Repeated Coordination with Privately Observed Preferences Shocks*

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Abstract

Abstract: We study a setting in which two privately informed agents have to take a commonly agreed decision over time, and cannot resort to side payments. If players interact just once, they cannot implement a Pareto Efficient rule. By endowing the players with future, the we show that the extent to which the best Public Perfect Equilibrium improves upon the D-D mechanism depends on whether asymmetric continuation values can be provided. If the latter cannot be provided, the best equilibrium replicates the DD mechanism at every single period irrespective of the player's discount factor. If asymmetric continuation values can be provided, improvements can be attained. First best can be approximated (but not achieved) when the players are extremely patient. We also show that the provision of intertemporal incentives necessarily leads to a dictatorial mechanism: the optimal scheme converges to the adoption of one player's favorite action.

1 Introduction

(THIS IS WHAT WE HAVE WRITTEN BEFORE... HAVE TO BE CHANGED)

Many situations of interest require two agents to take a joint action repeatedly over time. Examples abound– managers of two different divisions within a firm, parents deciding on some aspects of their children's education, Monetary Union members deciding on monetary policy, parties in a political coalition, and others. This paper analyzes such types of situations for the case in which the players' preferences over actions are private information, utility is non-transferable and the action to be chosen belongs to an interval.[1] Under repetition, even though players cannot transfer money between them, they can potentially transfer continuation utilities. We ask ourselves (i) if this allows for better per period allocations, and (ii) how does the implicit contract between the players evolve over time

Our approach to the analysis of the optimal allocation rule in the repeated game relies on the factorization results of Abreu, Pearce and Stacchetti (1990) which split the agent's payoff in a current value and a

*We have benefited from conversations with Simon Board, Woulter Dessein, Sergio Firpo, Niko Matoucheck, Roger Myerson, Alessandro Pavan, Phil Reny, Leo Rezende, Yuliy Sannikov, Andy Skrzypacz, and Balazs Szentes. We are particularly thankful to Ferdinando Monte for careful comments on an early draft. Antonio Sodre provided excellent research assistance.

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[1] If the decision were contractible and choices binary, a simple voting mechanism is able to attain the efficient outcome. See for example XXX. Alternatively, if transfers are possible and players have quasilinear utility, the problem could be easily solved using the expected externality mechanisms proposed by Arrow (1979), and d'Aspremont and Gerard-Varet (1979).

continuation value. In our setting, the continuation values of an agent play the same role as the number of remaining votes in Casella and Palfrey (2003) since we show that agents with larger continuation values get more future decision power. Furthermore, we show that in the limit one of the agents will eventually "run out of votes" which implies that the other agent essentially dictates all future allocations. This result is similar to the immiseration results found in many dynamic insurance problems, see for example Thomas and Worral (1990). An agent that reports to have an extreme type in a given period is like an agent that reports to have a low income realization, the optimal mechanism will respond by giving that agent more weight in the current allocation decision (similarly a higher transfer today) and in order to preserve incentives he "pays" for this by forgoing future weight in the allocation decision (future consumption). Our environment has a few differences with that of Thomas and Worral (1990). First, the problem is symmetric, we have two agents rather than a principal and an agent. Second, dictatorship is an absorbing state in the sense that once an agent is granted all the decision rights incentive constraints will not bind any longer and the continuation values will be constant rather than constantly drifting towards $-\infty$ (*immiseration*).

The other main result we establish in this section is that as the discount factor δ goes towards 1 we can attain values arbitrary close to those achievable with the first best allocation. What happens is that as $\delta \to 1$ the value of the current period which is weighted by $(1 - \delta)$ becomes insignificant relative to the continuation values. Hence, in order to guarantee truth-telling in the current period continuation values have to vary only minimally. Since even without transfers the utility possibility frontier is locally linear, this implies that the associated losses from the variation in continuation values become negligible. Hence, we have a similar result to Jackson and Sonnenschein (2006) Corollary 2 in which they prove that for any $\varepsilon > 0$ there is a δ large enough and finite but large number of periods such that there is less than ε inefficiency relative to the first best. The sources of the inefficiency are quite different though. In their setting, what happens when you get close to the last periods is that agents are not allowed to report truthfully since they might have run out of their budgeted reports for a particular type. In particular both agents could have the same type yet the allocation be different. Instead, in our setting the inefficiency arrises from two sources. On one hand we slowly drift towards one of the agents becoming a dictator and on the other relative to the first best the allocation is biased towards the center in order to provide incentives.

Higher continuation values for an agent are delivered by having his preferences have more weight on the determination of future allocations. Continuation values are used to provide incentives and vary from period to period. They tend to increase for the agent that reports a less extreme type, and this asymptotically leads to one of the agents becoming a dictator. That is, eventually only the preferences of one agent are taken into account to determine allocations. Dictatorship is incentive compatible hence there is no need to vary continuation values to implement it. This implies it is an absorbing state. Therefore, once an agent gets all the decision rights he will have them for ever.

Repetition is key in many of the settings in which agents have to agree on a joint action. It is natural then to ask if repetition allows for better per period allocations, and what are the properties of the best allocation rule. Similar questions have been risen in the social choice literature. For example, Jackson and Sonnenschein (2006) have shown in a more general setting that the inefficiencies arising from incentive compatibility considerations become arbitrary small when there are a large number of independent decisions to be taken simultaneously. This is established by defining a mechanism in which agents must budget their representations of preferences so that the frequency of preferences across problems mirrors the underlying distribution of preferences, and then arguing that agents' incentives are to satisfy their budget by being as truthful as possible. In a related paper Casella and Palfrey (2003) consider a repeated setting in which a binary choice must be taken each period and agents can different intensity of preferences then given them a number of vote to use across decisions increases efficiency since the agents in equilibrium use their votes mainly when they feel strongly about an issue.

2 The Model: Stage Game

We consider a setting in which two ex-ante symmetric players, $i = 1, 2$, have to take a joint action a. Player $i\H{}s$, favorite action is determined by a random variable $\theta_i \in [0,1]$, which is independently distributed across agents according to $F(.)$, which is symmetric around $\frac{1}{2}$. Players are privately informed about their favorite action.

Their (Bernoulli) utility function is

$$u_i(a, \theta_i)$$

with $\frac{\partial^2 u_i(a,\theta_i)}{\partial a \partial \theta_i} > 0 > \frac{\partial^2 u_i(a,\theta_i)}{\partial a^2}$, and

$$u_i(\theta_i, \theta_i) \geq u_i(a, \theta_i).$$

To save on notation, we normalize $u_i(\theta_i, \theta_i)$ to zero.

[to be completed]

[WHAT CAN WE SAY HERE? FIRST OBVIOUS POINT IS: WITHOUT SIDE PAYMENTS, ONE CANNOT IMPLEMENT EFFICIENT SCHEDULES OTHER THAN THE DICTATORIAL ONES. MORE SPECIFICALLy, THE SET OF ALL PARETO EFFICIENT ALLOCATIONS SOLVE

$$\max_a \beta_1 u_1(a, \theta_1) + \beta_2 u_2(a, \theta_2)$$

FOR $\beta_1, \beta_2 \geq 0$. *AMONG THOSE ALLOCATIONS, ONLY THE ONES ASSOCIATED WITH* $\beta_1 = 0$, *AND* $\beta_2 = 0$ *CAN BE IMPLEMENTED IN AN IC FASHION*).

3 The Infinitely Repeated Game

The Setup:

Consider repeating the game described in Section **??** an infinite number of periods. At each period $t \in \{0, 1, ...\}$ the two agents receive new i.i.d. preference shocks θ_i drawn from a uniform distribution over $[0,1]$. After they observe their preference shocks, they make reports $\tilde{\theta}_1$ and $\tilde{\theta}_2$. Given the reports and the history of the game denoted by h^t and defined below, a history dependent allocation is chosen according to a sequence of functions

$$\left\{a_t\left(\tilde{\theta}_i, \tilde{\theta}_{-i}, x^t, h^{t-1}\right) : [0,1]^3 \times [0,1]^{3(t-1)} \to [0,1]\right\}_{t=1}^{\infty}.$$

This allocation rule is chosen a priori before the agents learn their preference shocks.

Public Histories, Public Strategies and Equilibrium Payoffs

The public history at time t, h^t, is a sequence of (i) past announcements of the two players, (ii) past realized actions, and (iii) the realizations of a public random device x that serves convexification purposes:

$$h^t = \{\varnothing, \left(\tilde{\theta}_1^1, \tilde{\theta}_2^1, a^1, x^1\right), ..., \left(\tilde{\theta}_1^{t-1}, \tilde{\theta}_2^{t-1}, a^{t-1}, x^{t-1}\right)\}.$$

Let H^t be the set of all public histories of length t. A public strategy for player i is a sequence of functions $\{\tilde{\theta}_i^t(.,.)\}_t,$ where

$$\tilde{\theta}_i^t : H^t \times [0,1] \to [0,1].$$

Each pair of strategies $\tilde{\theta} = \left(\left\{\tilde{\theta}_1^t\right\}, \left\{\tilde{\theta}_2^t\right\}\right)$ defines a probability distribution over public histories and, as consequence, player i's expected payoff is given by:

$$E\left[(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i^t(a(\tilde{\theta}^t);\theta_i^t) \mid \widehat{\theta}\right].$$

We analyze this game using the recursive methods developed by Abreu, Pearce and Stacchetti (1990). More specifically, letting $W \subset \Re^2$ be the set of Public Pure Strategy Equilibria Payoffs for the Players[2], we can decompose the payoff into a current utility $u_i(a, \theta)$ and a continuation value $v_i(\tilde{\theta}) \in W$,

$$E_\theta[(1-\delta)u_i(a, \theta_i) + \delta v_i(\tilde{\theta}_i, \tilde{\theta}_{-i})],$$

In other words, any PPS equilibrium can be summarized by the actions to be taken in the current period and equilibrium continuation values as function of the announcements.

We can use this decomposition to write the Bellman equation that characterizes the frontier of equilibrium values that can be attained in this environment. Let $V(v)$ be the highest value to player 1 given that player 2 expected value is v. Clearly, the expected value for a player that always picks his favorite allocation is zero. Now, define v^s as the expected value for a player when the other one always chooses the allocation; that is, $E_\theta[u_i(\theta_i, \theta_{-i})]$.

We can write $V(v)$ as:

$$V(v) = \max_{\{a(\theta,x), w(\theta,x)\}_\theta} E_\theta[(1-\delta)u_1(a(\theta), \theta_1) + \delta V(w(\theta, x))]$$

s.t.

$$E_\theta[(1-\delta)u_2(a(\theta), \theta_2) + \delta w(\theta, x)] = v \quad \text{(Promise Keeping)}$$

$$E_{\theta_1}\left(\left[\frac{du_2(a(\theta), \theta_2)}{da}\frac{da(\theta, x)}{d\theta_2}|\theta_2\right] + \delta\frac{d}{d\theta_2}(E[w(\theta, x)]|\theta_2)\right) = 0. \quad \text{(IC Local)}$$

$$E\left(\left[\frac{du_1(a(\theta), \theta_1)}{da}\frac{da(\theta, x)}{d\theta_1}|\theta_1\right] + \delta\frac{d}{d\theta_1}(E[V(w(\theta, x))|\theta_1])\right) = 0 \quad \text{(IC Local)}$$

$$(V(w(\theta)), w(\theta)) \in W \ for\ all\ \theta, \quad \text{(Feasibility)}$$

$$E_{\theta_{-i}}[a(\theta_i, \theta_{-i})] \ is\ non-decreasing \quad \text{(Expected Monotonicity)}$$

[2] This set is not empty as the infinite repetition of the actions taken in the static game constitutes a PPS equilibrium, and there are plenty of equilibria in the stage game.

where, as standard, Incentive compatibility is substituted by the first order conditions that capture local incentive compatibility together with expected monotonicity which, together, are sufficient to guarantee global incentive compatibility.

Properties of the Optimal Allocation Rules

When Agents interact repeatedly, continuation values can play a similar role to the one side payments play in standard incentive problems. The difference between side payments and continuation values is that the latter can only imperfectly transfer utility across players. In particular, to transfer continuation utility from player 1 to player 2 in any period t, *allocations* for periods $\tau > t$ must be altered. When $\delta \to 1$ what happens is that the value of the current period which is weighted by $(1-\delta)$ becomes insignificant relative to the continuation values. Hence, in order to guarantee truth-telling in the current period continuation values have to vary only minimally. Since locally $V(v)$ is linear, this implies that the associated losses from the variation in continuation values becomes negligible.

Proposition 1 *As $\delta \to 1$, the first best payoffs can be arbitrarily approximated by PPE payoffs.*

In practice, higher continuation values for an Agent are delivered by having his preferences have more weight on the determination of the future allocations. In general, the continuation values are used to provide incentives and vary from period to period. Continuation values tend to increase for the Agent that reports a less extreme type. This leads to the following result: asymptotically, one of the Agents becomes a dictator, that is, eventually only his preferences are taken into account to determine the current and future allocations.

Proposition 2 (Dictatorship in the limit) *The provision of intertemporal incentives necessarily leads to a dictatorial mechanism: In the limit as $t \to \infty$, either v or $V(v)$ converge to 0 almost surely.*

The proof is detailed in the appendix but the main ingredients are as follows. In solving for the optimal allocation rule and continuation values one of the conditions for optimality is:

$$E\left[V'\left(w\left(\theta\right)\right)\right] = V'\left(v\right),$$

so that the marginal value for player 1 follows a non-positive martingale.

The optimal provision of incentives also requires that, for any given $w \in \left(E_{\theta}\left[u_i\left(\theta_{-i}, \theta_i\right)\right], 0\right)$, there is a positive mass of types for which $w(\theta) > v$, and another for which $w(\theta) < v$.[3]. This implies that, with probability 1, eventually either

$$w(\theta) \to 0, \text{ or } V\left(w\left(\theta\right)\right) \to 0.$$

Whenever an Agent is promised continuation values of 0, it must be the case that his most favorite action is taken from then on.

This result is similar to the immiseration results found in many dynamic insurance problems, see for example Thomas and Worral (1990). The difference with the hidden income models is that instead of having incentive to claim to be poorer than one actually is, here the incentives are to claim one is more extreme than

[3]As a side remark, we point here that it is optimal to provide continuation values that not only depend on the realization of types but also depend on the identity of the players. Indeed, it can be easily shown that the best equilibrium for the case in which continuation values are symmetric across players' identity replicates the DD mechanism at every period irrespective of the discount factor.

one actually is. To restore incentive compatibility, extreme types lose future decision rights and eventually the point is reached where they have no say at all on the future allocations. Endogenous participation constraints as those considered in the Monetary Union model of Fuchs and Lippi (2005), which we abstract from in the present setting, would provide a countervailing force against one of the agents becoming too powerful.

4 APPENDIX: The Dictator Result

(THE LEMMAS ARE WRITTEN IN THE WAY I THINK MAKES MORE SENSE... THIS SEEMS TO ME TO BE THE NATURAL ORDER. SOME OF THE RESULTS CAN BE IN THE TEXT)

The agent´s Bernoulli utilty are given by $u_i(a,\theta_i)$ with $\frac{\partial^2 u_i(a,\theta_i)}{\partial a \partial \theta_i} > 0 > \frac{\partial^2 u_i(a,\theta_i)}{\partial a_i^2}$, and

$$u_i(\theta_i,\theta_i) \geq u_i(a,\theta_i) \text{ for all } a.$$

Moreover, θ_i is drawn from a general $F(\theta_i)$ which is symmetric around $\frac{1}{2}$.

Our program of interest can be read as

$$V(v) = \max_{\{a(\theta,x),w(\theta,x)\}_\theta} E_\theta\left[(1-\delta)u_1(a(\theta),\theta_1) + \delta V(w(\theta,x))\right]$$

s.t.

$$E_\theta\left[(1-\delta)u_2(a(\theta),\theta_2) + \delta w(\theta,x)\right] = v \qquad \text{(Promise Keeping)}$$

$$E_{\theta_1}\left[(1-\delta)u_2(a(\theta),\theta_2) + \delta w(\theta)\right] \geq E_{\theta_1}\left[(1-\delta)u_2\left(a\left(\widehat{\theta}_2,\theta_1\right),\theta_2\right) + \delta w\left(\widehat{\theta}_2,\theta_1\right)\right] \qquad \text{(IC2)}$$

$$E_{\theta_2}\left[(1-\delta)u_1(a(\theta),\theta_1) + \delta V(w(\theta))\right] \geq E_{\theta_1}\left[(1-\delta)u_2\left(a\left(\widehat{\theta}_2,\theta_1\right),\theta_2\right) + \delta V\left(w\left(\widehat{\theta}_1,\theta_2\right)\right)\right] \qquad \text{(IC1)}$$

$$(V(w(\theta)), w(\theta)) \in W \textit{ for all } \theta, \qquad \text{(Feasibility)}$$

4.1 Preliminaries

We first prove the following result that will allow us, in search of an optimum, make use of Lagrangian methods.

Lemma 1 $V(.)$ *is strictly concave*

Proof. Let $(a_1(\theta), w_1(\theta))$ and $(a_2(\theta), w_2(\theta))$ be, respectively, solutions of the problem when the promise keeping constraint is indexed by w_1 and w_2, respectively.

If it were feasible to implement $a^{\alpha}(\theta) = \alpha a_1(\theta) + (1-\alpha) a_2(\theta)$, where $\alpha \in (0,1)$, with continuation values $w^{\alpha}(\theta) = \alpha w_1(\theta) + (1-\alpha) w_2(\theta)$, we would have

$$\begin{aligned} & E_{\theta}\left[(1-\delta) u_2\left(a^{\alpha}(\theta), \theta_2\right) + \delta w^{\alpha}(\theta)\right] \\ > \; & \alpha E_{\theta}\left[(1-\delta) u_2\left(a_1(\theta), \theta_2\right) + \delta w_1(\theta)\right] + (1-\alpha) E_{\theta}\left[(1-\delta) u_2\left(a_2(\theta), \theta_2\right) + \delta w_2(\theta)\right] \\ = \; & \alpha w_1 + (1-\alpha) w_2 = w^{\alpha}, \end{aligned}$$

where the first inequality follows from the strict concavity of $u_2(., \theta_2)$, and the equality follows from the definition of $(a_1(\theta), w_1(\theta))$ and $(a_2(\theta), w_2(\theta))$.

Now, since the inequality above is strict, we can find, for some $\overline{w} \geq w^{\alpha}$, a non-negative function $g(\theta) \equiv g_1(\theta_1) + g_2(\theta_2)$ such that

$$E_{\theta}\left[u_2\left(a^{\alpha}(\theta), \theta_2\right) + \delta\left[w^{\alpha}(\theta) - g(\theta)\right]\right] = \overline{w}, \tag{1}$$

and at the same time

$$\begin{aligned} & \delta E_{\theta_1}\left[w^{\alpha}(\theta) - g(\theta)\right] \\ = \; & -E_{\theta_1}\left[(1-\delta) u_2\left(a^{\alpha}(\theta), \theta_2\right)\right] + \left[\begin{array}{c} E_{\theta_1}\left[u_2\left(a^{\alpha}(0, \theta_1), \theta_2\right) + \delta\left[w^{\alpha}(0, \theta_1) - g(0, \theta_1)\right]\right] \\ + (1-\delta) \int_0^{\theta_2} E_{\theta_1}\left[\frac{\partial u_2(a(\tau, \theta_1), \theta_1)}{\partial \tau}\right] d\tau \end{array}\right] \end{aligned} \tag{2}$$

and

$$\begin{aligned} & \delta E_{\theta_2}\left[V\left(w^{\alpha}(\theta) - g(\theta)\right)\right] \\ = \; & -E_{\theta_2}\left[(1-\delta) u_1\left(a^{\alpha}(\theta), \theta_1\right)\right] + \left[\begin{array}{c} E_{\theta_2}\left[u_1\left(a^{\alpha}(0, \theta_2), 0\right) + \delta V\left(w(0, \theta_2) - g(0, \theta_2)\right)\right] \\ + (1-\delta) \int_0^{\theta_1} E_{\theta_2}\left[\frac{\partial u_1(a(\tau, \theta_2), \theta_2)}{\partial \tau}\right] d\tau \end{array}\right] \end{aligned} \tag{3}$$

Conditions 2 and 3 guarantee Incentive Compatibility (indeed, with these values, the integral formula implied by the Envelope Theorem holds. Also, since both $a_1(\theta)$ and $a_2(\theta)$ satisfy expected monotonicity, so will $a^{\alpha}(\theta)$).

Therefore, $a^{\alpha}(\theta)$, coupled with continuation values $w^{\alpha}(\theta) - g(\theta)$, is feasible when the promised value for player 2 is $\overline{w} \geq w^{\alpha}$

It then follows

$$\begin{aligned} V\left(w^{\alpha}\right) &= V\left(\alpha w_1 + (1-\alpha) w_2\right) \geq V(\overline{w}) \\ &\geq E_{\theta}\left[(1-\delta) u_1\left(a^{\alpha}(\theta), \theta_1\right) + \delta V\left(w^{\alpha}(\theta) - g(\theta)\right)\right] \\ &> \alpha\left[E_{\theta}\left[(1-\delta) u_1\left(a_1(\theta), \theta_1\right)\right]\right] + (1-\alpha)\left[E_{\theta}\left[u_1\left(a_2(\theta), \theta_1\right)\right]\right] + \delta V\left(w^{\alpha}(\theta)\right) \\ &\geq \left(\begin{array}{c} \alpha\left[E_{\theta}\left[(1-\delta) u_1\left(a_1(\theta), \theta_1\right)\right] + \delta V\left(w_1(\theta)\right)\right] \\ + (1-\alpha)\left[E_{\theta}\left[(1-\delta) u_1\left(a_2(\theta), \theta_1\right)\right] + \delta V\left(w_2(\theta)\right)\right] \end{array}\right) \\ &= \alpha V\left(w_1\right) + (1-\alpha) V\left(w_2\right), \end{aligned}$$

where the first inequality follow from the fact that $V(.)$ is decreasing, the second inequality follows from the fact that $a^{\alpha}(\theta)$ along with $w^{\alpha}(\theta) - g(\theta)$ is feasible when the promised value for player 2 is $\overline{w}$, the third inequality follow from strict concavity of Player 1 instantaneous payoff, and the fact that $V(.)$ is decreasing), which proves the result. ■

Another technical result that will be of use is

Lemma 2 *$V(.)$ is continuously differentiable.*

Proof. Since $V(.)$ is strictly concave, $V'(w)$ exists almost everywhere. We now argue that it is everywhere continuous. As $V(.)$ is strictly concave, $V'(w)$ strictly decreasing, and, therefore, continuous except possibly at a countable set. Now, if $V'(w)$ were discontinuous at one point, that would mean that $V(.)$ had a kink at that point. This, in turn, would imply that more than a solution to the maximization problem would exist. Proceeding exactly like in the Lemma above, one can rule this possibility out. ■

Now, as standard, we substitute the IC constraints by their local counterparts along with a monotonicity condition

Lemma 3 *A pair $(a(.), w(.))_\theta$ is Incentive Compatible if, and only if, they satisfy*

$$E_{\theta_1}\left(\left[\frac{du_2(a(\theta),\theta_2)}{da}\frac{da(\theta,x)}{d\theta_2}|\theta_2\right]+\delta\frac{d}{d\theta_2}\left(E\left[w(\theta,x)\right]|\theta_2\right)\right)=0. \qquad \text{(IC Local)}$$

$$E\left(\left[\frac{du_1(a(\theta),\theta_1)}{da}\frac{da(\theta,x)}{d\theta_1}|\theta_1\right]+\delta\frac{d}{d\theta_1}\left(E\left[V\left(w(\theta,x)\right)|\theta_1\right]\right)\right)=0 \qquad \text{(IC Local)}$$

$$E_{\theta_{-i}}\left[\frac{\partial u_i(a(\tau,\theta_{-i}),\theta_i)}{\partial\theta_i}\right] \ \textit{is non-decreasing in } \tau. \qquad \text{(Expected Monotonicity)}$$

Proof. Standard. ■

Now, ignoring (Expected Monotonicity), we construct the Lagrangian by assigning multipliers to the IC $(\lambda_i(\theta_i))_{i=1,2,\theta_i\in[0,1]}$, and PK (γ) constraints:

$$V(v)=\max_{\{a(.),w(.)\}}\left[\begin{array}{c}E_\theta\left[(1-\delta)u_1(a(\theta),\theta_1)+\delta V(w(\theta,x))\right]\\ \gamma\left(E_\theta\left[(1-\delta)u_2(a(\theta),\theta_2)+\delta w(\theta,x)\right]-v\right)\\ +\int_0^1\left[\lambda_2(\theta_2)\left((1-\delta)E_{\theta_1}\left(\left[\frac{du_2(a(\theta),\theta_2)}{da}\frac{da(\theta,x)}{d\theta_2}|\theta_2\right]+\delta\frac{d}{d\theta_2}\left(E\left[w(\theta,x)\right]|\theta_2\right)\right)\right)\right]d\theta_2\\ +\int_0^1\left[\lambda_1(\theta_1)\left((1-\delta)E\left[\frac{du_1(a(\theta),\theta_1)}{da}\frac{da(\theta,x)}{d\theta_1}|\theta_1\right]+\delta\frac{d}{d\theta_1}\left(E\left[V\left(w(\theta,x)\right)|\theta_1\right]\right)\right)\right]d\theta_1\end{array}\right]$$

Lemma 4 *Take an optimal $\{a^*(\theta), w^*(\theta)\}_\theta$, and assume that $a^*(.)$ satisfies expected monotonicity. Then, $\{a^*(\theta), w^*(\theta)\}_\theta$ maximizes the above Lagrangian for monotone multipliers $\{\lambda_i(\theta_i)\}$.*

Proof. This follows from Theorems 1 and 2 in sections 8.3-8.4 of Luenberger (1969) along with the fact that $V(.)$ is concave – the concavity of $V(.)$ plays the role that Proposition 1 in section 8.3 of Luenberger (1969) has for Theorems 1 and 2. ■

4.2 The Lagrangian Representation and the Result

We now move on to rewrite the Lagrangian in a more convenient way. Note that

$$\begin{aligned}&\frac{d\left[(1-\delta)E_{\theta_2}\left[u_1\left(a\left(\theta\right),\theta_1\right)\right]-\int\limits_0^{\theta_1}(1-\delta)E_{\theta_2}\left[\frac{\partial u_1(a(\tau,\theta_2),\tau)}{\partial\tau}\right]d\tau\right]}{d\theta_1}\\ =\;&(1-\delta)E\left[\frac{du_1\left(a\left(\theta\right),\theta_1\right)}{da}\frac{da\left(\theta,x\right)}{d\theta_1}|\theta_1\right].\end{aligned}$$

Hence

$$\begin{aligned}&\int\limits_0^1\lambda_2\left(\theta_2\right)\left((1-\delta)E_{\theta_1}\left(\left[\frac{du_2\left(a\left(\theta\right),\theta_2\right)}{da}\frac{da\left(\theta,x\right)}{d\theta_2}|\theta_2\right]\right)\right)d\theta_2\\
=\;&\lambda_2\left(\theta_2\right)\left[(1-\delta)E_{\theta_1}\left[u_2\left(a\left(\theta\right),\theta_1\right)\right]-\int\limits_0^{\theta_2}(1-\delta)E_{\theta_1}\left[\frac{\partial u_2\left(a\left(\tau,\theta_1\right),\tau\right)}{\partial\tau}\right]d\tau\right]|_0^1\\
&-\int\limits_0^1\frac{d\lambda_2\left(\theta_2\right)}{d\theta_2}\left[(1-\delta)E_{\theta_1}\left[u_2\left(a\left(\theta\right),\theta_1\right)\right]-\int\limits_0^{\theta_2}(1-\delta)E_{\theta_1}\left[\frac{\partial u_2\left(a\left(\tau,\theta_1\right),\tau\right)}{\partial\tau}\right]d\tau\right]d\theta_2\\
=\;&\lambda_2\left(\theta_2\right)\left[(1-\delta)E_{\theta_1}\left[u_2\left(a\left(\theta\right),\theta_1\right)\right]-\int\limits_0^{\theta_2}(1-\delta)E_{\theta_1}\left[\frac{\partial u_2\left(a\left(\tau,\theta_1\right),\tau\right)}{\partial\tau}\right]d\tau\right]|_0^1\\
&-\int\limits_0^1\frac{d\lambda_2\left(\theta_2\right)}{d\theta_2}(1-\delta)E_{\theta_1}\left[u_2\left(a\left(\theta\right),\theta_1\right)\right]d\theta_2\\
&+\int\limits_0^1\frac{d\lambda_2\left(\theta_2\right)}{d\theta_2}\left[\int\limits_0^{\theta_2}(1-\delta)E_{\theta_1}\left[\frac{\partial u_2\left(a\left(\tau,\theta_1\right),\tau\right)}{\partial\tau}\right]d\tau\right]d\theta_2\\
=\;&\lambda_2\left(\theta_2\right)\left[(1-\delta)E_{\theta_1}\left[u_2\left(a\left(\theta\right),\theta_1\right)\right]-\int\limits_0^{\theta_2}(1-\delta)E_{\theta_1}\left[\frac{\partial u_2\left(a\left(\tau,\theta_1\right),\tau\right)}{\partial\tau}\right]d\tau\right]|_0^1\\
&-\int\limits_0^1\frac{d\lambda_2\left(\theta_2\right)}{d\theta_2}(1-\delta)E_{\theta_1}\left[u_2\left(a\left(\theta\right),\theta_1\right)\right]d\theta_2\\
&+\lambda_2\left(\theta_2\right)\left[\int\limits_0^{\theta_2}(1-\delta)E_{\theta_1}\left[\frac{\partial u_2\left(a\left(\tau,\theta_1\right),\tau\right)}{\partial\tau}\right]d\tau\right]|_0^1-\int\limits_0^1\lambda_2\left(\theta_2\right)(1-\delta)E_{\theta_1}\left[\frac{\partial u_2\left(a\left(\theta_2,\theta_1\right),\theta_2\right)}{\partial\theta_2}\right]d\theta_2\\
=\;&\lambda_2\left(\theta_2\right)\left[(1-\delta)E_{\theta_1}\left[u_2\left(a\left(\theta\right),\theta_1\right)\right]\right]|_0^1-\int\limits_0^1\frac{d\lambda_2\left(\theta_2\right)}{d\theta_2}(1-\delta)E_{\theta_1}\left[u_2\left(a\left(\theta\right),\theta_1\right)\right]d\theta_2-\int\limits_0^1\lambda_2\left(\theta_2\right)(1-\delta)E_{\theta_1}\left[\frac{\partial u_2\left(a\left(\theta_2,\right.\right.}{\partial\theta_2}\right.
\end{aligned}$$

Moreover,

$$\delta\int_0^1 \left[\lambda_2(\theta_2)\frac{d}{d\theta_2}\left(E\left[w(\theta,x)\right]|\theta_2\right)\right]d\theta_2$$

$$= \delta\left[\lambda_2(\theta_2)\left(E\left[w(\theta,x)\right]|\theta_2\right)|_0^1 - \int_0^1 \frac{d\lambda_2(\theta_2)}{d\theta_2}\left(E\left[w(\theta,x)\right]|\theta_2\right)d\theta_2\right]$$

Using the expressions above (and the analogous ones for player 1), one can rewrite the Lagrangian as

$$V(v) = \max_{\{a(.),w(.)\}}\left[\begin{array}{c} E_\theta\left[(1-\delta)u_1(a(\theta),\theta_1)+\delta V(w(\theta,x))\right] \\ \gamma\left(E_\theta\left[(1-\delta)u_2(a(\theta),\theta_2)+\delta w(\theta,x)\right]-v\right) \\ +\lambda_2(\theta_2)\left[(1-\delta)E_{\theta_1}\left[(1-\delta)u_2(a(\theta),\theta_2)\right]\right]+\delta E_{\theta_1}\left[w(\theta)\right]|_0^1 \\ -\int_0^1\left[\frac{d\lambda_2(\theta_2)}{d\theta_2}\left[(1-\delta)E_{\theta_1}\left[u_2(a(\theta),\theta_2)\right]\right]\right]d\theta_2 - \int_0^1\lambda_2(\theta_2)\left[(1-\delta)E_{\theta_1}\left[\frac{du_2(a(\theta),\theta_2)}{d\theta_2}\right]\right]d\theta_2 \\ -\delta\int_{\frac{1}{2}}^1\left[\frac{d\lambda_2(\theta_2)}{d\theta_2}E_{\theta_1}\left[w(\theta)\right]\right]d\theta_2 \\ +\lambda_1(\theta_1)\left[(1-\delta)E_{\theta_2}\left[(1-\delta)u_1(a(\theta),\theta_1)\right]\right]+\delta E_{\theta_2}\left[w(\theta)\right]|_0^1 \\ -\int_0^1\left[\frac{d\lambda_1(\theta_1)}{d\theta_1}\left[(1-\delta)E_{\theta_2}\left[u_1(a(\theta),\theta_1)\right]\right]\right]d\theta_1 - \int_0^1\lambda_1(\theta_1)\left[(1-\delta)E_{\theta_2}\left[\frac{du_1(a(\theta),\theta_1)}{d\theta_1}\right]\right]d\theta_1 \\ -\delta\int_0^1\left[\frac{d\lambda_1(\theta_1)}{d\theta_1}E_{\theta_2}\left[V(w(\theta))\right]\right]d\theta_1. \end{array}\right]$$

The FOC wrt $a(.)$, and $w(.)$ are, respectively, for $\theta\in(0,1)^2$

$$\left(\begin{array}{c} \left[\frac{\partial u_1(a(\theta),\theta_1)}{\partial a}f(\theta_1)-\frac{d\lambda_1(\theta_1)}{d\theta_1}\frac{\partial u_1(a(\theta),\theta_1)}{\partial a}-\lambda_1(\theta_1)\frac{\partial^2 u_1(a(\theta),\theta_1)}{\partial\theta_1\partial a}\right]f(\theta_2) \\ +f(\theta_1)\left[\gamma\frac{\partial u_2(a(\theta),\theta_2)}{\partial a}-\frac{d\lambda_2(\theta_2)}{d\theta_2}\frac{\partial u_2(a(\theta),\theta_2)}{\partial a}-\lambda_2(\theta_2)\frac{\partial^2 u_2(a(\theta),\theta_2)}{\partial\theta_2\partial a}\right] \end{array}\right)=0,$$

and

$$V'(w(\theta,x))f(\theta_1)f(\theta_2)+\gamma f(\theta_1)f(\theta_2)-\frac{d\lambda_2(\theta_2)}{d\theta_2}f(\theta_1)-\frac{d\lambda_1(\theta_1)}{d\theta_1}V'(w(\theta,x))f(\theta_2)=0.$$

We then have

Lemma 5 (Martingale Lemma) *Assume that*

$$\frac{d\lambda_2(\theta_2)}{d\theta_2}=\gamma\frac{d\lambda_1(\theta_2)}{d\theta_2},$$

then, there exists a mesure $\mathcal{Q}$ such that Player 1´s marginal value follows a martingale, i.e.

$$E^{\mathbb{Q}}\left[V'(w(\theta,x))\right]=V'(w).$$

Proof. Under the assumption that

$$\frac{d\lambda_2(\theta_2)}{d\theta_2} = \gamma \frac{d\lambda_1(\theta_2)}{d\theta_2},$$

we can write the FOC wrt to $w(.)$ as

$$\begin{aligned} V'(w(\theta,x)) f(\theta_2)\left[f(\theta_1) - \frac{d\lambda_1(\theta_1)}{d\theta_1}\right] + \gamma f(\theta_1)\left[f(\theta_2) - \frac{d\lambda_1(\theta_2)}{d\theta_2}\right] &= 0 \\ V'(w(\theta,x)) \frac{f(\theta_2)\left[f(\theta_1) - \frac{d\lambda_1(\theta_1)}{d\theta_1}\right]}{f(\theta_1)\left[f(\theta_2) - \frac{d\lambda_1(\theta_2)}{d\theta_2}\right]} &= -\gamma \end{aligned}$$

Dividing both sides of the equality by

$$\int_0^1 \int_0^1 \left[\frac{f(\theta_2)\left[f(\theta_1) - \frac{d\lambda_1(\theta_1)}{d\theta_1}\right]}{f(\theta_1)\left[f(\theta_2) - \frac{d\lambda_1(\theta_2)}{d\theta_2}\right]}\right] d\theta_1 d\theta_2,$$

and integrating over (θ_1, θ_2), one gets

$$E^{\mathbb{Q}}\left[V'(w(\theta,x))\right] = -\gamma,$$

where, as suggested by the notation, the expected value is taken with respect to the density

$$q(\theta_1,\theta_2) = \frac{\frac{f(\theta_2)\left[f(\theta_1) - \frac{d\lambda_1(\theta_1)}{d\theta_1}\right]}{f(\theta_1)\left[f(\theta_2) - \frac{d\lambda_1(\theta_2)}{d\theta_2}\right]}}{\int_0^1 \int_0^1 \left[\frac{f(\theta_2)\left[f(\theta_1) - \frac{d\lambda_1(\theta_1)}{d\theta_1}\right]}{f(\theta_1)\left[f(\theta_2) - \frac{d\lambda_1(\theta_2)}{d\theta_2}\right]}\right] d\theta_1 d\theta_2},$$

whicht, through the derivatives of the multipliers, takes into account the players' Incentive Compatibility constraints[4]

Moreover, by the Envelope Theorem,

$$V'(v) = -\gamma.$$

Hence,

$$V'(v) = E^{\mathbb{Q}}\left[V'(w(\theta,x))\right]$$

as claimed. ■

We also have

Lemma 6 (Spreading of Values) *For all $w \in (\underline{w}, 0)$, there is positive probability of both $w(\theta) > w$, and $w(\theta) < w$.*

Proof. We prove that $w(\theta) > w$ with positive probability. The proof of the other case follows analogous reasoning.

[4] Note that $q(\theta_1,\theta_2)$ van be made to be always non-negative by a proper normalization of the multipliers (e.g., multiply $\lambda_1(\theta_1)$ by $\frac{\inf f(\theta_i)}{\sup\left|\left|\frac{d}{d\theta_1}\lambda_1(\theta_1)\right|\right|}$).

If

$$w(\theta) \leq w$$

for almost all θ, one has, from the concavity of $V(.)$, that

$$V'(w(\theta)) \geq V'(w) \text{ for almost all } \theta.$$

From the Martingale Lemma, this can hold only if

$$w(\theta) = w \text{ for almost all } \theta.$$

Plugging this in the first order condition for $w(.)$, we get

$$V'(w) - \frac{d\lambda_2(\theta_2)}{d\theta_2} f(\theta_2) - \frac{d\lambda_1(\theta_1)}{d\theta_1} V'(w) f(\theta_1) = V'(w) \text{ a.e.}$$

Hence,

$$\frac{d\lambda_2(\theta_2)}{d\theta_2} f(\theta_2) = -\frac{d\lambda_1(\theta_1)}{d\theta_1} V'(w) f(\theta_1) \text{ a.e.,}$$

which, in turn, calls for

$$\frac{d\lambda_2(\theta_2)}{d\theta_2} = \frac{d\lambda_1(\theta_1)}{d\theta_1} = 0 \text{ a.e.,}$$

so that $\lambda_i(\theta_i) = 0$ for all i, and θ_i. Plugging this in the FOC for $a(.)$, one gets.

$$\left[\frac{\partial u_1(a(\theta), \theta_1)}{\partial a} + \gamma \frac{\partial u_2(a(\theta), \theta_2)}{\partial a}\right] = 0.$$

It is easy to see that the $a(.)$ implicitly defined by the above equation is not IC when continuation values are constant, unless $\gamma = 0$, or $\gamma = \infty$).

We finally have ■

Proposition 3 (Dictatorship in the limit) *The provision of intertemporal incentives necessarily leads to a dictatorial mechanism: In the limit as $t \to \infty$ either v or $V(v)$ converge to 0 almost surely.*

Proof. Since $V'(w)$ is a non-positive martingale by Dobb's convergence Theorem (see Dobb (1953)) it converges almost surely to some random variable, R. Next we show by contradiction that R cannot have any positive likelihood for values $\in (0, \infty)$. Hence, all the probability is concentrated where $R = 0$ or $R = -\infty$.implying that one of the two players becomes a dictator in the limit. In search of a contradiction suppose there existed a positive probability of finding a path $V'(w_t)$ with the property that $lim_{t\to\infty}$ $V'(w_t) = C$ where $0 < C < \infty$. Since $V'(w)$ is continuous there must exist a convergent sequence of w_t. Call $lim_{t\to\infty}$ $w_t = \bar{w} \in \left(-\frac{1}{6}, 0\right)$ the limit of the player's continuation value. Let $W(w, \theta)$ denote the next period's continuation value given the current promised value w and reported state θ. For w_t to converge it must be that $W(\bar{w}, \theta) = \bar{w}$ for all θ. This however contradicts Lemma (6). ■

It is also easy to prove that

Proposition 4 *As $\delta \to 1$, the first best payoffs can be arbitrarily approximated by PPE payoffs.*

In practice, higher continuation values for an Agent are delivered by having his preferences have more weight on the determination of the future allocations. In general, the continuation values are used to provide incentives and vary from period to period. Continuation values tend to increase for the Agent that reports a less extreme type. This leads to the following result: asymptotically, one of the Agents becomes a dictator, that is, eventually only his preferences are taken into account to determine the current and future allocations.

5 The Approximate Efficiency Result

Proof of Approximate Efficiency. We prove that, for any $\epsilon > 0$, there exists $\delta^* \in (0,1)$ such that for $\delta > \delta^*$, the sum of the players equilibrium payoff is within ϵ of the first best payoff.

Given equal weights, the optimal allocation would be

$$a_t^*(\theta) = \frac{\theta_{1t} + \theta_{2t}}{2}, \ \textit{for all } t.$$

Simple calculations show that the discounted expected value for player i under this allocation is $-\left[\frac{1}{24}\right]$, so that the sum of the players' payoff is $-\frac{1}{12}$

Consider the following candidates for continuation values

$$\begin{aligned} v_1(\theta_1,\theta_2) &= \begin{pmatrix} -\left(\frac{1-\delta}{\delta}\right) E_{\theta_2}\left[\left(a^*(\theta_1,\theta_2)-\theta_2\right)^2\right] \\ +\frac{1-\delta}{\delta} E_{\theta_1}\left[\left(a^*(\theta_1,\theta_2)-\theta_1\right)^2\right] - \left[\frac{1}{24}\right] - \frac{\epsilon(\delta)}{2} \end{pmatrix}, \\ v_2(\theta_1,\theta_2) &= \begin{pmatrix} -\left(\frac{1-\delta}{\delta}\right) E_{\theta_1}\left[\left(a^*(\theta_1,\theta_2)-\theta_1\right)^2\right] \\ +\frac{1-\delta}{\delta} E_{\theta_2}\left[\left(a^*(\theta_1,\theta_2)-\theta_2\right)^2\right] \\ -\left[\frac{1}{24}\right] - \frac{\epsilon(\delta)}{2} \end{pmatrix}. \end{aligned}$$

where $\epsilon(\delta)$ is the constant that guarantees that, for all θ, $(v_i(\theta_1,\theta_2))_{i=1}^2$ is an pair of Equilibrium Payoffs.

It is easy to see that $v_1(\theta_1,\theta_2) + v_2(\theta_1,\theta_2) = -\frac{1}{12} - \epsilon(\delta)$ for all (θ_1,θ_2). We now show that, with these continuation values, one can implement $a^*(\theta_1,\theta_2)$ in an interim incentive compatible way. Toward that, note that, if player 2 is being truthful, Player 1's problem if a^* is implemented, and if he faces $v_1(\theta_1,\theta_2)$ as a continuation value is

$$\max_{\widehat{\theta}_1} -(1-\delta) E_{\theta_2}\left(a^*(\widehat{\theta}_1,\theta_2)-\theta_1\right)^2 + \delta E_{\theta_2} v_1(\widehat{\theta}_1,\theta_2)$$

which has the same solution as the one associated with the program[5]

$$\max_{\widehat{\theta}_1} -(1-\delta) E_{\theta_2}\left(a^*(\widehat{\theta}_1,\theta_2)-\theta_1\right)^2 - (1-\delta) E_{\theta_2}\left[\left(a^*(\widehat{\theta}_1,\theta_2)-\theta_2\right)^2\right].$$

Since

$$a^*(\theta_1,\theta_2) = \arg\max_a -(a-\theta_1)^2 - (a-\theta_2)^2,$$

the announcement $\widehat{\theta}_1 = \theta_1$ is optimal. An analogous reasoning applies to player 2.

We show that $\epsilon(\delta)$ goes to zero as $\delta \to 1$.

The magnitude of $\epsilon(\delta)$ depends on the variability of the promised continuation values. This variability in turn is bounded by

$$d(\delta) = \max_{1,2}\{d_1(\delta), d_2(\delta)\},$$

where

$$d_i(\delta) = \left(\frac{1-\delta}{\delta}\right)\begin{bmatrix} \max_\theta\left[-E_{\theta_{-i}}\left[\left(a^*(\theta_1,\theta_2)-\theta_i\right)^2\right] + E_{\theta_i}\left[\left(a^*(\theta_1,\theta_2)-\theta_i\right)^2\right]\right] \\ -\min_\theta\left[-E_{\theta_{-i}}\left[\left(a^*(\theta_1,\theta_2)-\theta_i\right)^2\right] + E_{\theta_i}\left[\left(a^*(\theta_1,\theta_2)-\theta_i\right)^2\right]\right] \end{bmatrix}.$$

Clearly, $d_i(\delta) \to 0$ as $\delta \to 1$. Hence, $\epsilon(\delta)$ also goes to zero as $\delta \to 1$ ■

[5] The term

$$\frac{1-\delta}{\delta} E_{\theta_1}\left[\left(a^*(\theta_1,\theta_2)-\theta_1\right)^2\right] - \left[\frac{1}{24}\right] - \frac{\epsilon(\delta)}{2}$$

does not affect incentives.