid a, k\right)^{2}$.
Notice also that:
$\left(E\left(u_{p}\left(r_{p}\right) \mid a, k\right)-E\left(u_{c}\left(r_{c}\right) \mid a, k\right)\right)^{2}=E\left(u_{p}\left(r_{p}\right) \mid a, k\right)^{2}+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)^{2}-(a+1)$.
The expression above implies that a $k \in\{1, \ldots a\}$ maximizes $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)^{2}+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)^{2}$ if and only if it maximizes $\left(E\left(u_{p}\left(r_{p}\right) \mid a, k\right)-E\left(u_{c}\left(r_{c}\right) \mid a, k\right)\right)^{2}$.
Therefore, a $k \in\{1, \ldots a\}$ maximizes $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)$ if and only if minimizes $\left.\mid E\left(u_{p}\left(r_{p}\right) \mid a, k\right)-E\left(u_{c}\left(r_{c}\right) \mid a, k\right)\right) \mid$.

Lemma $3 A k \in\{1, \ldots a\}$ maximizes $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)$ if and only if it also maximizes $\left(E\left(u_{p}\left(r_{p}\right) \mid a, k\right)-d\right)\left(E\left(u_{c}\left(r_{c}\right) \mid a, k\right)-d\right)$ where $d<0$.

Proof of Lemma 3. First notice that for every $k$ we have that:
$E\left(u_{p}\left(r_{p}\right) \mid a, k\right) E\left(u_{c}\left(r_{c}\right) \mid a, k\right)=\frac{a+1}{2}$.
Thus, $\left(E\left(u_{p}\left(r_{p}\right) \mid a, k\right)-d\right)\left(E\left(u_{c}\left(r_{c}\right) \mid a, k\right)-d\right)=\frac{a+1}{2}+d^{2}-d\left(E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)\right)$
Given that $d<0$, the expression above implies that $k$ maximizes $E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)$ if and only if it maximizes $\left(E\left(u_{p}\left(r_{p}\right) \mid a, k\right)-d\right)\left(E\left(u_{c}\left(r_{c}\right) \mid a, k\right)-d\right)$.

Lemma 4 Consider any a:

1) $E\left(u_{p}\left(r_{p}\right) \mid k, a\right)+E\left(u_{c}\left(r_{c}\right) \mid k, a\right)>E\left(u_{p}\left(r_{p}\right) \mid k-1, a\right)+E\left(u_{c}\left(r_{c}\right) \mid k-1, a\right)$ for every $k<$ $a+\frac{5}{2}-\sqrt{2 a+\frac{9}{4}}$;
2) $E\left(u_{p}\left(r_{p}\right) \mid k, a\right)+E\left(u_{c}\left(r_{c}\right) \mid k, a\right)=E\left(u_{p}\left(r_{p}\right) \mid k-1, a\right)+E\left(u_{c}\left(r_{c}\right) \mid k-1\right.$, a) if $k=a+\frac{5}{2}-$ $\sqrt{2 a+\frac{9}{4}}$;
3) $E\left(u_{p}\left(r_{p}\right) \mid k, a\right)+E\left(u_{c}\left(r_{c}\right) \mid k, a\right)<E\left(u_{p}\left(r_{p}\right) \mid k-1, a\right)+E\left(u_{c}\left(r_{c}\right) \mid k-1, a\right)$ for every $k>$ $a+\frac{5}{2}-\sqrt{2 a+\frac{9}{4}}$.

Proof of Lemma 4. For every $k \in\{2, . . a\}$ we have the following equality:
$E\left(u_{p}\left(r_{p}\right) \mid a, k\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k\right)-\left(E\left(u_{p}\left(r_{p}\right) \mid a, k-1\right)+E\left(u_{c}\left(r_{c}\right) \mid a, k-1\right)\right)=\frac{a+1}{(a-k+2)(a-k+3)}-\frac{1}{2}$. Notice $\frac{a+1}{(a-k+2)(a-k+3)}$ is decreasing with $k$ and $\frac{a+1}{(a-k+2)(a-k+3)}=\frac{1}{2}$ when $k=a-\frac{1}{2} \sqrt{8 a+9}+$ $\frac{5}{2}$. Let $P(k)=\frac{a+1}{(a-k+2)(a-k+3)}-\frac{1}{2}$ and $k^{*}=\frac{a+1}{(a-k+2)(a-k+3)}-\frac{1}{2}$. Thus, $P\left(k^{*}\right)=0, P(k)>0$ for any $k<k^{*}$ and $P(k)<0$ for any $k>k^{*}$.

Proof of Proposition 6. Proposition 6 is a direct consequence of Proposition 5 and the assumption that agent's preferences are randomly drawn from a uniform distribution over the domain of preferences.

1) Consider $m \geq q_{k}$. By Proposition 5, the equilibrium outcome is the best alternative of the majoritarian group out of the chooser's $(a-k+1)$-top candidates. Thus, $R_{c}$ has the same distribution than a discrete random variable uniformly distributed over $\{1,2, \ldots, a-k+1\} . R_{G 1}$ has the same distribution as the smallest element of a random sample with size $s=a-k+1$ drawn without replacement from a uniformly distributed population $D=\{1,2, \ldots, a\}$. $R_{G 2}$ has the same distribution as the biggest element of a random sample with size $s=a-k+1$ drawn without replacement from a uniformly distributed population. Therefore, we have:
$E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)=-\left(\frac{a-k+2}{2}\right)$
$E\left(u_{G 1}\left(r_{G 1}\right) \mid a, k, v\right)=-\frac{(a+1)}{(a-k+2)}$
$E\left(u_{G 2}\left(r_{G 2}\right) \mid a, k, v\right)=-\frac{(a+1)(a-k+1)}{(a-k+2)}$
$E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=-\frac{m}{n} \frac{(a+1)}{(a-k+2)}-\frac{n-m}{n} \frac{(a+1)(a-k+1)}{(a-k+2)}$,
2) Consider $q_{k}^{v}>m \geq q_{1}^{v}>n-m$. By Proposition 5, the equilibrium outcome is the chooser's best alternative out of the majoritarian group's $k$-top candidates. Thus, $R_{c}$ has the same distribution as the smallest element of a random sample with size $s=k$ drawn without replacement from a uniformly distributed population $D=\{1,2, \ldots, a\}$. $R_{G 1}$ has the same distribution than a discrete random variable uniformly distributed over $\{1,2, \ldots, k\} . R_{G 2}$ has the same distribution than a discrete random variable uniformly distributed over $\{a-k+1, \ldots, a\}$. Therefore, we have:
$E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)=-\frac{(a+1)}{(k+1)}$
$E\left(u_{G 1}\left(r_{G 1}\right) \mid a, k, v\right)=-\frac{(k+1)}{2}$
$E\left(u_{G 2}\left(r_{G 2}\right) \mid a, k, v\right)=-\frac{(2 a-k+1)}{2}$
$E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=-\frac{m}{n} \frac{(k+1)}{2}-\frac{n-m}{n} \frac{(2 a-k+1)}{2}$
3) Consider $q_{k}^{v}>m>n-m \geq q_{1}^{v}$. By Proposition 5, the equilibrium outcome is the chooser's best candidate. Thus, $R_{c}$ is a constant and it is equal to $1 . R_{G 1}$ and $R_{G 2}$ have the same distribution than a discrete random variable uniformly distributed over $\{1,2, \ldots, a\}$.
$E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)=-1$
$E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=-\frac{(a+1)}{2}$.
Proof of Proposition 7. Proposition 7 is a direct consequence of Proposition 6 and lemmas 5 and 6 below.

Lemma 5 In the domain of all pairs $(k, v)$ such that $m \geq q_{k}>n-m$, we have that:

1) $E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)>E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)$ for every $k<\tau_{1}$;
2) $E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)$ if $k=\tau_{1}$ is an integer number;
3) $E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)<E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)$ for every $k>\tau_{1}$.
where $\tau_{1}=\frac{m}{n}\left(\frac{n}{m}+(a+1)-\sqrt{\frac{n}{m}\left(2-\frac{n}{m}\right)+(2 a+1)+a^{2}\left(\frac{n}{m}-1\right)^{2}}\right)$

Proof of Lemma 5. Given that $m \geq q_{k}>n-m$, by Proposition 6, $E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=$ $-\frac{m}{n} \frac{(a+1)}{(a-k+2)}-\frac{n-m}{n} \frac{(a+1)(a-k+1)}{(a-k+2)}$,
and $E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)=-\left(\frac{a-k+2}{2}\right)$.
Notice that:
$\left|E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)-E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)\right|$ is single dipped and reaches the minimum when $k=\tau_{1}$. When $k=\tau_{1}$, we have that: $\left|E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)-E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)\right|=0$.

Lemma 6 In the domain of all pairs $(k, v)$ such that $q_{k}>m \geq q_{1}>n-m$, we have that:

1) $E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)>E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)$ for every $k<\tau_{2}$;
2) $E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)$ if $k=\tau_{2}$ is an integer number;
3) $E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)<E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)$ for every $k>\tau_{2}$.
where $\tau_{2}=\frac{\frac{m}{n}}{2 \frac{m}{n}-1}\left((a-1)-a \frac{n}{m}+\sqrt{\frac{n}{m}\left(2-\frac{n}{m}\right)+(2 a+1)+a^{2}\left(\frac{n}{m}-1\right)^{2}}\right)$

Proof of Lemma 6. Given that $q_{k}>m \geq q_{1}>n-m$, by Proposition 6, we have that $E\left(u_{p}\left(r_{p}\right) \mid a, k, v\right)=-\frac{m}{n} \frac{(k+1)}{2}-\frac{n-m}{n} \frac{(2 a-k+1)}{2}$
$E\left(u_{c}\left(r_{c}\right) \mid a, k, v\right)=-\frac{(a+1)}{(k+1)}$.
Notice that:
$\left|E\left(R_{p} \mid k, v, a\right)-E\left(R_{c} \mid k, v, a\right)\right|$ is single dipped and reaches the minimum when $k=\tau_{2}$. When $k=\tau_{2}$, we have that $\left|E\left(R_{p} \mid k, v, a\right)-E\left(R_{c} \mid k, v, a\right)\right|=0$.

## Appendix B

Remark 3 Consider any $v$-screening rule for $k$ names and any $x \in A$, if $x$ is one of the $k$-top candidates according to the tie breaking criterion then $q_{k}^{v}(\{x\})=q_{1}^{v}-1$ or $q_{k}^{v}(\{x\})=q_{1}^{v}$. If $x$ is not one of $k$-top candidates according to the tie breaking criterion then $q_{k}^{v}(\{x\})=q_{1}^{v}$.

Remark 4 Consider any v-screening rule for $k$ names and any $X \in A_{k}$, if the set $X$ is formed by the $k$-top candidates according to the tie breaking criterion then $q_{k}^{v}(X)=q_{k}^{v}-1$ or $q_{k}^{v}(X)=q_{k}^{v}$. If $X$ is not formed by the $k$-top candidates according to the tie breaking criterion then $q_{k}^{v}(X)=q_{k}^{v}$.

Proposition 8 If a screening rule for $k$ names is a $v$-screening rule for $k$ names then $q_{k}^{v}=\left\lceil\frac{k n}{(k+v)}\right\rceil+\mathcal{I}\left(\left\lfloor\frac{v\left\lceil\frac{k n}{(k+v)}\right\rceil}{k}\right\rfloor \leq n-\left\lceil\left\lceil\frac{k n}{(k+v)}\right\rceil\right)\right.$ and $q_{1}^{v}=\left\lceil\frac{v n}{(k+v)}\right\rceil+\mathcal{I}\left(\frac{v n}{(k+v)}=\left\lceil\frac{v n}{(k+v)}\right\rceil\right)$, where $\mathcal{I}$ denotes the indicator function. ${ }^{18}$

Proof . Take any coalition $Q \subseteq N$ with $\#|Q|=q$ and any subset of candidates $B \subseteq A$ with $\#|B|=k$. Suppose that the members of coalition $Q$ coordinate their votes in order to elect $B$. The worst scenario is the one where all members of the complementary coalition $N \backslash Q$ vote together for some $x \in A / B$, and $x$ receives $n-q$ votes. Given this worst scenario, the best that coalition $Q$ can do to ensure the selection of $B$ is to spread equally, as much as possible, their $v \cdot q$ votes among the $k$ candidates in $B$. Though this strategy, the number of votes that any candidate in $B$ receives is $\left\lfloor\frac{q v}{k}\right\rfloor$ or $\frac{q v}{k}$. Thus, the set $B$ will be elected if $\left\lfloor\frac{q v}{k}\right\rfloor>n-q$. By definition, $q_{k}^{v}$ is the minimum $q$ that the following inequality holds: $\left\lfloor\frac{q v}{k}\right\rfloor>n-q$. It implies that $q_{k}^{v}=\left\lceil\frac{k n}{(k+v)}\right\rceil+\mathcal{I}\left(\left\lfloor\frac{v\left\lceil\frac{k n}{(k+v)}\right.}{k}\right\rfloor \leq n-\left\lceil\frac{k n}{(k+v)}\right\rceil\right)$, where $\mathcal{I}$ denotes the indicator function. Thus the first part of the proposition is established Take any coalition $Q \subseteq N$ with $\#|Q|=q$ and any candidate $x \in A$. Suppose that the members of coalition $Q$ coordinate their votes in order to ensure that $x$ is one of the $k$ selected names. The worst scenario is the one where the members of the complementary coalition $N \backslash Q$ distribute their $v(n-q)$ votes as equally as possible, among some set $B \subseteq A /\{a\}$ with $\#|B|=k$. This implies that at least one candidate in $B$ receives $\left\lfloor\frac{(n-q) v}{k}\right\rfloor$ votes. Given this worst scenario, the best response of coalition $Q$ to ensure the inclusion of $x$ in the list is to have all its members vote for $x$. Given this strategy, a

[^10]candidate $x$ will receives $q$ votes. Thus, $x$ will be one of the $k$ listed names if $q>\left\lfloor\frac{(n-q) v}{k}\right\rfloor$. By definition, $q_{1}^{v}$ is the minimum $q$ for which this inequality holds. This implies that $q_{1}^{v}=\left\lceil\frac{v n}{(k+v)}\right\rceil+\mathcal{I}\left(\frac{v n}{(k+v)}=\left\lceil\frac{v n}{(k+v)}\right\rceil\right)$, where $\mathcal{I}$ denotes the indicator function.

## Appendix C1

The three propositions in this appendix provide different sufficient conditions on the distribution of preferences guaranteeing that a candidate will be the unique strong Nash equilibrium outcome.

Proposition 9 Consider a v-rule of $k$ names. If candidate $x$ satisfies the two conditions below then it is the unique strong Nash equilibrium outcome of the Constrained Chooser Game:

1. It is a chooser's $(a-k+1)$-top candidate.
2. There exists $X \in A_{k}$ such that $x$ is the chooser's best candidate and there exist $q_{k}^{v}(X)$ proposers that rank $x$ first.

Proof . We must show that there is a strategy profile that sustains $x$ as a strong Nash equilibrium outcome. Let $C \subset N$ be the set of proposers that rank $x$ first, so $\# C \geq q_{k}^{v}(X)$. By definition of $q_{k}^{v}(Y)$, there exists $m_{C} \in M^{C}$ such that for every profile of the complementary coalition $m_{\mathbf{N} \backslash C} \in M^{\mathbf{N} \backslash C}$ we have that $S_{k}\left(m_{C}, m_{\mathbf{N} \backslash C}\right)=X$. Consider any strategy profile, where the coalition $C$ uses $m_{C}$. At this strategy profile, $X$ will be selected and $x$ will be the winning candidate independently of the actions of the complementary coalition. Thus, there is no coalition of proposers that has incentives to deviate. Therefore, this strategy profile sustains $x$ as a strong Nash equilibrium outcome. Now let us show that $x$ is the unique strong Nash equilibrium outcome. Suppose by contradiction that there is a strategy profile that sustains $y \in A \backslash\{x\}$ as a strong Nash equilibrium outcome. By Proposition 1, this implies $\#\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(X)$, and $y$ is a chooser's $(a-k+1)$-top candidate. This is a contradiction since $x$ is a $q_{k}^{v}(X)$-Condorcet winner over the set of the chooser's $(a-k+1)$-top candidates.

Proposition 10 Consider a v-rule of $k$ names. If candidate $x$ satisfies the two conditions below then it is the unique strong Nash equilibrium outcome of the Constrained Chooser Game:

1. It is a chooser's $(a-k+1)$-top candidate.
2. If $y$ is a chooser's $(a-k+1)$-top candidate then $\#\left\{i \in N \mid y \succ_{i} x\right\}<\left\lfloor\frac{n v}{2 k}\right\rfloor$.

Proof. Let us show that there is a strategy profile that sustains $x$ as a strong Nash equilibrium outcome. Take any set $B \subseteq A$ with $\#|B|=k$, where $x$ is chooser' best candidate in the set (this set exists because $x$ is a chooser's $(a-k+1)$-top candidate). Consider a strategy profile where each candidate in $B$ receives at least $\left\lfloor\frac{n v}{k}\right\rfloor$ votes and all the candidates in $A \backslash B$ receive zero votes. Notice that the candidates in $B$ will form the chosen list. Then, candidate $x$ will be elected, since he is the best candidate for the chooser in the list. In order to change this result, the only way is to avoid the inclusion of $x$ in the list or to substitute another listed name by a candidate that the chooser would prefer to $x$. A necessary condition to make this change would be to transfer at least $\left\lfloor\frac{n v}{2 k}\right\rfloor$ votes of a candidate in $B$ to another candidate in $A /\{B\}$. Hence, no coalition with size smaller than $\left\lfloor\frac{n v}{2 k}\right\rfloor$ can avoid the inclusion of $x$ in the chosen list. Notice that no coalition with size higher or equal to $\left\lfloor\frac{n v}{2 k}\right\rfloor$ has incentive to deviate, since there is no $y \in A \backslash\{x\}$ among the chooser's $(a-k+1)$-top candidates such that $\#\left\{i \in N \mid y \succ_{i} x\right\} \geq\left\lfloor\frac{n v}{2 k}\right\rfloor$ (recall that only the chooser's $(a-k+1)$-top candidates can be the chooser's best name among the candidates of a set with cardinality $k$ ). Therefore, this strategy profile is a strong Nash equilibrium of the Constrained Chooser Game.

Now let us show that $x$ is the unique strong Nash equilibrium outcome. First notice that $n-\left\lfloor\frac{n v}{2 k}\right\rfloor+1 \geq q_{k}^{v}$ (Because any coalition with size higher than $n-\left\lfloor\frac{n v}{2 k}\right\rfloor+1$ can impose all the $k$ names in the list), so if $y$ is a chooser's $(a-k+1)$-top candidate then $\#\left\{i \in N \mid x \succ_{i} y\right\} \geq q_{k}^{v}$. Suppose by contradiction that there is a strategy profile that sustains $y \in A \backslash\{x\}$ as strong Nash equilibrium outcome. By Proposition 1, this implies that $\#\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}$ and $y$ is a chooser's $(a-k+1)$-top candidate, a contradiction.

Proposition 11 Consider a $v$-rule of $k$ names. If candidate $x$ satisfies the two conditions below, then it is the unique strong Nash equilibrium outcome of the Constrained Chooser Game:

1. It is the chooser's best candidate.
2. If $y$ is a chooser's $(a-k+1)$-top candidate then $\#\left\{i \in N \mid x \succ_{i} y\right\} \geq q_{k}^{v}(\{x\})$.

Proof. First let us show that there is a strategy profile that sustains $x$ as a strong Nash equilibrium outcome. Consider the strategy profile where all proposers votes for $x$.

Notice that $x$ will be in the chosen list.
Then, candidate $x$ will be elected since he will be in the list and he is the chooser's top candidate. The only way to change this result is to avoid the inclusion of $x$ in the chosen list. So, no coalition with size smaller than $n-q_{k}^{v}(\{x\})$ can avoid the inclusion of $x$ in the chosen list, because the complementary coalition would have size higher than $q_{k}^{v}(\{x\})$. Notice that no coalition with size higher or equal to $n-q_{k}^{v}(\{x\})+1$ has incentive to deviate, since there is no $y \in A \backslash\{x\}$ among the chooser's $(a-k+1)$-top candidates that is considered better than $x$ by all proposers in the coalition (recall that only the chooser's ( $a-k+1$ )-top candidates can be the chooser' best name among the candidates of a set with cardinality $k$ ). Therefore, this strategy profile is a strong Nash equilibrium of the Constrained Chooser Game.
Now let us show that $x$ is the unique strong Nash equilibrium outcome. Suppose by contradiction that there is a strategy profile that sustains $y \in A \backslash\{x\}$ as strong Nash equilibrium outcome. By Proposition 1, this implies that $y$ is a chooser's ( $a-k+1$ )-top candidate and $\#\left\{i \in N \mid x \succ_{i} y\right\}<q_{k}^{v}(\{x\})$, a contradiction.

Proposition 12 If the chooser's best candidate is a strong Nash equilibrium outcome of the Constrained Chooser Game under a $v^{\prime}$-rule of $k$ names, it is also a strong Nash equilibrium outcome of the Constrained Chooser Game for any $\widetilde{v}-$ rule of $k$ names whenever $\widetilde{v}<v^{\prime}$ (provided that both rules share the same tie breaking criterion).

Proof . First notice that since the chooser's best candidate is a strong Nash equilibrium outcome under a $v^{\prime}$-votes screening rule for $k$ names, it implies that any strategy profile where all proposers vote for $x$ is a strong Nash equilibrium.
Take any strategy profile where all voters vote for $x$, and call it $m^{\prime}$. Since it is a strong Nash equilibrium, no coalition of voters that can make any profitable deviation. The voters that would wish to avoid the election of $x$ are those that prefer another of the chooser's $(a-k+1)$-top candidates to $x$ (recall that only the chooser's $(a-k+1)$-top candidates can be the chooser' best name among the candidates of a set with cardinality $k$ ).The only way to avoid the election of $x$ would be to avoid the inclusion of $x$ in the chosen list. Take any chooser's $(a-k+1)$-top candidate and call it $y$. Even if all the voters that prefer $y$ to $x$ deviate from $m^{\prime}$ by not voting for $x, x$ would still have enough votes to be among the $k$ listed names. Otherwise, the strategy profile where all the voters
vote for $x$ would not be a strong Nash equilibrium.
Now let us show that $x$ is also a strong Nash equilibrium under any $\widetilde{v}$-votes screening rule for $k$ names for any $\widetilde{v}<v$.
Take any strategy profile where all voters vote for $x$ and call this strategy $\widetilde{m}$. So, $x$ will be one of $k$ listed names, and it will be the elected candidate. We need to show that there is no coalition of voters that can make a profitable deviation under $m^{\prime}$. Given $m^{\prime}$ and $\widetilde{m}$, notice that it is more difficult to make a profitable deviation under a $\widetilde{v}$-rule of $k$ names than $v^{\prime}$-rule of $k$ names. Because, under a $\widetilde{v}$-rule of $k$ names, any coalition of voters that would have incentive to avoid the election of $x$ has less votes to distribute among the $k$ candidates in order to avoid the inclusion of $x$ in the list. Thus, given that there exists no coalition that can make a profitable deviation under $m^{\prime}$, there exists no coalition that can make a profitable deviation under $\widetilde{m}$. Therefore, $x$ is a strong Nash equilibrium outcome under $\widetilde{v}$-votes screening rule for $k$ names.

Proposition 13 If the chooser 1-top-candidate is a strong Nash equilibrium outcome of the Constrained Chooser Game under v-rule of $k$ names then it is also a strong Nash equilibrium outcome of the Constrained Chooser Game under any v-rule of $\widetilde{k}$ names whenever $k^{\prime}<\widetilde{k}$ provided that both rules share the same tie breaking criterion.

## Appendix C2

The example below shows that without Assumption 4 the Polarized Proposers Model may not have a strong Nash equilibrium outcome.

Example 6 Let $\mathbf{A}=\{a, b, c, d, e, f\}$ and let $\mathbf{N}=\{1,2,3\}$. The proposers use the rule of 4 names, $(k=4, v=1)$, with the following tie breaking rule when needed: $e \succ d \succ c \succ b \succ a \succ f$. The preferences of the chooser and the committee members are as follows:

## Preference Profile

Proposer 1 Proposer 2 Proposer 3 Proposer 4 Proposer 5 Chooser

| $d$ | $d$ | $d$ | $d$ | $f$ | $f$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $e$ | $e$ | $e$ | $e$ | $b$ | $a$ |
| $c$ | $c$ | $c$ | $c$ | $a$ | $b$ |
| $a$ | $a$ | $a$ | $a$ | $c$ | $c$ |
| $b$ | $b$ | $b$ | $b$ | $e$ | $d$ |
| $f$ | $f$ | $f$ | $f$ | $d$ | $e$ |

First, notice that $q_{k}^{v}(\{x\})=1$ for any $x \in\{b, c, d, e\}, q_{k}^{v}(\{x\})=2$ for any $x \in A \backslash\{b, c, d, e\}$ and $q_{k}^{v}(X)=5$ for any $X \in A_{k} \backslash\{b, c, d, e\}$ and $q_{k}^{v}(\{b, c, d, e\})=4$. Notice that proposers 1, 2, 3, and 4 form the majoritarian group of proposers, so $m=4$. Notice also that the tie breaking rule is equal to the reverse of the chooser's preference over the set of candidates. The first step in describing the equilibrium outcomes is to identify those candidates that satisfy the three necessary conditions established in Proposition 1.
Inspecting the preference profile above, we have that:

1. Condition 1: $\{a, b, f\}$.
2. Condition 2: $\{a, b, c, d, e\}$.
3. Condition 3: $\{a, b, c, d, e, f\}$.
4. Condition 4: $\{a, b, c, d, e, f\}$.

So, only candidates a and b satisfy all four conditions. However, there exists no strategy profile that can sustain them as an strong Nash equilibrium outcome of the Constrained Chooser Game.


[^0]:    *Some materials in this paper have been circulated in preliminary drafts under the title "Choosing among rules of $k$ names".
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[^1]:    ${ }^{1}$ For precise difinitions see Section 2.

[^2]:    ${ }^{2}$ Transitive: For all $x, y, z \in A:(x \succ y$ and $y \succ z)$ implies that $x \succ z$.
    ${ }^{3}$ Asymmetric: For all $x, y \in A: x \succ y$ implies that $\neg(y \succ x)$.
    ${ }^{4}$ Irreflexive: For all $x \in A, \neg(x \succ x)$.
    ${ }^{5}$ Complete: For all $x, y \in A: x \neq y$ implies that $(y \succ x$ or $x \succ y)$.

[^3]:    ${ }^{6}$ Notice if the number of candidates with positive votes is lower than k , the tie breaking criterion must be used to break the ties among candidates with zero votes.
    ${ }^{7}$ See other examples of screening rules in Barbera and Coelho (2008), Ratliff (2003) and Gehrlein (1985).
    ${ }^{8}$ In a preceding paper (Barberà and Coelho, 2010) we already noticed that there is a substantial difference between rules of k names, depending on the power that screening rules assign to majorities. Specifically, there are screening rules where the majority can always impose the full list, if it agrees to do so, and others where its power is more limited.

[^4]:    ${ }^{9}$ The analysis of the homogeneous proposer's case is similar, though not identical, to that of one proposer case (see Subsection 5.1)
    ${ }^{10}$ Notice that definitions 6 an 7 are closely linked to that of effectivity functions studied by, among others, Peleg (1984), Abdou and Keiding (1991) and Sertel and Sanver (2004). These concepts of effectivity refer to the ability of agents to ensure an outcome, under the given rule.

[^5]:    ${ }^{11}$ In Appendix B, we provides explicit formulas for these two parameters.
    ${ }^{12}$ The proofs of the propositions are in the Appendix A.

[^6]:    ${ }^{13}$ Without this assumption, the existence result may not hold. See an example in Appendix C, Section C2.

[^7]:    ${ }^{14}$ The analysis of this case is available from the authors.

[^8]:    ${ }^{15}$ The corollary above follows by Proposition 1 and from the fact that if $a-\sqrt{2 a+2}+2$ is an interger number then it is equal to $\left\lfloor a+\frac{5}{2}-\sqrt{2 a+\frac{9}{4}}\right\rfloor$. To see it, let $x=a+\frac{5}{2}-\sqrt{2 a+\frac{9}{4}}$ and $y=a-\sqrt{2 a+2}+2$. Notice that
    $x-y=\sqrt{2 a+2}+\frac{1}{2}-\sqrt{2 a+2+\frac{1}{4}}$. Thus $1>x-y>0$ for every $a>0$. Therefore if $a-\sqrt{2 a+2}+2$ is an interger number then we have that $\left\lfloor a+\frac{5}{2}-\sqrt{2 a+\frac{9}{4}}\right\rfloor=a-\sqrt{2 a+2}+2$.

[^9]:    ${ }^{16}$ Notice $S_{2}$ can be empty. But $S_{1}$ is never empty since $\left\{\left(\left\lfloor\tau_{1}\right\rfloor,\left\lfloor\tau_{1}\right\rfloor\right),\left(\left\lceil\tau_{1}\right\rceil,\left\lceil\tau_{1}\right\rceil\right)\right\} \subseteq S_{1}$.
    ${ }^{17}$ These calculations are elementary and available from the authors.

[^10]:    ${ }^{18}$ The indicator function, $\mathcal{I}(\cdot)$, takes value 1 if the expression in brackts is true, and 0 otherwise.

