Outside Options and the Limiting Distribution of Power in Repeated Decision Taking with Private Information*

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Abstract

We introduce ex-post participation constraints in a setting in which, repeatedly, two agents have to take a joint action, cannot resort to side payments, and each period are privately informed about their favorite actions. We derive a number of results. First, we show that, irrespective of how patient the agents are, any mechanism satisfying ex-post participation constraints delivers outcomes that are bounded away from efficiency. Second, for an agent whose outside option became tempting, the optimal mechanism (i) gives, relatively to a forced participation setting, *less* weight on current actions and, so to allow the agents to continue to trading decision rights in the future, (ii) always promises continuation values that are higher than the value of his outside option. Finally, we derive properties of the dynamics of relative bargain power, and prove that it leads to a unique limiting distribution of power. This limiting distribution is non-degenerate, memoryless and such that power continually changes hands in the limit, meaning that the weight agents have on decisions necessarily varies from period to period.

Keywords: Repeated Decision Taking, Mechanism Design, Private Information, Limiting Distribution of Power.

J.E.L. Classifications: C7, D7, D82

1 Introduction

Instances of repeated decision taking are pervasive in Economics. Members of partnerships, employers and employees, and couples in a household, among others, are subject to the problem of taking a common action repeatedly over time. In all these examples, the agents involved have some outside option. An employee can quit his job and look for an alternative one, whereas an employer can fire a worker and search for replacement. Partnerships are often dissolved by members that decide to implement individual business strategies. More prosaically, couples can always divorce and look for new matches.

In this paper, we introduce outside options in a setting in which, repeatedly, two agents have to take a joint action, cannot resort to side payments, and each period are privately informed about their favorite actions. We are specially interested in understanding how the outside options affect the dynamics of the actions taken by the agents and their relative bargain power in the partnership.

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Outside options take a very simple form in the model: for any given contingency (current and past realization of private information), and any period of time, agent i can collect life-time utility of $\underline{w}_i$ outside the partnership. As a consequence, any feasible mechanism must satisfy ex-post participation constraints: for every contingency and period of time, it must provide to each agent expected lifetime utility of no less than what they can get exercising their outside options.

We derive a number of results. First off, in contrast to settings in which agents are forced to participate, we show that a mechanism satisfying ex-post participation constraints cannot approximate an (ex-ante) efficient outcome irrespective of how patient the agents are. The reason is simple. As shown by Jackson and Sonnenschein (2007) and Carrasco and Fuchs (2008) (we discuss these papers in detail when we review the literature), an efficient incentive compatible mechanism links current decisions to future ones: an agent who is given relatively more weight on a current decision has to relinquish future decision power. The way through which the mechanism grants an agent a lower future decision power is by promising him lower continuation values. The outside options place a lower bound on what a mechanism can promise to any single agent, impeding the mechanism to implement the efficient intertemporal trade of decision power.

A mechanism (or contract) determines, for each period of time and every contingency, the whole sequence of actions to be taken. In any given period of time, the current action taken depends on whether or not participation constraints are binding. When the participation constraints slack, the relative weights on current actions are determined by two effects. The first one is the realization of the agents' current private information. The second one, summarized by a time varying weight the agents are given on decisions, is related to past actions: the agent who had more weight on past actions will have less weight on the current one. Both of these effects are already known from previous work on repeated action taking under forced participation. When a participation constraint binds, however, a third, new, effect comes into play. The player whose participation constraint binds is given, relatively to a forced participation setting, *less* weight on the current action. Although this may seem surprising, the reason why this is the case is simple. In states for which the participation constraint binds, it would be optimal to give an agent more weight on current decisions in exchange for less weight on future ones. Since one cannot promise values that are lower than the agent's outside option value, this intertemporal exchange of decision rights cannot be implemented. The agent is then given less weight relatively to what would happen in a forced participation environment.

Future actions and decision power are fully embedded in promised continuation values. In an optimal mechanism, the dynamics of continuation values has two main features. First, whenever current values are higher than outside option values, there is positive probability of next period's promised values being higher or smaller than current values. Put differently, "off-corners", continuation values are continually spread to provide incentives for truthful reporting by the agents. In fact, an agent who is given more (less) weight on current decisions is promised lower (higher) continuation values. The spreading of values is a force toward promising extreme values for the agents. Second, whenever current values equal outside option values for an agent, the optimal mechanism assigns positive probability of next period's promised value for that agent being higher than his outside option value. This is mean reversing force for promised values, which allows the members of the partnership to continue trading decisions in the future when one of the participation constraints binds. Both of these features imply that values must continually vary over time: there are no absorbing states.

When combined, the two features discussed above lead to our main results. Whenever both agents have outside options, the stochastic process that governs the agents' promised values converge to a unique invariant limiting distribution. The limiting distribution is non-degenerate, and assigns positive probability to all feasible values. We derive a close link between promised continuation values and the time varying weights on decisions the optimal mechanism assigns to the agents. Agent 2's time varying weight on actions – which is the relevant measure of decision power in our setting – is, at an optimal, given by the derivative of agent 1's value (given the promised value for agent 2). Therefore, the convergence of promised values implies that (i) there will be a unique limiting distribution for those weights, (ii) this distribution will be itself non-degenerate, and will assign positive probability to all weights that are compatible with the participation constraints.

Two important properties of the limiting distribution of power are the following. First, it is memoryless: even if the partnership starts with, say, agent 1 having all the bargain power (meaning, the initial promised value to agent 2 is $\underline{w}_2$), in the far future the relative bargain power will have no dependence whatsoever on this fact. Second, power continually changes of hands in the limit, meaning that the weight agents have in decisions will continually vary.

Unfortunately, it is not possible to derive analytically the limiting distribution of power. To get a grasp of how the agents' outside options affect the limiting distribution of power, we consider the situation in which only one of the agents has an outside option. This is a simple and tractable way to capture the effect of differences in outside options on the limiting distribution of power in a partnership. For this case, we show that the relative weight on decisions of the agent *who has* an outside option is, on average, increasing over time.[1] This dynamics leads to a dictatorship in the limit: the agent who has an outside option will eventually take all decisions. Therefore, as intuition suggests, a player who has an outside option will have more power in the partnership at some point. More surprising is the fact that, eventually, he will be the only one with power. This result also shows that both players having outside options is necessary (as well as sufficient) for the limiting distribution of power being non-degenerate.

There are few papers that show that, by linking unrelated decisions (i.e., common actions), efficiency gains can be attained. In an environment in which there is a binary choice each period and agents can have either have weak or strong preferences for either option, Casella (2005) proposes a mechanism in which agents are given a vote every period which they can use over time. The possibility of shifting votes intertemporally allows agents to concentrate their votes when preferences are more intense, leading to efficiency gains. In a two-player decision problem setting with two binary issues, but a continuum of preference intensities, Hortala-Vallve (2007a) shows that if the players are allowed to freely distribute a given number of votes across the two issues, the ex-ante efficient decision can be attained.

Jackson and Sonnenschein (2007) propose a simple "budgeting mechanism", in which each agent is allowed to report a possible type (they have a discrete type space) a fixed number of times. The number of times an agent can report certain type is chosen to replicate the frequency with which that type should be realized. They show that, if players are patient, their mechanism leads to approximate efficiency. In a setting similar to the one in this paper, Carrasco and Fuchs (2008) show that the optimal mechanism involves the trading of decisions power over time, and that such trading will arbitrarily approximate (but never attain) efficiency when participation is forced.

In our setting, and in contrast to those results, we show that the imposition of ex-post participation constraints bound the attainable payoffs away from the efficient ones irrespective of how patient the agents

[1] More precisely, when, say, agent 2 is the one with an outside option, his weight on decisions is a sub-martingale.

are. Therefore, much as in static mechanism design theory (e.g., Myerson and Satterwhaite (1983)), there is a trade-off between efficiency and ex-post participation in a repeated common action setting: repetition does not slack ex-post participation constraints.[2]

The optimal mechanism in Carrasco and Fuchs (2008) will eventually lead to a dictatorship. Hence, an ex-post consequence of an optimal mechanism that approximates efficiency ex-ante is that the limiting distribution of power is degenerate, with all decisions being taken by a single agent from a certain period of time onwards. Our paper shows that, while preventing approximate efficiency ex-ante, the possibility that agents have to exercise ex-post their outside options allows for a non-degenerate limiting distribution of power. Moreover, the weights the agents have on decisions will continually vary over time. Therefore, not only decisions will necessarily be shared but, also, the stakes the agents have on decisions will continually vary.

Lastly, our work also relates to the dynamic insurance literature, specially Atkenson and Lucas (1995). In a general equilibrium setting in which a planner has to provide insurance for agents whose binary effort toward finding jobs are unobservable, they show that, when those agents have outside options, a unique invariant distribution of wealth exists. Moreover, under certain conditions related to the interest rate of the Economy, this distribution may be non-degenerate.[3] In our paper, an agent who reports to have an extreme type in a given period is like an agent that reports to have not found a job. The optimal mechanism will respond by giving that agent more weight in the current allocation decision (similarly a higher transfer today). Incentive compatibility then calls for the agent to "pay" for this by forgoing future weight in the allocation decision (future consumption). In their setting, a Principal designing an optimal insurance policy trades-off risk-sharing (which calls for a constant consumption stream) and the provision of incentives – through varying continuation values. In both papers, the outside options put a bound on the provision of incentives, which – in some cases for them and always for us – leads to non-degenerate distributions of wealth and power.

The paper is organized as follows. We introduce the model in section 2. The optimal mechanism and some of its basic properties are characterized in Section 3. In section 4, we discuss the dynamics of decisions taking. Section 5 deals with the case in which only one of the agents has an outside option. Section 6 draws the concluding remarks. All proofs are relegated to the Appendix.

2 The Model

Two agents, $i = 1, 2$, have to take, repeatedly, a joint action, a, over time. At each period $t \in \{0, 1, ...\}$, they receive privately preference shocks $\theta_i \in [0, 1]$. The preference shocks are i.i.d. over time and across players, and are drawn from a distribution $F(.)$, with density $f(\theta_i) > 0$, which is symmetric around $\frac{1}{2}$.

Player i's instantaneous (Bernoulli) utility function is

$$u(a, \theta_i),$$

[2] A related point is made by Hortala-Vallve (2007b), who shows that there is a trade-off between efficiency and ex-post Incentive Compatibility constraints in a voting environment in which decisions can be linked.

[3] In contrast to what happens in our model, in their paper, for certain values of the interest rate, the dynamics of wealth in their model has absorbing states, and the limiting distribution of wealth is degenerate.

with

$$u(\theta_i, \theta_i) \geq u(a, \theta_i) \ \ for\ all\ a,$$

and

$$\frac{\partial^2 u(a, \theta_i)}{\partial a \partial \theta_i} > 0 > \frac{\partial^2 u(a, \theta_i)}{\partial a^2}.$$

Put in words, their preferences are single peaked, and θ_i represents agent i's favorite action. We also impose that their preferences are symmetric around $\frac{1}{2}$: for all $a, \theta_i \in [0,1]$,

$$u(a, \theta_i) = u(1-a, 1-\theta_i).^4$$

As preferences and the distribution of types are symmetric around $\frac{1}{2}$, the problem itself is symmetric around $\frac{1}{2}$. Hence, one can measure how extreme a preference shock θ_i is in terms of its distance from $\frac{1}{2}$.

After the players observe their preference shocks, they make reports $\tilde{\theta}_i$, $i = 1, 2$. A public history at time t, h^t, is a sequence of (i) past announcements of all players, and (ii) past realized actions:

$$h^t = \{\varnothing, \left(\tilde{\theta}_1^1, \tilde{\theta}_2^1, a^1\right), ..., \left(\tilde{\theta}_1^{t-1}, \tilde{\theta}_2^{t-1}, a^{t-1}\right)\}.$$

Given the reports and the history of the game, a history dependent allocation is determined according to a contract, which is a sequence of functions of the form

$$\left\{a_t\left(\tilde{\theta}_i, \tilde{\theta}_{-i}, h^{t-1}\right) : [0,1]^2 \times [0,1]^{3(t-1)} \to [0,1]\right\}_{t=1}^{\infty}.$$

This contract is chosen a priori before the agents learn their preference shocks.

Let H^t be the set of all public histories of length t. A public strategy for player i is a sequence of functions $\{\tilde{\theta}_i^t(.,.)\}_t$, where

$$\tilde{\theta}_i^t : H^t \times [0,1] \to [0,1].$$

Each profile of strategies $\tilde{\theta} = \left(\left\{\tilde{\theta}_1^t(.)\right\}_t, \left\{\tilde{\theta}_2^t(.)\right\}_t\right)$ defines a probability distribution over public histories. Let $\delta \in [0,1)$ denote the common discount factor. Player i's discounted expected payoff is given by

$$E\left[(1-\delta)\sum_{t=0}^{\infty} \delta^t u^t(a(\tilde{\theta}^t); \theta_i^t)\right],$$

where the expectation is taken with respect to the probability distribution over public histories induced by the strategy profile.

We analyze this game using the recursive methods developed by Abreu, Pearce and Stacchetti (1990). More specifically, letting $W \subset \Re^2$ be the set of Public Pure Strategy Equilibria (PPSE) payoffs for the agents, we can decompose the payoffs into a current utility $u(a, \theta_i)$ and a continuation value $w_i(\tilde{\theta}) \in W$:

$$E_\theta[(1-\delta)u(a\left(\tilde{\theta}\right), \theta_i) + \delta w_i(\tilde{\theta}_i, \tilde{\theta}_{-i})].$$

In other words, any PPSE can be summarized by the actions to be taken in the current period and equilibrium continuation values as a function of the announcements.

[4] Note that, in particular, this holds whenever an agent with type θ_i is indifferent between any two actions a and b that are equidistant from θ_i.

Player i has an outside option which grants him, for any given contingency, and any period of time, life-time utility of $\underline{w}_i$. Hence, for each of the agents it must be the case that

$$w_i(\theta) \geq \underline{w}_i \text{ for all } \theta.$$

We interpret $(\underline{w}_1, \underline{w}_2)$ as defining the initial distribution of power within the partnership, and seek to derive its implications for the joint actions chosen and the long-run distribution of power.

3 The Optimal Mechanism

We can use the decomposition implied by Abreu, Pearce and Stacchetti (1990) to write the Bellman equation that characterizes the frontier of equilibrium values that can be attained in this environment.

Define $\underline{v}$ as the expected value for a player when the *other* one always chooses the allocation; that is, $\underline{v} = E_\theta \left[u\left(\theta_i, \theta_{-i}\right)\right]$. Analogously, define $\overline{v}$ as the players' payoff when their preferred action is always taken, $\overline{v} = E_\theta \left[u\left(\theta_i, \theta_i\right)\right]$. We assume that there exists a $\kappa > 0$ so that $\underline{w}_i - \kappa > \underline{v}$, $i = 1, 2$. In words, the payoff a player collects in case he exercises his outside option is bounded away from the payoff he gets when his opponent takes all decisions.

Let $V(v)$ be the highest value to player 1 given that player $2's$ expected lifetime utility is v. Letting $\theta = (\theta_1, \theta_2)$, we can write $V(v)$ as

$$V(v) = \max_{\{a(\theta), w(\theta)\}_{\theta \in [0,1]^2}} E_\theta \left[(1-\delta) u\left(a(\theta), \theta_1\right) + \delta V\left(w(\theta)\right)\right]$$

subject to

$$E_\theta \left[(1-\delta) u\left(a(\theta), \theta_2\right) + \delta w(\theta)\right] = v \qquad \text{(Promise Keeping)}$$

$$E_{\theta_2} \left[(1-\delta) u\left(a(\theta), \theta_1\right) + \delta V\left(w(\theta)\right)\right] \geq E_{\theta_1} \left[(1-\delta) u\left(a\left(\widehat{\theta}_1, \theta_2\right), \theta_1\right) + \delta V\left(w\left(\widehat{\theta}_1, \theta_2\right)\right)\right] \; for\ all\ \theta_1, \hat{\theta}_1 \qquad \text{(IC1)}$$

$$E_{\theta_1} \left[(1-\delta) u\left(a(\theta), \theta_2\right) + \delta w(\theta)\right] \geq E_{\theta_2} \left[(1-\delta) u\left(a\left(\theta_1, \widehat{\theta}_2\right), \theta_2\right) + \delta w\left(\theta_1, \widehat{\theta}_2\right)\right] \; for\ all\ \theta_2, \hat{\theta}_2 \qquad \text{(IC2)}$$

$$w(\theta) \geq \underline{w}_2 \; for\ all\ \theta \qquad \text{(IR agent 2)}$$

$$V\left(w(\theta)\right) \geq \underline{w}_1 \; for\ all\ \theta. \qquad \text{(IR agent 1)}$$

The constraints are standard. The Promise Keeping constraint requires that, if agent 2 is promised discounted expected utility of v, the mechanism must choose an action $a(.)$ and continuation values $w(.)$ that deliver such promise. (IC1) and (IC2) state that, given a truthful report of the other agent, it must be optimal for agent i to report truthfully his preference shock.[5] Finally, the last two constraints are the participation constraints for agent 2 and 1, respectively.

Defining $\overline{w}_2 = V^{-1}\left(\underline{w}_1\right)$, we can write the Participation Constraints simply as

$$w(\theta) \in \left[\underline{w}_2, \overline{w}_2\right] \; for\ all\ \theta. \qquad \text{(IR')}$$

[5] We make use of the Revelation Principle (Myerson, 1981).

Early work (e.g., Casella (2005) and Jackson and Sonnenschein (2007)) has shown that the repeated taking of joint actions allows for significant improvements over a one shot framework. The efficiency gains are attained by allowing the actions to be linked over time. A player who reports to have a more extreme preference shock – as measured by its distance from $\frac{1}{2}$ – is granted, relatively to a one shot case, more weight on the current action, relinquishing future decision power.

Define

$$a^*(\theta) = \arg\max_a E\left[u(a, \theta_1) + u(a, \theta_2)\right],$$

to be the (ex-ante) Pareto efficient allocation, and let $v^{FB} = E\left[u(a^*(\theta), \theta_1) + u(a^*(\theta), \theta_2)\right]$ be the total surplus when action $a^*(\theta)$ is taken in all periods.

Under repeated decision taking, if either the Participation Constraint is ex-ante or agents are forced to participate, v^{FB} can be arbitrarily approximated – but not attained – by equilibrium payoffs when agents become patient. Carrasco and Fuchs (2008), however, show that this can only be accomplished through the continuing variation in decision power. This variation, in turn, will necessarily lead to one of the players becoming the dictator: in the long run, one of the players will be promised $\overline{v}$.

In the current setting, this is not feasible because, whenever a player is promised sufficiently low continuation values, he will exercise his outside option. As the mechanism cannot grant unbounded power to a player ex-post, it will not approximate efficiency ex-ante.

Theorem 1 (Inefficiency) *There exists a $\epsilon > 0$ such that, for all $\delta \in [0, 1)$, the sum of the agents' payoffs is no larger than $v^{FB} - \epsilon$.*

Therefore, irrespective of how patient the agents are, any feasible mechanism that satisfies the ex-post participation constraints will deliver outcomes that are bounded away from the efficient ones. Indeed, efficiency calls for intertemporal decisions to be linked: an agent who is given relatively more weight on a current decision has to relinquish future bargain power. The way through which the mechanism grants an agent a lower future bargain power is by promising him lower continuation values. The outside options place a lower bound on what a mechanism can promise to any single agent, impeding the mechanism to implement the efficient intertemporal trade of decision power.

4 The Dynamics of Decision Taking

Having shown that the introduction of Participation Constraints bound equilibrium payoffs away from the ex-ante efficient payoffs, we turn to the characterization of the dynamics of decision taking. We consider first how current decisions are taken, and then discuss the dynamics of continuation values (future decisions). Before that, however, we establish two results that will be of use later on.

Its is well known from the mechanism design literature that the incentive constraints imposed by (IC1) and (IC2) can be equivalently re-stated in terms of a "first order condition" for an optimal truthful announcement and a monotonicity condition, which guarantees that if a local deviation from truthtelling is not optimal, the same will be true for a global deviation. The formal statement of these conditions are presented in the following result

Lemma 1 *A pair* $(a(.), w(.))_\theta$ *is Incentive Compatible if, and only if, they satisfy*

$$E_{\theta_1}\left(\left[\frac{du(a(\theta),\theta_2)}{da}\frac{da(\theta)}{d\theta_2}\right]+\delta\frac{d}{d\theta_2}w(\theta)\right)=0 \qquad \text{(IC Local)}$$

$$E_{\theta_2}\left(\frac{du(a(\theta),\theta_1)}{da}\frac{da(\theta)}{d\theta_1}+\delta\frac{d}{d\theta_1}V(w(\theta))\right)=0, \qquad \text{(IC Local)}$$

and

$$E_{\theta_{-i}}\left[\frac{\partial u(a(\tau,\theta_{-i}),\theta_i)}{\partial\theta_i}\right] \text{ is non}-\text{decreasing in } \tau \text{ for } i=1,2. \qquad \text{(Expected Monotonicity)}$$

The second result states that the value function $V(.)$ is (strictly) concave when players are patient. Hence, we can use Lagrangian methods to solve for the optimal contract (Luenberger (1969)).

Lemma 2 *For* δ *large,* $V(.)$ *is strictly concave*

With these two results in hands, we are able to turn to the analysis of the optimal actions and continuation values picked by an optimal mechanism

4.1 Actions

Assigning multipliers $\{\lambda_i(\theta_i)\}_{\theta_i\in[0,1]}$, $i=1,2$, to, respectively, the first order condition counterparts of (IC1), and (IC2), and multiplier γ to the Promise Keeping constraint, the first order necessary condition for an optimal action $a(\theta)$ is (this is shown in the Appendix)

$$\left(\begin{array}{c}\left[\frac{\partial u(a(\theta),\theta_1)}{\partial a}\left[f(\theta_1)-\frac{d\lambda_1(\theta_1)}{d\theta_1}\right]-\lambda_1(\theta_1)\frac{\partial^2 u(a(\theta),\theta_1)}{\partial\theta_1\partial a}\right]f(\theta_2)\\ +f(\theta_1)\left[\frac{\partial u(a(\theta),\theta_2)}{\partial a}\left[\gamma f(\theta_2)-\frac{d\lambda_2(\theta_2)}{d\theta_2}\right]\frac{\partial u(a(\theta),\theta_2)}{\partial a}-\lambda_2(\theta_2)\frac{\partial^2 u(a(\theta),\theta_2)}{\partial\theta_2\partial a}\right]\end{array}\right)=0. \qquad \text{(FOC } a(\theta)\text{)}$$

The optimal action is determined by a weighted average of the agents' preferences, where the weights take into account the Incentive Compatibility and Participation constraints. Indeed, the multiplier on the Promise Keeping constraint, γ, works as a time varying Pareto weight that defines the relative weight of agent 2 on the current allocation. This weight will depend on the history of actions taken, and on whether the Participation constraints bind. The multipliers on the local counterparts of the IC constraints and their derivatives, in turn, determine how sensitive the current allocation is to the players' announcement.

Analogously, by assigning multipliers $\{\zeta(\theta)\}_{\theta\in[0,1]^2}$ to

$$w(\theta)\leq\overline{w}_2$$

and $\{\xi(\theta)\}_{\theta\in[0,1]^2}$ to

$$w(\theta)\geq\underline{w}_2,$$

which are the Participation Constraints in (IR'), we can write the first order condition for $w(\theta)$ as

$$V'(w(\theta))f(\theta_1)f(\theta_2)+\gamma f(\theta_1)f(\theta_2)-\frac{d\lambda_2(\theta_2)}{d\theta_2}f(\theta_1)-\frac{d\lambda_1(\theta_1)}{d\theta_1}V'(w(\theta))f(\theta_2)-\zeta(\theta)+\xi(\theta)=0. \qquad \text{(FOC } w(\theta)\text{)}$$

Using the above conditions, we can divide the analysis of how current decisions are taken according to two sets of states:

1. **States for which neither of the Participation Constraints bind**: The action chosen is the same as the one derived in Carrasco and Fuchs (2008). In particular, as argued by them, compared to a one-shot setting (or to the case in which $\delta = 0$), more weight is given to a relatively extreme player, who forgoes future decision power.

2. **States for which one Participation Constraint binds:** This is a more interesting case. Note, first, that whenever neither of the IRs bind (or, alternatively, when participation is forced), the weight given to player 2 when $a(.)$ is chosen, γ, can be written as

$$\gamma = -V'(w(\theta)) + \frac{\frac{d\lambda_2(\theta_2)}{d\theta_2}}{f(\theta_2)} + \frac{\frac{d\lambda_1(\theta_1)}{d\theta_1}}{f(\theta_1)} V'(w(\theta)).$$

Consider the case in which agent 2's participation constraint is binding: $w(\theta) = \underline{w}_2$ and $\xi(\theta) > 0$ (the analysis for the other case is analogous). The first order condition for $w(\theta)$ reads

$$V'(\underline{w}_2) f(\theta_1) f(\theta_2) + \gamma f(\theta_1) f(\theta_2) - \frac{d\lambda_2(\theta_2)}{d\theta_2} f(\theta_1) - \frac{d\lambda_1(\theta_1)}{d\theta_1} V'(\underline{w}_2) f(\theta_2) + \xi(\theta) = 0,$$

so that

$$\gamma = -V'(\underline{w}_2) + \frac{\frac{d\lambda_2(\theta_2)}{d\theta_2}}{f(\theta_2)} + \frac{\frac{d\lambda_1(\theta_1)}{d\theta_1}}{f(\theta_1)} V'(\underline{w}_2) - \frac{\xi(\theta)}{f(\theta_1) f(\theta_2)}.$$

Since $\frac{\xi(\theta)}{f(\theta_1)f(\theta_2)} > 0$, whenever agent 2's participation constraint is binding, the optimal mechanism incorporates an additional, negative, term to the weight given to agent 2 on the current decision. The reason why this is the case is simple. In states for which the participation constraint binds, it is not feasible to give an agent more weight on current decisions in exchange for less weight on future ones. This intertemporal exchange of decision rights would have been implemented in a forced participation setting, however. Therefore, the agent is given less weight relatively to what would prevail in a forced participation environment.

Whenever none of the agents are tempted to exercise their outside options, the allocation will be such that a player with extreme preferences – as measured by its distance to $\frac{1}{2}$ – will trade more weight on the *current* allocation for less weight in *future* allocation (Carrasco and Fuchs (2008)). In states for which one of the participation constraints is binding, the intertemporal exchange of decision power has to take an additional factor into account: it is not feasible anymore for the player whose constraint binds to forego future decision power in exchange for a higher weight on current decisions.

4.2 Continuation Values

The equation (FOC $w(\theta)$) implicitly defines promised continuation values $w(\theta)$ as a function of current value, w. Let the relationship between these values be given by

$$w' = g(w, \theta). \qquad \text{(Transition)}$$

At an optimum, continuation values must vary from period to period to reflect the agents' weights in the allocation rule. Continuation values tend to increase (higher future decision power) for an agent with less weight on the current decision, and to decrease (lower future decision power) for a player that is given more weight on the current decision. The next result states this precisely

Proposition 1 (Spreading Values) *If $w \in (\underline{w}_2, \overline{w}_2)$, there is strictly positive of probability of both $w' > w$ and $w' < w$.*

The variation in continuation values allows for agents to get more weight in the current decision in exchange for forgoing decision rights in the future. This is the mean by which *ex-ante* efficiency gains are attained. We seek to derive the *ex-post* implications of such variation in values on the agents' relative bargain power.

In a setting without participation constraints, the variation of values implied by Proposition 1 is a force toward a degenerate limiting distribution of power: whenever participation is forced, the continuing variation in values necessarily leads to a dictatorship in the limit (Carrasco and Fuchs (2008)).

In the current setting, however, outside options induce mean reversion: whenever one of the extreme values, $\{\underline{w}_2, \overline{w}_2\}$, is hit, a force toward intermediate values kicks in. When one of the agents is taken to his outside option payoff at a period t, there will be strictly positive probability of him being promised strictly higher continuation values for period $t+1$.

Proposition 2 (Mean Reversion) *Whenever $w = \overline{w}_2$ (respectively, $w = \underline{w}_2$), there is strictly positive probability of $w' < \overline{w}_2$ (respectively, $w' > \underline{w}_2$).*

When participation is forced, from the moment a player is promised dictatorship values, it is not feasible anymore to implement any exchange of decision rights. In other words, dictatorship is an absorbing state. In the current setting, in contrast, even when taken to his outside option, a player can exchange some current decision rights for more stake in future decisions. Implementing such exchange is optimal, as it allows both players to continue trading decisions rights over time.

A joint implication of Propositions 1 and 2 is that, whenever ex-post participation constraints have to be satisfied for both players, there are no absorbing states. So, if a limiting (invariant) distribution exists, it will necessarily be non-degenerate.

4.2.1 The Limiting Distribution of Power

We now move on to show that an unique invariant distribution of power exists. Toward that, define, for any set $A \subset [\underline{w}_2, \overline{w}_2]$, the inverse of $g(.)$ as

$$\Gamma(A, w) = \{(w, \theta) : g(w, \theta) \in A\}.$$

Then, the probability of next period's value lying in A given that current value is w,

$$Q(w, A) = \Pr[\Gamma(A) | w],$$

is a transition function (see Stokey et al (1989), p.212).

For an arbitrary distribution φ over $[\underline{w}_2, \overline{w}_2]$, define the operator T^* as follows

$$(T^*\varphi)(A) = \int Q(w, A)\, \varphi(dw).$$

This operator gives the probability of agent 2's next period promised values lying in A given that the current period promised value is drawn according to φ.

In the appendix, we show that T^* is a contraction in the total variation norm. Hence, starting with any initial distribution φ^0, the sequence defined by

$$\varphi^n = \left(T^* \varphi^{n-1}\right)$$

converges to a *unique* invariant φ^*. This leads to the following result.

Theorem 2 *The Markov Process that governs* w *has an unique invariant distribution* φ^* *over* $[\underline{w}_2, \overline{w}_2]$ *.* φ^* *is non-degenerate and assign positive likelihood to all* w *in* $[\underline{w}_2, \overline{w}_2]$*.*

In the limit, the distribution φ^* fully determines the probability of agent 2 being promised any given value. Using the Envelope Theorem, one has

$$-V'(w) = \gamma.$$

Therefore, the time varying weight of agent 2 on the current allocation is equal to the negative of agent 1's marginal value at an optimal. Hence, as $V'(.)$ is continuous (this is shown in the Appendix), an invariant distribution φ^* for continuation values will imply an invariant distribution γ^* for the weights. This distribution is itself non-degenerate. It follows that, when both agents have outside options, each will always have stake on current decisions, with their relative weights being fully determined by γ^*.

Two properties of the dynamics of decision power are worth mentioning. First off, the limiting distribution of power is memoryless: even if the partnership starts with, say, agent 1 having all the bargain power (meaning, the initial promised value to agent 2 is $\underline{w}_2$), in the far future, the relative bargain power will have no dependence whatsoever on this fact. This holds because the sequence $\{\varphi^n\}_n$ converges to φ^* for any φ^0.

Second, power continually changes hands in the limit, meaning that the weight agents have on decisions varies from period to period. This last property is a consequence of Propositions 1 and 2.

We summarize the above discussion in the following result

Theorem 3 *There exists a unique limiting distribution of power. This distribution is non-degenerate, memoryless and such that the weights agents have on decisions continually vary from period to period.*

5 The Case of a Single Participation Constraint

It would be interesting to evaluate analytically how the agents' outside options shape the invariant distribution φ^*, and the resulting distribution of power implied by $-V'(.)$. Unfortunately, we are not able to derive the distribution φ^* analytically, so we cannot fully tackle such type of question.

We are nevertheless able to deal with a particular (and extreme) case that sheds some light on it. This section analyzes the case in which only one of the players has an outside option.

Assume at first that, while agent 1 is forced to participate in the Partnership, agent 2 has a potentially tempting outside option. Our first result for this case is

Proposition 3 *When only agent 2 has an outside option, there exists a measure* $\mathcal{Q}$ *such that:*

$$E^{\mathcal{Q}}\left[V'(w(\theta))\right] \leq V'(w).$$

Hence, $V'(.)$ *is a supermartingale.*

Using $-V'(w) = \gamma$, the above result says that, on average, agent 2's weight on decisions cannot decrease over time. Using slightly modified versions of Proposition 1 and 2, it can be shown that $w(\theta)$ must vary continually over time, so that agent 2's weight on decisions must, in fact, *increase* on average over time. This will necessarily lead to agent 2 becoming a dictator in the long-run.

More formally, since $V'(.)$ follows a supermartingale, it must converge almost surely to a random variable (Dobb (1953)). As, in order to provide incentives, values must vary continually, we show in the Appendix that, in the limit, $V'()$ cannot assign positive likelihood to any point in $(-\infty, V'(\underline{w}_2)]$. Therefore, $V'(.)$ must converge almost surely to $-\infty$, so that $w(.)$ must converge to $\overline{v}$. Hence, agent 2 will eventually become a dictator.

The analysis for the case in which agent 1 has a tempting outside option, while agent 2 is forced to participate, is virtually the same. Indeed, we have

Proposition 4 *When only agent 1 has an outside option, there exists a measure $\mathcal{Q}$ such that:*

$$E^{\mathcal{Q}}\left[V'(w(\theta))\right] \geq V'(w).$$

Hence, $V'(w(\theta))$ is a submartingale.

Therefore, on average, agent 2's weight on decisions cannot increase over time. Since in order to provide incentives, values must vary continually, agent 2's weight must, in fact, decrease over time. This process will lead to agent 1 becoming a dictator.

The following result summarizes the above discussion.

Theorem 4 *If only one of the agents has an outside option, dictatorship will ensue eventually. The dictator will be the agent who has the outside option.*

On top of illustrating how differences in outside options may affect the shape of the limiting distribution of power, the above theorem also shows that both players having outside options is necessary (as well as sufficient) for such distribution being non-degenerate.

6 Conclusions

In this paper, we introduced outside options in a setting in which, repeatedly, two agents take a joint action over time, cannot resort to side payments, and each period are privately informed about their favorite actions. The main results are as follows. Irrespective of how patient agents are, a mechanism that satisfies ex-post participation constraints can only attain outcomes that are bounded away from efficiency. In terms of current decisions, ex-post participation constraints introduces a new effect on the relative weights the agents have on them. Whenever a participation constraint binds, the player whose outside option became tempting is given, relatively to a forced participation setting, *less* weight on the current action.

Regarding the dynamics of bargain power, we proved that there will be a unique limiting distribution of power. This distribution is non-degenerate, memoryless and such that power continually changes of hands in the limit, meaning that the weight agents have on decisions varies from period to period. As a tractable way to analyze how differences in outside options shape the limiting distribution of power, we considered the extreme case in which only one of the players has an outside option. For that case, and in contrast to the case in which both players have outside options, the limiting distribution of power will be degenerate, and the player for which the mechanism has to satisfy participation constraints will eventually become a dictator.

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7 Appendix:

7.1 The Lagrangian Representation

We first prove that the function $V(.)$ is strictly concave. This will allow us to make use of Lagrangian methods. In order to do so, we start by using a standard result from the Mechanism Design literature. Since the agents' preferences satisfy a single crossing condition, Incentive Compatibility can be replaced by a first order condition for truthtelling and a monotonicity condition.

Lemma 3 *A pair $(a(.), w(.))_\theta$ is Incentive Compatible if, and only if, they satisfy*

$$E_{\theta_1}\left(\left[\frac{du(a(\theta),\theta_2)}{da}\frac{da(\theta)}{d\theta_2}\right]+\delta\frac{d}{d\theta_2}w(\theta)\right)=0 \tag{IC Local}$$

$$E_{\theta_2}\left(\frac{du(a(\theta),\theta_1)}{da}\frac{da(\theta)}{d\theta_1}+\delta\frac{d}{d\theta_1}V(w(\theta))\right)=0, \tag{IC Local}$$

and

$$E_{\theta_{-i}}\left[\frac{\partial u(a(\tau,\theta_{-i}),\theta_i)}{\partial\theta_i}\right] \; is\ non-decreasing\ in\ \tau\ for\ i=1,2. \tag{Expected Monotonicity}$$

Proof. The proof is standard, and, therefore, omitted. ■

As a first step toward showing that $V(.)$ is strictly concave, we have

Lemma 4 *For any $w \in [\underline{w}_2, \overline{w}_2]$, define, for a given $V_0(.)$ strictly concave, the sequence $\{V_k(w)\}_{k\geq 1, w\in[\underline{v},\overline{v}]}$ recursively as follows*

$$V_k(w)=\max_{\{a(\theta),w(\theta)\}_\theta} E_\theta\left[(1-\delta)u(a(\theta),\theta_1)+\delta V_{k-1}(w(\theta))\right]$$

subject to

$$E_\theta\left[(1-\delta)u(a(\theta),\theta_2)+\delta w(\theta)\right]=w$$

$$E_{\theta_2}\left[(1-\delta)u(a(\theta),\theta_1)+\delta V_{k-1}(w(\theta))\right]\geq E_{\theta_2}\left[(1-\delta)u\left(a\left(\widehat{\theta}_1,\theta_2\right),\theta_1\right)+\delta V_{k-1}\left(w\left(\widehat{\theta}_1,\theta_2\right)\right)\right] \; for\ all\ \theta_1,\hat{\theta}_1$$

$$E_{\theta_1}\left[(1-\delta)u(a(\theta),\theta_2)+\delta w(\theta)\right]\geq E_{\theta_1}\left[(1-\delta)u\left(a\left(\theta_1,\widehat{\theta}_2\right),\theta_2\right)+\delta w\left(\theta_1,\widehat{\theta}_2\right)\right] \; for\ all\ \theta_2,\hat{\theta}_2$$

$$w(\theta)\in[\underline{w}_2,\overline{w}_2] \; for\ all\ \theta.$$

Then, if δ is large, $V_k(.)$ is strictly concave for all k.

Proof. We make an induction argument. By hypothesis, $V_0(.)$ is strictly concave. Assume that $V_{k-1}(.)$ is strictly concave.

Let $(a_1(\theta), w_1(\theta))$ and $(a_2(\theta), w_2(\theta))$ be, respectively, solutions of the problem in the statement of the Lemma (the one that defines $V_k(,)$) when the promise keeping constraint is indexed by w_1 and w_2. Denote $\alpha w_1 + (1-\alpha) w_2$ by w^α.

If it were feasible to implement $a^\alpha(\theta) = \alpha a_1(\theta) + (1-\alpha) a_2(\theta)$, where $\alpha \in (0,1)$, with continuation values $w^\alpha(\theta) = \alpha w_1(\theta) + (1-\alpha) w_2(\theta)$, we would have, for player 2,

$$\begin{aligned} & E_\theta\left[(1-\delta)u(a^\alpha(\theta),\theta_2)+\delta w^\alpha(\theta)\right] \qquad \text{(Ineq)} \\ > \; & \alpha E_\theta\left[(1-\delta)u(a_1(\theta),\theta_2)+\delta w_1(\theta)\right]+(1-\alpha)E_\theta\left[(1-\delta)u(a_2(\theta),\theta_2)+\delta w_2(\theta)\right] \\ = \; & \alpha w_1+(1-\alpha)w_2\equiv w^\alpha, \end{aligned}$$

where the first inequality follows from the strict concavity of $u(., \theta_2)$, and the equality follows from the definition of $(a_1(\theta), w_1(\theta))$ and $(a_2(\theta), w_2(\theta))$.

We consider two cases:

Case 1: There exists $\epsilon > 0$ such that $w^\alpha(\theta) - \epsilon > \underline{w}_2$ for all $\theta \in [0,1]^2$ and

$$E_{\theta_{-i}}\left[\frac{du(a^\alpha(\tau, \theta_{-i}), \theta_i)}{d\theta_i}\right] \text{ non-decreasing in } \tau \text{ for all } i.$$

Since the inequality in (Ineq) is strict, we can find, for some $\overline{w} > w^\alpha$, a non-negative function $g_2(\theta_2, \theta_1) \equiv h_2(\theta_2) + h_1(\theta_1)$ such that

$$E_\theta[(1-\delta)u(a^\alpha(\theta), \theta_2) + \delta[w^\alpha(\theta) - g_2(\theta_2, \theta_1)]] = \overline{w},$$

$$w^\alpha(\theta) - g_2(\theta_2, \theta_1) \geq \underline{w}_2$$

and, at the same time, for large δ,

$$-\frac{\delta}{(1-\delta)}\frac{dh_2(\theta_2)}{d\theta_2} = E_{\theta_1}\left[\begin{array}{c} \alpha\frac{du(a_1(\theta),\theta_2)}{da}\frac{da_1(\theta)}{d\theta_2} + (1-\alpha)\frac{du(a_2(\theta),\theta_2)}{da}\frac{da_2(\theta)}{d\theta_2} \\ -\frac{du(a^\alpha(\theta),\theta_2)}{da}\left[\alpha\frac{da_1(\theta)}{d\theta_2} + (1-\alpha)\frac{da_2(\theta)}{d\theta_2}\right] \end{array}\right] \quad (2)$$

and

$$E_{\theta_2}\left[\frac{du(a^\alpha(\theta), \theta_1)}{da}\left[\alpha\frac{da_1(\theta)}{d\theta_1} + (1-\alpha)\frac{da_2(\theta)}{d\theta_1}\right]\right] + \frac{\delta}{1-\delta}E_{\theta_2}\left[\frac{d}{d\theta_1}V_{k-1}(w^\alpha(\theta) - g_2(\theta_2, \theta_1))\right] = 0. \quad (3)$$

Conditions (2) and (3) above guarantee Incentive Compatibility. Indeed, with these continuation values, one can see by inspection that the first order condition for truthtelling is satisfied for both players. Moreover, by assumption, $a^\alpha(\theta)$ satisfies expected monotonicity. Therefore, $a^\alpha(\theta)$, coupled with continuation values $w^\alpha(\theta) - g(\theta)$, is feasible when the promised value for player 2 is $\overline{w} \geq w^\alpha$.

We then have

$$\begin{aligned}
V_k(w^\alpha) &= V_k(\alpha w_1 + (1-\alpha)w_2) > V_k(\overline{w}) \qquad \text{(Concavity)} \\
&\geq E_\theta[(1-\delta)u(a^\alpha(\theta), \theta_1) + \delta V_{k-1}(w^\alpha(\theta) - g(\theta))] \\
&> \alpha[E_\theta[(1-\delta)u(a_1(\theta), \theta_1)]] + (1-\alpha)[E_\theta[(1-\delta)u(a_2(\theta), \theta_1)]] + \delta E_\theta[V_{k-1}(w^\alpha(\theta))] \\
&> \left(\begin{array}{c} \alpha[E_\theta[(1-\delta)u(a_1(\theta), \theta_1)] + \delta V_{k-1}(w_1(\theta))] \\ +(1-\alpha)[E_\theta[(1-\delta)u(a_2(\theta), \theta_1)] + \delta V_{k-1}(w_2(\theta))] \end{array}\right) \\
&= \alpha V_k(w_1) + (1-\alpha)V_k(w_2),
\end{aligned}$$

where the first inequality follows from the fact that $V_k(.)$ is strictly decreasing, the second inequality follows from the fact that $a^\alpha(\theta)$ along with $w^\alpha(\theta) - g(\theta)$ is feasible when the promised value for player 2 is $\overline{w}$, the third inequality follows from strict concavity of Player 1's instantaneous payoff and from the fact that $V_{k-1}(.)$ is decreasing, and the forth inequality follows from the strict concavity of V_{k-1}. It follows that $V_k(.)$ is strictly concave.

Case 2: $w^\alpha(\theta) = \underline{w}_2$ for all θ belonging to a (positive probability) set $A \subset [0,1]^2$ and/or $a^\alpha(\theta)$ does not satisfy expected monotonicity.

The same procedure (changing $a^\alpha(\theta)$) applies to both cases, so we focus on the situation in which $w^\alpha(\theta) = \underline{w}_2$ for all θ in some $A \subset [0,1]^2$ and assume that expected monotonicity holds. Since the inequality in 1 is strict, we can find, for some $\overline{w} > w^\alpha$ a function $l(\theta) = l_2(\theta_2) + l_1(\theta_1)$ such that

$$E_\theta[(1-\delta)u(a^\alpha(\theta) + l(\theta), \theta_2) + \delta w^\alpha(\theta)] = \overline{w},$$

$$E_{\theta_1}\left[\frac{du\left(a^{\alpha}\left(\theta\right)+l\left(\theta\right),\theta_2\right)}{da}\left[\frac{da^{\alpha}\left(\theta\right)+l\left(\theta\right)}{d\theta_2}\right]-\left[\alpha\frac{du\left(a_1\left(\theta\right),\theta_2\right)}{da}\frac{da_1\left(\theta\right)}{d\theta_2}-\left(1-\alpha\right)\frac{du\left(a_2\left(\theta\right),\theta_2\right)}{da}\frac{da_2\left(\theta\right)}{d\theta_2}\right]\right]=0,$$

and

$$E_{\theta_2}\left[\frac{du\left(a^{\alpha}\left(\theta\right)+l\left(\theta\right),\theta_1\right)}{da}\left[\alpha\frac{da_1\left(\theta\right)}{d\theta_1}+\left(1-\alpha\right)\frac{da_2\left(\theta\right)}{d\theta_1}+\frac{dl\left(\theta\right)}{d\theta_1}\right]\right]+\frac{\delta}{1-\delta}E_{\theta_2}\left[\frac{d}{d\theta_1}V_{k-1}\left(w^{\alpha}\left(\theta\right)\right)\right]=0.$$

Hence, $a^{\alpha}(.)+l(.)$, coupled with continuation values $w^{\alpha}(.)$, is feasible when promised values are $\overline{w}>w^{\alpha}$. It then follows that, for large δ,

$$\begin{aligned}
V_k\left(w^{\alpha}\right) &= V_k\left(\alpha w_1+\left(1-\alpha\right)w_2\right)>V_k\left(\overline{w}\right)\\
&\geq E_{\theta}\left[\left(1-\delta\right)u\left(a^{\alpha}\left(\theta\right)+l\left(\theta\right),\theta_1\right)+\delta V_{k-1}\left(w^{\alpha}\left(\theta\right)\right)\right]\\
&\geq E_{\theta}\left[\left(1-\delta\right)u\left(a^{\alpha}\left(\theta\right),\theta_1\right)+\delta\left[\alpha V_{k-1}\left(w_1\left(\theta\right)\right)+\left(1-\alpha\right)V_{k-1}\left(w_2\left(\theta\right)\right)\right]\right]\\
&> \begin{pmatrix}\alpha\left[E_{\theta}\left[\left(1-\delta\right)u\left(a_1\left(\theta\right),\theta_1\right)\right]+\delta V_{k-1}\left(w_1\left(\theta\right)\right)\right]\\ +\left(1-\alpha\right)\left[E_{\theta}\left[\left(1-\delta\right)u\left(a_2\left(\theta\right),\theta_1\right)\right]+\delta V_{k-1}\left(w_2\left(\theta\right)\right)\right]\end{pmatrix}\\
&= \alpha V_k\left(w_1\right)+\left(1-\alpha\right)V_k\left(w_2\right),
\end{aligned}$$

where the first inequality follows from $V_k(.)$ being strictly decreasing, the second follows because $a^{\alpha}(.)+l(.)$ and $\widetilde{w}(.)$ are feasible when promised values are $\overline{w}$, the third inequality, which is the key one, holds when δ is large because, since V_{k-1} is strictly concave,

$$V_{k-1}\left(w^{\alpha}\left(\theta\right)\right)=V_{k-1}\left(\alpha w_1\left(\theta\right)+\left(1-\alpha\right)w_2\left(\theta\right)\right)>\alpha V_{k-1}\left(w_1\left(\theta\right)\right)+\left(1-\alpha\right)V_{k-1}\left(w_2\left(\theta\right)\right),$$

and the forth inequality follows from the strict concavity of $u(.,\theta_i)$. Hence, $V_k(.)$ is strictly concave. ■

Proposition 5 *For δ large, $V(.)$ is strictly concave*

Proof. We prove the result in five steps.

Define

$$T\left(V\right)\left(w\right)=\max_{\left\{a(\theta),w(\theta)\right\}_{\theta}}E_{\theta}\left[\left(1-\delta\right)u\left(a\left(\theta\right),\theta_n\right)+\delta V\left(w\left(\theta\right)\right)\right]$$

subject to

$$E_{\theta}\left[\left(1-\delta\right)u\left(a\left(\theta\right),\theta_1\right)+\delta w\left(\theta\right)\right]=w$$

$$E_{\theta_2}\left[\left(1-\delta\right)u\left(a\left(\theta\right),\theta_1\right)+\delta V_{k-1}\left(w\left(\theta\right)\right)\right]\geq E_{\theta_2}\left[\left(1-\delta\right)u\left(a\left(\widehat{\theta}_1,\theta_2\right),\theta_1\right)+\delta V_{k-1}\left(w\left(\widehat{\theta}_1,\theta_2\right)\right)\right]\; for\ all\ \theta_1,\hat{\theta}_1$$

$$E_{\theta_1}\left[\left(1-\delta\right)u\left(a\left(\theta\right),\theta_2\right)+\delta w\left(\theta\right)\right]\geq E_{\theta_1}\left[\left(1-\delta\right)u\left(a\left(\theta_1,\widehat{\theta}_2\right),\theta_2\right)+\delta w\left(\theta_1,\widehat{\theta}_2\right)\right]\; for\ all\ \theta_2,\hat{\theta}_2$$

$$w\left(\theta\right)\in\left[\underline{v},\overline{v}\right]\;\; for\ all\ \theta.$$

STEP 1: For any δ, the set of $\left\{a\left(\theta\right),w\left(\theta\right)\right\}_{\theta}$ that satisfies the constraints of the above problem is compact and upper hemi-continuous.

We prove compactness. The proof of upper hemi-continuity of the constraint set follows similar steps.

Note that $\left\{a\left(\theta\right),w\left(\theta\right)\right\}_{\theta}$ satisfies the Incentive Compatibility constraints if, and only if:

$$\begin{aligned}
&\left(1-\delta\right)E_{\theta_1}\left[u\left(a\left(\theta\right),\theta_2\right)\right]+\delta E_{\theta_1}\left[w\left(\theta\right)\right] \qquad \text{(Envelope)}\\
=&\left(1-\delta\right)E_{\theta_1}\left[u\left(a\left(0,\theta_2\right),0\right)\right]+\delta E_{\theta_1}\left[w\left(0,\theta_1\right)\right]\\
&+\left(1-\delta\right)\int_0^{\theta_1}E_{\theta_1}\left[u_{\theta_2}\left(a\left(\tau,\theta_1\right),\tau\right)\right]d\tau,
\end{aligned}$$

and

$$\begin{aligned} & (1-\delta) E_{\theta_2}\left[u\left(a\left(\theta\right),\theta_1\right)\right]+\delta E_{\theta_2}\left[V\left(w\left(\theta\right)\right)\right] \qquad \text{(Envelope 1)} \\ = & (1-\delta) E_{\theta_2}\left[u\left(a\left(0,\theta_1\right),0\right)\right]+\delta E_{\theta_2}\left[V\left(w\left(0,\theta_2\right)\right)\right] \\ & +(1-\delta)\int_0^{\theta_1} E_{\theta_2}\left[u_{\theta_1}\left(a\left(\tau,\theta_2\right),\tau\right)\right] d\tau. \end{aligned}$$

and $E_{\theta_{-i}}\left[u_{\theta_i}\left(a\left(\tau,\theta_{-i}\right),\theta_i\right)\right]$ being non-decreasing in τ. (The Envelope Conditions follow from the first order condition for truthtelling. Indeed, letting

$$U_1\left(\theta_1\right)=E_{\theta_2}\left[(1-\delta) u\left(a\left(\theta\right),\theta_1\right)+\delta V_{k-1}\left(w\left(\theta\right)\right)\right],$$

be agent 1's payoff and using (IC Local), one has that

$$\frac{dU_1\left(\theta_1\right)}{d\theta_1}=E_{\theta_2}\left[u_{\theta_1}\left(a\left(\theta_1,\theta_2\right),\theta_1\right)\right].$$

Integrating such condition, the Envelope condition holds. The argument for agent 2 is virtually the same).

Using (Envelope), after some integration by parts, the Promise Keeping constraints can be written as

$$(1-\delta) E_{\theta_1}\left[u\left(a\left(0,\theta_1\right),0\right)\right]+\delta E_{\theta_1}\left[w\left(0,\theta_1\right)\right]+(1-\delta) E_{\theta}\left[u_{\theta_2}\left(a\left(\theta_2,\theta_1\right),\theta_1\right)\frac{\left(1-F\left(\theta_2\right)\right)}{f\left(\theta_2\right)}\right]=w.$$

Feasibility, in turn, calls for

$$w\left(\theta\right)\in\left[\underline{v},\overline{v}\right] \text{ for all } \theta$$

Now, take a sequence $\left\{a_k\left(.\right),w_k\left(.\right)\right\}_k$ satisfying all those constraints. We show that there exists a convergent subsequence.

Since $E_{\theta_{-i}}\left[u_{\theta_i}\left(a_k\left(\tau,\theta_{-i}\right),\tau\right)\right]$ is non-decreasing, by Helly's Selection Theorem (Kolmogorov and Fomin, 1970, p. 373), there exists a subsequence $E_{\theta_{-i}}\left[u_{\theta_i}\left(a_{k_m}\left(\tau,\theta_{-i}\right),\tau\right)\right]$ that converges to a non-decreasing $E_{\theta_{-i}}\left[u_{\theta_i}\left(a\left(\tau,\theta_{-i}\right),\tau\right)\right]$. Moreover, for agent 2,

$$\begin{aligned} (1-\delta) E_{\theta_1}\left[u\left(a_k\left(\theta\right),\theta_2\right)\right]+\delta E_{\theta_1}\left[w_k\left(\theta\right)\right] &= \\ \left[\begin{array}{c} (1-\delta) E_{\theta_1}\left[u\left(a_k\left(0,\theta_1\right),0\right)\right]+\delta E_{\theta_1}\left[w_k\left(0,\theta_1\right)\right] \\ +(1-\delta)\int_0^{\theta_2} E_{\theta_1}\left[u_{\theta_2}\left(a_k\left(\tau,\theta_1\right),\tau\right)\right] d\tau \end{array}\right], \end{aligned}$$

and, for agent 1,

$$\begin{aligned} (1-\delta) E_{\theta_2}\left[u\left(a_k\left(\theta\right),\theta_1\right)\right]+\delta E_{\theta_2}\left[V\left(w_k\left(\theta\right)\right)\right] &= \\ \left[\begin{array}{c} (1-\delta) E_{\theta_2}\left[u\left(a_k\left(0,\theta_2\right),0\right)\right]+\delta E_{\theta_2}\left[V\left(w_k\left(0,\theta_2\right)\right)\right] \\ +(1-\delta)\int_0^{\theta_1} E_{\theta_2}\left[u_{\theta_1}\left(a_k\left(\tau,\theta_2\right),\tau\right)\right] d\tau \end{array}\right] \end{aligned}$$

We now argue that, for both expressions, the right hand side converges (possibly along subsequences). We show this for gent 2 (the analysis for player 1 is analogous).

Note that

$$(1-\delta) E_{\theta_1}[u(a_k(0,\theta_1),0)] + \delta E_{\theta_1}[w_k(0,\theta_1)]$$

is a sequence of real numbers that lies in a compact set. Therefore, there exists a subsequence that converges. Denote the limit of such convergent subsequence by

$$(1-\delta) E_{\theta_1}[u(a(0,\theta_1),0)] + \delta E_{\theta_1}[w(0,\theta_1)].$$

Also, since there exists a subsequence $E_{\theta_1}[u_{\theta_1}(a_{k_m}(\tau,\theta_1),\tau)]$ that converges to a non-decreasing $E_{\theta_1}[u_{\theta_i}(a(\tau,\theta_1),\tau)]$, one has, by the Dominated Converge Theorem, that

$$(1-\delta)\int_0^{\theta_2} E_{\theta_1}[u_{\theta_2}(a_{k_m}(\tau,\theta_1),\tau)]\,d\tau \to (1-\delta)\int_0^{\theta_2} E_{\theta_1}[u_{\theta_2}(a(\tau,\theta_1),\tau)]\,d\tau.$$

Therefore, letting $a(.)$ and $w(.)$ be the functions for which

$$\left[\begin{array}{c} (1-\delta) E_{\theta_{-i}}[u(a_k(0,\theta_{-i}),0)] + \delta E_{\theta_{-i}}[w_k(0,\theta_{-i})] \\ +(1-\delta)\int_0^{\theta_n} E_{\theta_{-n}}[u_{\theta_i}(a_{k_n}(\tau,\theta_{-n}),\tau)]\,d\tau \end{array}\right] \to \left[\begin{array}{c} (1-\delta) E_{\theta_{-i}}[u(a(0,\theta_{-i}),0)] + \delta E_{\theta_{-i}}[w(0,\theta_{-i})] \\ +(1-\delta)\int_0^{\theta_n} E_{\theta_{-n}}[u_{\theta_i}(a(\tau,\theta_{-n}),\tau)]\,d\tau. \end{array}\right],$$

it follows that the Envelope condition holds at $(a(.),w(.))$.

As for the Promise Keeping constraints, since

$$(1-\delta) E_{\theta_1}[u(a_k(0,\theta_1),0)] + \delta E_{\theta_1}[w_k(0,\theta_1)] + (1-\delta) E_{\theta_1}\left[u_{\theta_2}(a_k(\theta_2,\theta_1),\theta_2)\frac{(1-F(\theta_2))}{f(\theta_2)}\right] = w,$$

and, along a subsequence, the left hand side converges, one has, invoking again the Dominated Convergence Theorem, that

$$(1-\delta) E_{\theta_1}[u(a(0,\theta_1),0)] + \delta E_{\theta_1}[w(0,\theta_1)] + (1-\delta) E_{\theta 1}\left[u_{\theta_1}(a(\theta_2,\theta_1),\theta_1)\frac{(1-F(\theta_1))}{f(\theta_2)}\right] = w_i.$$

Finally, since, for all k,

$$w_k(\theta) \in [\underline{v},\overline{v}] \text{ for all } \theta$$

one must have

$$w(\theta) \in [\underline{v},\overline{v}] \text{ for all } \theta.$$

It follows that the choice set is compact.

STEP 2: If $V(.)$ is continuous, $T(V)$ is also continuous

Given STEP 1, this follows from Theorem 2 of Ausubel and Deneckere (1993).

STEP 3: $T(.)$ is a contraction of modulus δ.

Denote by $D(w)$ the set of feasible actions and continuation values given current values w. We have

$$\begin{aligned} T(V_1) &= \max_{\{a(.),w(.)\}\in D(w)} E_\theta[(1-\delta) u(a(\theta),\theta_n) + \delta V_1(w(\theta))] \\ &= \max_{\{a(.),w(.)\}\in D(w)} E_\theta[(1-\delta) u(a(\theta),\theta_n) + \delta V_2(w(\theta)) + \delta[V_1(w(\theta)) - V_2(w(\theta))]] \\ &\leq \max_{\{a(.),w(.)\}\in D(w)} E_\theta[(1-\delta) u(a(\theta),\theta_n) + \delta V_2(w(\theta))] + \delta\,||V_1 - V_2|| \\ &= T(V_2) + \delta\,||V_1 - V_2||, \end{aligned}$$

where $||.||$ is the sup norm.

Therefore,

$$||T(V_1) - T(V_2)|| \leq \delta \, ||V_1 - V_2|| \, .$$

STEP 4: The sequence $\{V_k(.)\}_{k\geq 1 w\in[\underline{v},\overline{v}]}$, with $V_0(w) = 0$ for all w, converges to $V(.)$

This follows from the fact that T is a contraction and the set, $C[\underline{v},\overline{v}]$, of continuous functions over $[\underline{v},\overline{v}]$ endowed with the sup norm is a complete metric space.

STEP 5: $V(.)$ is strictly concave if δ is large.

When δ is large, all elements in $\{V_k(.)\}$ are strictly concave. As each element in the sequence $\{V_k(.)\}$ is strictly concave, the limit must be concave. Now, using the concavity of $V(.)$ and proceeding exactly as in the proof of the above Lemma, it is easy show that $V(.)$ must be, in fact, strictly concave. ■

A property of $V(.)$ that we will be of use later on is

Lemma 5 *$V(.)$ is continuously differentiable over $(\underline{v},\overline{v})$.*

Proof. Since $V(.)$ is concave, this follows from Corollary 2 in Milgrom and Segal (2002). ■

Assigning multipliers $\{\lambda_i(\theta_i)\}_{\theta_i}$, $i = 1, 2$, to, respectively, the first order condition counterparts of IC1, and IC2, multiplier γ to Promise Keeping, and multipliers $\{\xi(\theta)\}_\theta$ and $\{\zeta(\theta)\}_\theta$ to the participation constraints $w(\theta) \in [\underline{w}_2, \overline{w}_2]$ for all θ, one can write $V(v)$ as

$$V(v) = \max_{\{a(.),w(.)\}} \left[\begin{array}{c} E_\theta\left[(1-\delta)\, u\left(a(\theta), \theta_1\right) + \delta V\left(w(\theta,x)\right)\right] \\ \gamma\left(E_\theta\left[(1-\delta)\, u\left(a(\theta), \theta_2\right) + \delta w(\theta,x)\right] - v\right) \\ + \int\limits_0^1 \left[\lambda_2(\theta_2)\left((1-\delta)\, E_{\theta_1}\left(\left[\frac{du(a(\theta),\theta_2)}{da}\frac{da(\theta,x)}{d\theta_2}|\theta_2\right] + \delta\frac{d}{d\theta_2}\left(E\left[w(\theta,x)\right]|\theta_2\right)\right)\right)\right] d\theta_2 \\ + \int\limits_0^1 \left[\lambda_1(\theta_1)\left((1-\delta)\, E\left[\frac{du(a(\theta),\theta_1)}{da}\frac{da(\theta,x)}{d\theta_1}|\theta_1\right] + \delta\frac{d}{d\theta_1}\left(E\left[V\left(w(\theta,x)\right)|\theta_1\right]\right)\right)\right] d\theta_1 \\ + \int\limits_0^1\int\limits_0^1 \xi(\theta)\left[w(\theta) - \underline{w}_2\right] d\theta_1 d\theta_2 - \int\limits_0^1\int\limits_0^1 \zeta(\theta)\left[w(\theta) - \overline{w}_2\right] d\theta_1 d\theta_2. \end{array} \right]$$

Some rounds of integration by parts allow us to re-write $V(v)$ as

$$V(v)=\max_{\{a(.),w(.)\}}\left[\begin{array}{c} E_\theta\left[(1-\delta)u(a(\theta),\theta_1)+\delta V(w(\theta))\right]+\gamma\left(E_\theta\left[(1-\delta)u(a(\theta),\theta_2)+\delta w(\theta)\right]-v\right) \\ +\lambda_2(\theta_2)\left[(1-\delta)E_{\theta_1}\left[u(a(\theta),\theta_2)\right]\right]+\delta E_{\theta_1}\left[w(\theta)\right]|_{\theta_i=0}^{\theta_i=1} \\ -\left(\int_0^1\left[\frac{d\lambda_2(\theta_2)}{d\theta_2}\left[(1-\delta)E_{\theta_1}\left[u(a(\theta),\theta_2)\right]\right]\right]d\theta_2+\int_0^1\lambda_2(\theta_2)\left[(1-\delta)E_{\theta_1}\left[\frac{du(a(\theta),\theta_2)}{d\theta_2}\right]\right]d\theta_2\right) \\ -\delta\left[\int_0^1\left[\frac{d\lambda_2(\theta_2)}{d\theta_2}E_{\theta_1}\left[w(\theta)\right]\right]d\theta_2\right] \\ +\left(\lambda_1(\theta_1)\left[(1-\delta)E_{\theta_2}\left[u(a(\theta),\theta_1)\right]\right]+\delta E_{\theta_2}\left[w(\theta)\right]|\right)_{\theta_1=0}^{\theta_1=1} \\ -\left(\int_0^1\left[\frac{d\lambda_1(\theta_1)}{d\theta_1}\left[(1-\delta)E_{\theta_2}\left[u(a(\theta),\theta_1)\right]\right]\right]d\theta_1+\int_0^1\lambda_1(\theta_1)\left[(1-\delta)E_{\theta_2}\left[\frac{du(a(\theta),\theta_1)}{d\theta_1}\right]\right]d\theta_1\right) \\ -\delta\int_0^1\left[\frac{d\lambda_1(\theta_1)}{d\theta_1}E_{\theta_1}\left[V(w(\theta))\right]\right]d\theta_1 \\ +\int_0^1\int_0^1\xi(\theta)\left[w(\theta)-\underline{w}_2\right]d\theta_1 d\theta_2-\int_0^1\int_0^1\zeta(\theta)\left[w(\theta)-\overline{w}_2\right]d\theta_1 d\theta_2. \end{array}\right] \quad \text{(Lagrangian)}$$

As is standard (see Theorems 1 and 2 in sections 8.3-8.4 of Luenberger (1969))[6], $\{a^*(\theta),w^*(\theta)\}_\theta$ – with $a^*(.)$ satisfying expected monotonicity strictly – is optimal if, and only if, there are multipliers $\{\lambda_i(\theta_i),\xi(\theta_i,\theta_{-i}),\zeta(\theta_i,\theta_{-i}),\gamma\}_{i=1,\theta_i}^2$ for which $\{a^*(\theta),w^*(\theta)\}_\theta$ maximizes the above Lagrangian. It is easy to see that the First Order Conditions for such problem are the ones in the text.

7.2 The Inefficiency Result

Proof of Theorem 1. Define $\widehat{V}(w;\delta)$ as the value function for the problem without participation constrains when the discount factor is δ; call this problem unrestricted. Let $V(w;\delta)$ be the value function for the problem with participation constraints; call this problem restricted. From Carrasco and Fuchs (2008), we know that, for all $w<\overline{v}$, there is strictly positive probability of next period's continuation value for player 2 being both higher and lower than the current value value w. Suppose that the current value for player 2 is $\overline{w}_2<\overline{v}$. In the unrestricted program, it would be optimal to make next period's promised value higher than $\overline{w}_2$ with strictly positive probability. However, in the restricted problem this is not possible. Thus, there exists $\epsilon'>0$ such that $\widehat{V}(\overline{w}_2;\delta)-\epsilon'>V(\overline{w}_2;\delta)$ for all $\delta\in[0,1)$. Furthermore, as will be shown in the proof of Theorem 2, in the restricted problem, starting from any value w, there is positive probability of $\overline{w}_2$ being hit in a finite number of steps. Therefore, for any w, there exists a $\epsilon>0$ – that may depend on w – such that $\hat{V}(w;\delta)-\epsilon>V(w;\delta)$ for all $\delta\in[0,1)$. Let w^* be the value promised to agent 2 by the mechanism in Carrasco and Fuchs (2008) that approximates efficiency. It then follows that, for all δ,

$$w^*+V(w^*,\delta)<w^*+\widehat{V}(w^*;\delta)-\epsilon\le v^{FB}-\epsilon.$$

■

[6] The concavity of $V(.)$ in our setting plays the role Proposition 1 in section 8.3 plays in Theorems 1 and 2 of Luenberger (1969).

7.3 The Limiting Distribution of Power

Lemma 6 *There exists a measure $\mathcal{Q}$ such that*

$$E^{\mathcal{Q}}\left[V'\left(w\left(\theta\right)\right)\right] = V'\left(w\right) - E^{\mathcal{Q}}\left[\left(\xi\left(\theta\right) - \zeta\left(\theta\right)\right)\left(\left[f\left(\theta_1\right)f\left(\theta_2\right) - \dot{\lambda}_1(\theta_2)f\left(\theta_1\right)\right]\right)\right]. \quad (1)$$

Proof. From the Lagrangian, the First Order Condition with respect to $w\left(\theta\right)$ is

$$\left[f(\theta_1) - \dot{\lambda}_1(\theta_1)\right]V'(w\left(\theta\right))f(\theta_2) + \left[\gamma f(\theta_2) - \dot{\lambda}_2(\theta_2)\right]f(\theta_1) + \xi\left(\theta\right) - \zeta\left(\theta\right) = 0$$

Using the Envelope Theorem, one has $\gamma = -V'\left(w\right)$. Moreover, using, as in Carrasco and Fuchs (2008), $\dot{\lambda}_2(\theta_2) = \gamma\dot{\lambda}_1(\theta_2)$ – i.e., the objective is a weighted average of the agents' virtual utility[7], with agent 2 having weight γ –, we have that

$$V'\left(w\left(\theta\right)\right)\left[f\left(\theta_1\right)f\left(\theta_2\right) - \dot{\lambda}_1(\theta_1)f\left(\theta_2\right)\right] = V'\left(w\right)\left[f\left(\theta_1\right)f\left(\theta_2\right) - \dot{\lambda}_1(\theta_2)f\left(\theta_1\right)\right] - \left[\xi\left(\theta\right) - \zeta\left(\theta\right)\right].$$

If one divides the left hand side of the above equality by

$$\int_0^1\int_0^1\left[f\left(\theta_2\right)\left[f\left(\theta_1\right) - \dot{\lambda}_1(\theta_1)\right]\right]d\theta_1 d\theta_2$$

and the right hand side by

$$\int_0^1\int_0^1\left[f\left(\theta_1\right)\left[f\left(\theta_2\right) - \dot{\lambda}_1(\theta_2)\right]\right]d\theta_2 d\theta_1,$$

the equality is preserved, as

$$\int_0^1\int_0^1\left[f\left(\theta_2\right)\left[f\left(\theta_1\right) - \dot{\lambda}_1(\theta_1)\right]\right]d\theta_1 d\theta_2 = \int_0^1\int_0^1\left[f\left(\theta_1\right)\left[f\left(\theta_2\right) - \dot{\lambda}_1(\theta_2)\right]\right]d\theta_2 d\theta_1.$$

Therefore,

$$\frac{V'\left(w\left(\theta\right)\right)\left[f\left(\theta_1\right)f\left(\theta_2\right) - \dot{\lambda}_1(\theta_1)f\left(\theta_2\right)\right]}{\int_0^1\int_0^1\left[f\left(\theta_2\right)\left[f\left(\theta_1\right) - \dot{\lambda}_1(\theta_1)\right]\right]d\theta_1 d\theta_2} = \frac{V'\left(w\right)\left[f\left(\theta_1\right)f\left(\theta_2\right) - \dot{\lambda}_1(\theta_2)f\left(\theta_1\right)\right] - \left[\xi\left(\theta\right) - \zeta\left(\theta\right)\right]}{\int_0^1\int_0^1\left[f\left(\theta_1\right)\left[f\left(\theta_2\right) - \dot{\lambda}_1(\theta_2)\right]\right]d\theta_2 d\theta_1}$$

Integrating both sides of the above expression over θ, we get the desired result with the measure $\mathcal{Q}$ being the one induced by the density[8]

$$q\left(\theta\right) = \frac{\left[f\left(\theta_1\right)f\left(\theta_2\right) - \dot{\lambda}_1(\theta_1)f\left(\theta_2\right)\right]}{\int_0^1\int_0^1\left[f\left(\theta_2\right)\left[f\left(\theta_1\right) - \dot{\lambda}_1(\theta_1)\right]\right]d\theta_1 d\theta_2} \geq 0, \forall\left(\theta_1, \theta_2\right).$$

■

[7] See Myerson (1981) and Myerson's notes on virtual utility at http://home.uchicago.edu/~rmyerson/research/virtual.pdf

[8] One can always normalize the multipliers to guarantee that $q\left(\theta\right)$ is non-negative.

Proof of Proposition 1. Suppose, toward a contradiction, that $w(\theta) \geq w > \underline{w}_2$ for almost all θ. Since $V(.)$ is strictly concave, it must be the case that $V'(w(\theta)) \leq V'(w)$. Moreover, if there is a positive probability set for which $w(\theta) > w$, it must be true that $E^{\mathcal{Q}}[V'(w(\theta))] < V'(w)$. Since

$$E^{\mathcal{Q}}[V'(w(\theta))] = V'(w) - E^{\mathcal{Q}}\left[(\xi(\theta) - \zeta(\theta))\left(\left[f(\theta_1) f(\theta_2) - \dot{\lambda}_1(\theta_2) f(\theta_1)\right]\right)\right],$$

one must have

$$E^{\mathcal{Q}}\left[(\xi(\theta) - \zeta(\theta))\left[f(\theta_1) f(\theta_2) - \dot{\lambda}_1(\theta_2) f(\theta_1)\right]\right] > 0,$$

which implies that $\xi(\theta) > 0$ for some θ. Hence, $w(\theta) = \underline{w}_2$ for those θ; a contradiction. Similar arguments can be used to show that one cannot have $w(\theta) \leq w < \overline{w}_2$ for all θ, with strict inequality with positive probability.

The only remaining possibility is that $w(\theta) = w \in (\underline{w}_2, \overline{w}_2)$ for almost all θ. Plugging this into the first order condition for $w(.)$, we get:

$$\begin{aligned} & V'(w) f(\theta_1) f(\theta_2) + \gamma f(\theta_1) f(\theta_2) - \dot{\lambda}_2(\theta_2) f(\theta_1) - \dot{\lambda}_1(\theta_1) V'(w) f(\theta_2) \\ = \; & V'(w) f(\theta_1) f(\theta_2) - V'(w) f(\theta_1) f(\theta_2) - \dot{\lambda}_2(\theta_2) f(\theta_1) - \dot{\lambda}_1(\theta_1) V'(w) f(\theta_2) = 0, \end{aligned}$$

where, from the first equation to the second, we have used that, by the Envelope Theorem, $V'(w) = -\gamma$.

Dividing both sides by $f(\theta_1) f(\theta_2) > 0$,

$$V'(w) - \frac{\dot{\lambda}_2(\theta_2)}{f(\theta_2)} - V'(w) \frac{\dot{\lambda}_1(\theta_1)}{f(\theta_1)} = V'(w) \text{ for almost all } (\theta_1, \theta_2).$$

Therefore, one must have

$$\frac{\dot{\lambda}_2(\theta_2)}{f(\theta_2)} = -\frac{\dot{\lambda}_1(\theta_1)}{f(\theta_1)} V'(w) \text{ for almost all } (\theta_1, \theta_2).$$

Since the left hand side depends only on θ_2, and the right hand side on θ_1, the above equality can hold for almost all (θ_1, θ_2) only if $\dot{\lambda}_1(\theta_1) = \dot{\lambda}_2(\theta_2) = 0$ for almost all (θ_1, θ_2).

Furthermore, since for all $s \in \left[\frac{1}{2}, 1\right]$, $\lambda_i\left(\frac{1}{2} - s\right) = -\lambda_i\left(\frac{1}{2} + s\right)$ – this follows from the symmetry of the problem around $\frac{1}{2}$ –, one must have $\lambda_i(\theta_i) = 0$ for all i, and θ_i. Plugging this in the FOC for $a(.)$, one gets:

$$\left[\frac{\partial u(a(\theta), \theta_1)}{\partial a} + \gamma \frac{\partial u(a(\theta), \theta_2)}{\partial a}\right] = 0.$$

It is easy to see that the action $a(.)$ implicitly defined by the above equation is not IC when continuation values are constant, unless $\gamma = 0$, or $\gamma = \infty$; which cannot hold as $\gamma \in [-V'(\underline{w}_2), -V'(\overline{w}_2)]$. ■

Proof of Proposition 2. Assume, toward a contradiction, that, once $w(\theta)$ hits $\underline{w}_2$, it stays there forever. The FOC for $w(\theta)$ evaluated at $\underline{w}_2$ is given by

$$V'(\underline{w}_2) f(\theta_2)\left[f(\theta_1) - \dot{\lambda}_1(\theta_1)\right] + \gamma f(\theta_1)\left[f(\theta_2) - \dot{\lambda}_1(\theta_2)\right] = -\xi(\theta)$$

where $\xi(\theta) \geq 0$ is the multiplier associated to the constraint

$$w(\theta) \geq \underline{w}_2.$$

.Using the Envelope Condition, $\gamma = -V'(\underline{w}_2)$, we have

$$V'(\underline{w}_2)\left(f(\theta_2)\left[f(\theta_1) - \dot{\lambda}_1(\theta_1)\right] - f(\theta_1)\left[f(\theta_2) - \dot{\lambda}_1(\theta_2)\right]\right) = -\xi(\theta).$$

Since $\xi(\theta) \geq 0$ and $V'(\underline{w}_2) < 0$,

$$f(\theta_2)\left[f(\theta_1) - \dot{\lambda}_1(\theta_1)\right] - f(\theta_1)\left[f(\theta_2) - \dot{\lambda}_1(\theta_2)\right] \geq 0 \; for \; all \; \theta.$$

In particular for θ' of the form (θ_2, θ_1), so

$$\left[f(\theta_1)\left[f(\theta_2) - \dot{\lambda}_2(\theta_2)\right] - f(\theta_2)\left[f(\theta_1) - \dot{\lambda}_2(\theta_1)\right]\right] \geq 0.$$

Hence,

$$\left[f(\theta_1)\left[f(\theta_2) - \dot{\lambda}_2(\theta_2)\right] - f(\theta_2)\left[f(\theta_1) - \dot{\lambda}_2(\theta_1)\right]\right] = 0,$$

implying that $\xi(\theta) = 0$ for all θ.

The proof now follows exactly the same steps as the proof of the Spreading Values Proposition. Indeed, plugging $\xi(\theta) = 0$ for all θ back in the FOC evaluated at $w(\theta) = \underline{w}_2$, we get

$$V'(\underline{w}_2) f(\theta_2)\left[f(\theta_1) - \dot{\lambda}_1(\theta_1)\right] - V'(\underline{w}_2) f(\theta_1)\left[f(\theta_2) - \dot{\lambda}_1(\theta_2)\right] = 0$$

which calls for

$$\dot{\lambda}_1(\theta_1) = \dot{\lambda}_1(\theta_2) \text{ for all } \theta.$$

Since the left hand side only depends on θ_1, and the right hand side on θ_2, this can only hold if $\dot{\lambda}_1(\theta_1) = \dot{\lambda}_2(\theta_2) = 0$ for all (θ_1, θ_2).

Furthermore, since for all $s \in \left[\frac{1}{2}, 1\right]$, $\lambda_i\left(\frac{1}{2} - s\right) = -\lambda_i\left(\frac{1}{2} + s\right)$ – this follows again from the symmetry of the problem around $\frac{1}{2}$ –, one must have $\lambda_i(\theta_i) = 0$ for all i, and θ_i. Plugging this in the FOC for $a(.)$, one gets:

$$\left[\frac{\partial u(a(\theta), \theta_1)}{\partial a} + \gamma \frac{\partial u(a(\theta), \theta_2)}{\partial a}\right] = 0.$$

It is easy to see that the policy $a(.)$ implicitly defined by the above equation is not IC when continuation values are constant ($w(\theta) = \underline{w}_2$ for all θ, and $V(w(\theta)) = V(\underline{w}_2)$ for all θ), unless $\gamma = 0$, or $\gamma = \infty$; which cannot hold as $\gamma = -V'(\underline{w}_2)$. ■

Proof of Theorem 2. We show that a sufficient condition for the operator T^*, defined in the text, being a contraction in the total variation norm is satisfied.

Proposition 2 implies that, if $w = \underline{w}_2$, there is a strictly positive probability of $w' > \underline{w}_2$. Moreover, Proposition 1 implies that, for all $w \in (\underline{w}_2, \overline{w}_2)$, there is a strictly positive probability of $w' > w$.

For any $w \in [\underline{w}_2, \overline{w}_2]$ and set A, define $Q^n(w, A)$ as the probability of, starting at w, getting to the set A in n periods. Since $[\underline{w}_2, \overline{w}_2]$ is compact, it follows from Propositions 1 and 2 and that there exists a finite $M > 0$ and a $\gamma(M) > 0$ so that, for all $w \in [\underline{w}_2, \overline{w}_2]$, $Q\left(w, w' \in \left[w + \frac{1}{M}, \overline{w}_2\right]\right) \geq \gamma(M) > 0$.

Define $w^1 = \underline{w}_2$ and $w^n = w^{n-1} + \frac{1}{M}$. Also, let $A_n = [w^n, \overline{w}_2]$. Note that $Q(w^1, A_2) \geq \gamma(M) > 0$.

Moreover, we have that

$$Q^2(w^1, A_3) = \int\limits_{z\in W} Q(z, A_3)Q(w^1, dz) \geq$$
$$\int\limits_{z\in A_2} Q(z, A_3)Q(w^1, dz) \geq \int\limits_{z\in A_2} \gamma(M)Q(w^1, dz) \equiv q_2 > 0.$$

The first equality is the definition of $Q^2(.,.)$. The first inequality follows because $A_2 \subset W$. The second inequality follows because, for all z in A_2, $Q(z, A_3) \geq \gamma(M)$. Finally, q_2 being strictly positive follows because, since $Q(w^1, A_2) \geq \gamma(M) > 0$, for a positive probability subset $B_2 \subset A_2$, $Q(w^1, B_2)$ is positive and bounded away from zero.

Proceeding inductively,

$$\begin{aligned} Q^n(w^1, A_{n+1}) &= \int\limits_{z\in W} Q(z, A_{n+1})Q^{n-1}(w^1, dz) \geq \int\limits_{z\in A_n} Q(z, A_{n+1})Q^{n-1}(w^1, dz) \\ &> \int\limits_{z\in A_n} \gamma(M)Q^{n-1}(w^1, dz) \equiv q_n > 0. \end{aligned}$$

where, again, the equality is the definition of $Q^n(w^1, A_{n+1})$, the first inequality follows because $A_n \subset W$, the second inequality because, for all $z \in A_n$, $Q(z, A_{n+1}) > \gamma(M)$. As before, q_n is strictly positive because for a positive probability subset $B_n \subset A_n$, $Q^{n-1}(w^1, B_n)$ is positive and bounded away from zero.

Now, pick $\overline{N}$ such that $w^{\overline{N}} + \frac{1}{M} \geq \overline{w}_2$. Clearly, this $\overline{N}$ is finite. Since $\overline{N}$ is finite, $Q^{\overline{N}}(\underline{w}_2, A^{\overline{N}}) \equiv q_{\overline{N}} > 0$. Setting $\epsilon = q_{\overline{N}}$, condition M of Stokey et al ((1989), p. 348) holds.

Theorem 11.12 of Stokey et al (1989, p. 350) then applies, implying that the operator T^*, defined in the text, is a contraction in the total variation norm. Hence, starting from any distribution φ_0, the sequence φ_n converges to a unique distribution φ^*, which is the unique fixed point of T^*. ■

7.4 Dictatorship under a single Participation Constraint

Proof of Proposition 3. Immediate from Lemma 6 and the fact that, when only agent 2 has an outside option, $\zeta(\theta) = 0$ for all θ. ■

Proof of Proposition 4. Immediate from Lemma 6 and the fact that, when only agent 1 has an outside option, $\xi(\theta) = 0$ for all θ. ■

Proof of Theorem 4. We prove the result for the case in which only agent 2 has an outside option; the other case is analogous. From Proposition 3, when only agent 2 has an outside option, $V'(w)$ is a supermartingale. By Dobb's Convergence Theorem (Dobb, 1953), the stochastic process $V'(w)$ converges to a random variable, R . Suppose there existed positive probability of finding a path $V'(w_t)$ with the property that $lim_{t\to\infty}\ V'(w_t) = C$, where $-\infty < C \leq V'(\underline{w}_2)$. Since $V'(w)$ is continuous for any $v \in (\underline{w}_2, \overline{v})$, the sequence v_t must converge. Let $lim_{t\to\infty}\ v_t = v' \in [\underline{v}_2, \overline{v})$, be the limit of agent 2's continuation values. Let $g(w, \theta)$ denote the next period's continuation value given the current promised value w and reported state θ. For w_t to converge to w', it must be that $g(w', \theta) = w'$ for all θ. This however contradicts Propositions 1 and 2. ■