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The Organization of Reserve Capacity

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Abstract

We study a model in which organizations jointly choose reserve capacity and directed relationships that provide access to reserves held by other organizations. Reserve capacity helps to absorb local shortfalls but is costly to hold, while access relationships are costly to establish and use. Strict Nash equilibria exhibit core-periphery structures in which core organizations hold more reserve capacity and peripheral organizations rely more heavily on access to others. We show that, in equilibrium, every organization can become more exposed to shortages even as aggregate reserves increase. We then characterize the optimal subsidy to reserve capacity and show that it can be either higher or lower than in a fixed-network benchmark, highlighting that endogenous network adjustment is central for policy design. Applications include hospital surge planning, emergency-response systems, and decentralized inventory management.

Keywords: endogenous network formation; reserve capacity; emergency response; lateral transshipment; optimal subsidies **JEL Codes:** C72, D85, L14, H23

1 Introduction

In many settings, organizations must decide how many resources to devote to covering local surges of demand and how much to rely on support from others. This matters because the ability to respond to local spikes depends not only on how many resources are available in the aggregate, but also on where they are held and who can access them when needed. Hospitals, for example, plan internal surge capacity (Hasan et al., 2023) and participate in

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inter-hospital transfer networks (Tsai et al., 2023).[1] The same pattern appears in emergency-response systems more broadly, such as among fire departments and emergency medical service providers, each of which maintains its own capacity while also relying on support from similar organizations when local needs intensify ((U.S. Fire Administration, 2012; Federal Emergency Management Agency, 2017)). The same tradeoff arises in decentralized inventory management with lateral transshipments, where firms hold stock locally but can also obtain supply from other firms when shortages arise (Paterson et al. (2011)).

This paper studies a model of endogenous network formation in which organizations allocate resources between ordinary use and reserve capacity, and establish directed access relationships that may vary in their strength. Reserve capacity allows organizations to absorb local shortfalls, but ties up resources that could otherwise be allocated to ordinary use. Directed access relationships allow organizations to draw on reserve capacity held by others, but they are costly to establish and to use. We focus on rare local events, modeled as demand shocks that hit one organization at a time. We thereby abstract from simultaneous requests for the same reserve capacities. Unmet local demand is assumed to incur a cost proportional to the shortfall. Organizations face a joint choice over internal reserve capacity and external access to reserves held by others. Access relationships are more valuable when they connect to organizations that hold more reserve capacity, while access to others' reserves reduces the incentive to hold reserves. Thus, equilibrium jointly determines both the allocation of reserve capacity across organizations and the structure of access relationships.

This interaction yields a sharp equilibrium characterization. Strict Nash equilibria exhibit core-periphery networks in which core organizations hold more reserve capacity, while peripheral organizations rely more heavily on access to others. The intuition is that organizations with higher reserve capacity are more attractive to establish relationships with, while organizations that rely more on access relationships optimally hold less reserve capacity themselves. As a result, decentralized relationship choices concentrate on organizations with higher reserve capacity, endogenously generating a core-periphery structure among ex ante identical organizations. In comparative statics, changes in fundamentals affect not only reserve choices but also the pattern of access relationships, and this interaction shapes how organizations respond in equilibrium. The analysis therefore explains not only how much reserve capacity is held in equilibrium, but also how it is distributed across organizations and how access is organized. Endogenous access also separates aggregate reserve holdings from organizations' total available capacity. Within weighted-star equilibria, a marginal increase in the return to ordinary activity can raise aggregate reserves while reducing every organization's total available reserve capacity. Peripheral organizations replace lost access

[1] We have in mind capacity- or surge-driven transfers rather than routine referrals.

with additional own reserves, but this replacement is insufficient to prevent greater exposure to shortages.

We then use the model to study policy and characterize the optimal uniform per-unit subsidy to reserve capacity when access relationships adjust endogenously. The endogeneity of the network creates an additional policy margin: subsidies can crowd out access relationships or reinforce them. As a result, the optimal subsidy can be either higher or lower than in a fixed-network benchmark: under the conditions of Proposition 9, it is weakly lower if, at the benchmark optimum, the subsidy weakens equilibrium access relationships and their weight is no higher than the benchmark weight, and weakly higher under the reverse conditions.[2] Moreover, under relevant conditions, the optimal subsidy decreases in the cost of unmet local demand.

The paper relates to the literature on endogenous network formation (Jackson and Wolinsky, 1996; Bala and Goyal, 2000), and in particular to models in which agents choose a costly effort and links that give access to others' effort. In a local public-good model of information acquisition, Galeotti and Goyal (2010) show that strict equilibria exhibit a core-periphery structure. Kinateder and Merlino (2017) extend the analysis to heterogeneous agents, obtaining different hierarchical network structures depending on the source of heterogeneity, while Kinateder and Merlino (2022) allow for weighted links and characterize core-periphery structures among players exerting positive effort.[3] Our model microfounds the joint choice of reserves and access through localized demand shocks: organizations choose reserve capacity and access relationships before shocks occur, and incur costs when drawing on external reserves. Own capacity therefore reduces expenditure on external support even after access has been established, making own and accessible reserves imperfect substitutes. These features lead to differences in equilibrium characterization despite the shared core-periphery structure. In particular, our model supports strict weighted-star equilibria in which peripheral organizations hold positive own reserves and choose interior access weights. We also study how endogenous access relationships affect a funding authority's optimal reserve subsidy.

Turning to the paper's applications, we relate to research on hospital transfers, emergency-response coordination, and lateral transshipment. In hospitals, transfer networks can exhibit pronounced structural asymmetries: nationwide emergency-department transfers in Taiwan exhibit a multiple hub-and-spoke, regionalized pattern (Tsai et al., 2023), while critical-care

[2]The derivative comparison applies when the benchmark optimum lies below the upper endpoint of the selected star branch, using a right derivative at zero.

[3]A related literature studies informal risk sharing in networks, where links sustain ex post transfers or mutual insurance after shocks are realized (Bramoullé and Kranton, 2007; Ambrus et al., 2014; Ambrus and Elliott, 2021). Our focus is different: organizations choose reserve capacity ex ante, and links determine who can access those reserves when a local shortfall occurs.

and emergency-general-surgery transfers tend to flow toward hospitals with greater resources or capabilities (Iwashyna et al., 2009; Teng et al., 2021). Interhospital patient-transfer networks have also been found to exhibit a core-periphery structure (Lomi and Pallotti, 2012). In emergency-response systems, empirical studies likewise document core-periphery organization in interorganizational response networks (Chen et al., 2020; Ku et al., 2024); Hu et al. (2022) provide a broader review of empirical emergency-management network research. In lateral transshipment, firms combine local inventories with the possibility of obtaining supply from other locations when shortages arise. Much of this literature takes the set of locations and transshipment opportunities as given and studies inventory and transshipment decisions (Paterson et al., 2011); for example, Rudi et al. (2001) analyze a two-location system. A notable exception is the theoretical analysis of Fang and Cho (2014), who study the stability and endogenous formation of inventory-transshipment networks. In their model, the relevant network is formed after uncertainty is resolved and governs ex post transshipment. By contrast, our paper treats access relationships themselves as ex ante strategic objects: they are organized before needs are realized and shape incentives to hold reserve capacity.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes equilibrium reserve capacity choices and network structure, and shows that strict equilibria exhibit a core-periphery architecture. We then turn to comparative statics, focusing first on the weighted-star case and then discuss how the insights extend to equilibria with multiple core organizations. Section 4 analyzes policy and characterizes the optimal subsidy to reserve capacity when access relationships adjust endogenously. Section 5 concludes.

2 Model Description

2.1 Environment

There are $n \geq 3$ organizations, indexed by $N = \{1, \ldots, n\}$. Each organization i has a total amount of resources $w_i > 0$, which can be allocated either to ordinary use or to reserve capacity. Each unit allocated to ordinary use generates a benefit $r^x > 0$. Resources allocated to reserve capacity yield no immediate benefit, but can be used to meet demand generated by a local shock. Let $z_i \geq 0$ denote organization i's reserve capacity.

Organizations can also establish access relationships. Organization i's access relationships are represented by $\mathbf{g}_i = (g_{i,1}, \ldots, g_{i,n}) \in G_i = [0,1]^n$, and the resulting access network is denoted by $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_n)$. We assume that $g_{i,i} = 0$ for all $i \in N$. We say that organization i has access to organization j if $g_{i,j} > 0$. When convenient, we refer to $g_{i,j}$ as a link. The

network is weighted: the value of $g_{i,j}$ captures the intensity of the access relationship and determines how much of j's reserve capacity can be used by i. The network is directed, since i may have access to j without j having access to i, i.e. access relationships need not be reciprocal. The network is empty, denoted by $\mathbf{g}^e$, if $g_{i,j} = 0$ for all $i, j \in N$. We also define the associated undirected network $\overline{\mathbf{g}}$ by $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\} = \overline{g}_{j,i}$.

An access relationship allows an organization to draw on reserve capacity held by others. More precisely, if organization i establishes a link $g_{i,j} \in [0, 1]$ to organization j, then i can access up to a fraction $\psi g_{i,j}$ of organization j's reserve capacity, where $\psi \in (0, 1]$. Therefore, ψ captures the baseline transferability of reserve capacity across organizations. The total amount of reserve capacity accessible to organization i through the network is

$$y_i(\mathbf{z}_{-i}, \mathbf{g}_i) \equiv \sum_{j \in N} \psi g_{i,j} z_j.$$

This formulation makes external support depend jointly on the access relationships chosen by i and the reserve capacity held by other organizations. Throughout the paper, y_i denotes this accessible reserve capacity. Establishing and maintaining access relationships is costly. If organization i establishes a link $g_{i,j} \in [0, 1]$, it pays a linear relationship cost $\tilde{\kappa} g_{i,j}$, where $\tilde{\kappa} > 0$.

Demand shocks are localized. A single organization is randomly hit by a shock. Since no other organization experiences a shock at the same time, the reserve capacity available through the network is determined by the shocked organization's access relationships and by the reserve capacity held by the organizations to which it has access. Let $\epsilon \geq 0$ denote the size of the shock, distributed according to the cumulative distribution function F. We assume that shocks are bounded above by an organization's total resources, so that any organization can, in principle, self-insure fully by choosing $z_i = w_i$. Let $\bar{\epsilon}$ denote the largest possible shock and assume that $\bar{\epsilon} \leq \min_{j \in N} w_j$. Together with the maintained payoff assumptions introduced below, this condition allows us to focus on equilibria in which every organization allocates some resources to ordinary use.

Using reserve capacity obtained through an access relationship entails a constant marginal access cost $c > 0$. This cost captures the logistical, administrative, or transactional burden of mobilizing reserve capacity held by another organization. We assume that the access cost is borne by the organization obtaining the resources, either directly or after being passed on by the provider.

Finally, organization i incurs a shortage cost $\gamma > c$ for each unit of demand that remains unmet after using its own reserve capacity and any reserve capacity obtained through the

network. The assumption $\gamma > c$ ensures that, whenever external reserve capacity is available, drawing on reserve capacity through the network is less costly than leaving demand unmet.

2.2 Applications and Model Interpretation

The formal analysis can be read independently of this subsection; the discussion here maps the model's main primitives into the motivating applications.

In hospitals, reserve capacity corresponds to resources used to expand treatment when patient volume rises abruptly due to emergencies or disasters, including staff, space, supplies, and organizational arrangements needed to accommodate surges (Petanidis et al., 2025; Hasan et al., 2023). In emergency-response systems, personnel, vehicles, equipment, and other response resources are used in day-to-day operations, but must also be planned, positioned, and managed so that they remain available for mobilization when local needs increase (U.S. Fire Administration, 2012; Federal Emergency Management Agency, 2017). In lateral transshipment settings, reserve capacity corresponds to additional inventory kept on hand so that it is available when local demand spikes (Paterson et al., 2011; Rudi et al., 2001).

Access relationships also have natural counterparts in these applications. In hospitals, they are transfer or referral arrangements through which patients can be directed to providers with available capacity (Tsai et al., 2023). In emergency-response systems, they correspond to operational support arrangements through which personnel, vehicles, equipment, or other resources can be mobilized across organizations (U.S. Fire Administration, 2012; Federal Emergency Management Agency, 2017). In lateral transshipment settings, they take the form of transshipment agreements through which inventory at one location can be made available to another (Paterson et al., 2011; Rudi et al., 2001). Weighted links allow for access relationships that may differ in intensity and the parameter ψ captures how effectively reserve capacity held elsewhere can be made usable once an access relationship exists.

The model distinguishes between the cost of creating access to the reserves of others and the cost of using it. The relationship cost $\tilde{\kappa}$ captures the administrative and organizational cost associated with creating access relationships and is incurred ex ante, before a shock is realized. The access cost c is incurred ex post when reserve capacity is obtained through the network. In the applications, this corresponds to transport and handoff costs in hospitals, mobilization costs in emergency-response systems, and logistics costs in lateral transshipment settings.

The shock assumptions capture rare, localized events rather than system-wide disruptions: local mass-casualty incidents or temporary provider-specific surges in hospitals; sud-

den local increases in emergency demand in emergency-response systems; and local demand spikes at an individual location in lateral transshipment settings.

Finally, the shortage cost captures the consequences of unmet demand. In hospitals, it corresponds to delayed or foregone care (Zarka et al., 2021). In emergency-response systems, it corresponds to longer response times or reduced incident coverage (U.S. Fire Administration, 2012; Federal Emergency Management Agency, 2017). In lateral transshipment settings, this corresponds to stockout or backorder costs (Paterson et al., 2011).

2.3 Strategies, Timing, and Payoffs

An organization i's set of strategies is given by $S_i = [0, w_i] \times G_i$. A strategy profile $\mathbf{s}_i = (z_i, \mathbf{g}_i) \in S_i$ specifies the amount of reserve capacity that player i holds, z_i, and its linking behavior $\mathbf{g}_i$. The set of strategies of all agents is $S = S_1 \times S_2 \times ... \times S_n$. Defining $\mathbf{z} = (z_1, z_2, ..., z_n)$, a strategy profile $\mathbf{s} = (\mathbf{z}, \mathbf{g}) \in S$ then specifies each agent's own reserve capacity, z_i, and links, $\mathbf{g}_i$.

Players choose their strategies simultaneously. Subsequently, a player is chosen with probability $1/n$ to be hit by the shock and ϵ is realized. If organization i is selected, its own reserve capacity z_i is used first, at no additional cost. Any remaining demand is met using accessible reserve capacity y_i at marginal cost c. If the shock exceeds $z_i + y_i$, the residual unmet demand is penalized at marginal cost γ, with $\gamma > c$. Thus, the realized cost of a shock to organization i is zero if $\epsilon \leq z_i$, equals $c(\epsilon - z_i)$ if $z_i < \epsilon \leq z_i + y_i$, and equals $cy_i + \gamma(\epsilon - z_i - y_i)$ if $\epsilon > z_i + y_i$.

Writing $\zeta = \bar{\epsilon}$, the expected loss conditional on organization i being hit is

$$\begin{aligned} \widehat{\Pi}_i(z_i, y_i) = & \int_{\min\{z_i, \zeta\}}^{\zeta} c \min\{\epsilon - z_i, y_i\} \, dF(\epsilon) \\ & + \int_{\min\{z_i + y_i, \zeta\}}^{\zeta} \gamma(\epsilon - z_i - y_i) \, dF(\epsilon). \end{aligned} \tag{1}$$

The first term is the expected cost of obtaining reserve capacity through the network, while the second term is the expected shortage cost from unmet demand.

The corresponding expected payoff is

$$\hat{u}_i(z_i, \mathbf{g}_i) = r^x(w_i - z_i) - \frac{1}{n}\widehat{\Pi}_i(z_i, y_i) - \tilde{\kappa} \sum_{j \in N} g_{i,j}. \tag{2}$$

The first term is the return from resources allocated to ordinary use, while the last term is the cost of maintaining access relationships.

For the analysis, we assume $\epsilon \sim \mathcal{U}[0, \zeta]$, with $0 < \zeta \leq \min_i w_i$.[4] Define

$$\beta = \gamma - nr^x, \qquad \lambda = \gamma - c, \qquad \kappa = n\zeta\tilde{\kappa}.$$

We analyze an auxiliary game with the same players and strategy spaces and payoff

$$u_i(z_i, \mathbf{g}_i) = \zeta\beta z_i - \frac{\gamma}{2}z_i^2 - \lambda z_i y_i + \zeta\lambda y_i - \frac{\lambda}{2}y_i^2 - \kappa \sum_{j \in N} g_{i,j}. \tag{3}$$

When $z_i + y_i \leq \zeta$, this expression equals $n\zeta$ times the original expected payoff, minus the choice-independent constant $n\zeta r^x w_i - \gamma\zeta^2/2$. For $z_i + y_i > \zeta$, we use the same quadratic formula as an auxiliary expression, but this equality need not hold. Lemma 1 shows that, under the maintained assumptions $r^x > 0$, $\tilde{\kappa} > 0$, and $\gamma > c > 0$, the original and auxiliary network-formation games have identical best-response sets for every profile of opponents' strategies. Hence they have the same Nash equilibria and strict Nash equilibria.[5]

The parameter definitions imply $\gamma > \beta$ and $\gamma > \lambda > 0$, inequalities that we use throughout the analysis. For the auxiliary payoff,

$$\frac{\partial^2 u_i}{\partial z_i \partial z_j} = -\lambda\psi g_{i,j} \leq 0, \qquad j \neq i,$$

with strict inequality when $g_{i,j} > 0$.

Organization i's reserve-capacity best response given accessible outside reserves y_i is

$$z_i(y_i) = \max\left\{0, \frac{\zeta\beta - \lambda y_i}{\gamma}\right\}. \tag{4}$$

The upper bound on own reserve capacity is nonbinding: since $y_i \geq 0$ and $\beta < \gamma$,

$$0 \leq z_i(y_i) \leq \max\left\{0, \frac{\zeta\beta}{\gamma}\right\} < \zeta \leq w_i.$$

[4]Uniformity yields a linear-quadratic representation of payoffs, facilitating explicit equilibrium and policy analysis. For common shock distributions with continuous, strictly positive densities on the same bounded support, the complete core-periphery characterization of strict equilibria continues to hold, as do several comparative-static results within regions where a strict weighted-star equilibrium exists.

[5]Strategies with $z_i + y_i > \zeta$ remain feasible but cannot be best responses: an organization can reduce its own reserve capacity or access relationships and save costs without increasing its emergency costs. Thus, $z_i + y_i \leq \zeta$ follows from optimal behavior rather than from a restriction on the strategy spaces.

Let $v(y_i)$ denote the value obtained by maximizing the auxiliary payoff before relationship costs over own reserve capacity, taking y_i as given. Then

$$v(y_i) = \zeta\lambda y_i - \frac{\lambda}{2}y_i^2 + \frac{\left(\max\{0, \zeta\beta - \lambda y_i\}\right)^2}{2\gamma}.$$

When $\zeta\beta - \lambda y_i > 0$, optimal reserve capacity is positive and the value function simplifies to

$$v(y_i) = \frac{\zeta^2\beta^2}{2\gamma} + \frac{\zeta\lambda(\gamma - \beta)}{\gamma}y_i - \frac{(\gamma - \lambda)\lambda}{2\gamma}y_i^2. \tag{5}$$

Over this range, $v(y_i)$ is strictly concave in y_i and strictly increasing whenever $z_i(y_i) + y_i \leq \zeta$.

3 Analysis

3.1 Equilibrium Characterization

We use the best-response and value-function properties to characterize equilibrium reserve capacities and access relationships

We first fix the access network and show that the induced reserve capacity-choice game has a unique interior Nash equilibrium. We then return to the full network formation game and show that every strict Nash equilibrium has a complete core-periphery structure. We provide sufficient conditions for the existence of a strict Nash equilibrium for every positive linking cost κ. Finally, we focus on weighted-star equilibria, which provide a tractable class for the comparative statics and policy analysis that follow.

Existence of a Nash Equilibrium on a Fixed Network

We first fix the access network and characterize reserve capacity choices.

Proposition 1. *If $\psi < \frac{\gamma}{\lambda(n-1)}$ and $0 < \beta < \lambda$, then there exists a unique and interior Nash equilibrium* $\mathbf{z}$ *for any given network* $\mathbf{g}$.

The result reflects the substitutability between own reserves and reserve capacity accessible through the network. Greater access reduces an organization's incentive to hold reserves itself, but the bound on ψ limits this substitution effect sufficiently that every organization continues to hold positive reserve capacity. At the same time, $0 < \beta < \lambda < \gamma$ implies that optimal reserve holdings remain strictly below $\zeta \leq w_i$. Hence the unique equilibrium is interior. The Appendix uses the contraction mapping theorem to establish existence and

uniqueness, derives interiority from the equilibrium reserve bounds, and verifies that the auxiliary and original fixed-network games have the same reserve-capacity best responses.

Core-Periphery Network and Equilibrium Characterization

Below, we define complete core-periphery networks, which are central to the equilibrium characterization that follows. Related core-periphery structures have been documented in empirical studies of hospital patient-transfer and emergency-response networks (Lomi and Pallotti, 2012; Ku et al., 2024). Recall that we obtain the undirected network $\overline{\mathbf{g}}$ from $\mathbf{g}$ by taking $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\} = \overline{g}_{j,i}$.

Definition 1. *A core-periphery network $\overline{\mathbf{g}}$ is such that there exists a partition of N into two sets, $P(\overline{\mathbf{g}})$ (the periphery) and $C(\overline{\mathbf{g}})$ (the core), such that $\overline{g}_{i,j} = 0\ \forall i, j \in P(\overline{\mathbf{g}})$ and $\overline{g}_{i,j} > 0\ \forall i \neq j \in C(\overline{\mathbf{g}})$. A complete core-periphery network is a core-periphery network such that $\overline{g}_{i,j} > 0\ \forall i, j :\ i \in P(\overline{\mathbf{g}}),\ j \in C(\overline{\mathbf{g}})$.*

A core-periphery network is complete if every peripheral organization is connected to every core organization. Note that the complete network is a complete core-periphery network, since we can partition N into $C(\overline{\mathbf{g}}) = N$ and $P(\overline{\mathbf{g}}) = \emptyset$. Similarly, the empty network is a complete core-periphery network, with associated partition $C(\overline{\mathbf{g}}) = \emptyset$ and $P(\overline{\mathbf{g}}) = N$.

Proposition 2. *In any strict Nash equilibrium $\mathbf{s} = (\mathbf{z}, \mathbf{g})$, $\overline{\mathbf{g}}$ admits a complete core-periphery partition such that $z_i = z_j\ \forall i, j \in P(\overline{\mathbf{g}})$ and $z_i > z_j\ \forall i, j :\ i \in C(\overline{\mathbf{g}}),\ j \in P(\overline{\mathbf{g}})$. Moreover, $g_{i,j} = 0\ \forall i, j :\ i \in C(\overline{\mathbf{g}}),\ j \in P(\overline{\mathbf{g}})$.*

Proposition 2 shows that every strict Nash equilibrium admits a complete core-periphery partition in which peripheral organizations hold identical reserves and, whenever both groups are nonempty, core organizations hold strictly greater reserves. For this partition, no core organization extends a link to the periphery. Empty and complete networks are included by taking the core and the periphery, respectively, to be empty.

The argument begins with the observation that organizations holding greater reserves provide more accessible reserve capacity per unit of relationship cost. In a strict equilibrium, whenever an organization links to another organization, its links to all other organizations with weakly greater reserves must have weight one. Otherwise, reallocating link weights could preserve accessible reserve capacity without increasing costs. In particular, each organization extends at most one link with weight strictly between zero and one.

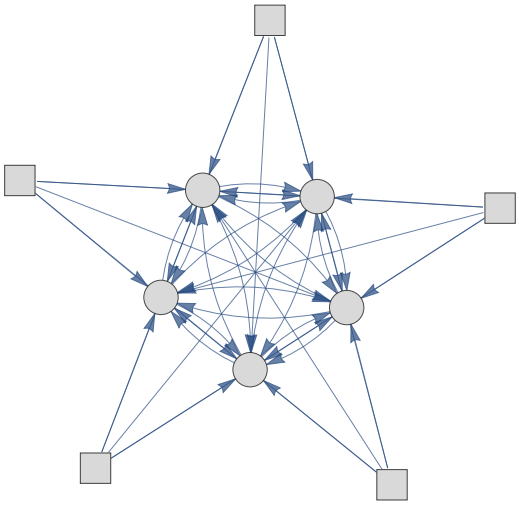
To see how these properties generate a core-periphery structure, suppose that the associated undirected graph is not complete. The organizations holding minimum reserves can then be taken as the periphery, which must contain at least two organizations: if there were only

one minimum-reserve organization, it would link to every higher-reserve organization, and the equilibrium linking properties would also connect the higher-reserve organizations to one another, making the associated undirected graph complete. There can be no links among them, since a link between two minimum-reserve organizations, together with the linking properties above, would make the associated graph complete. Since every peripheral organization holds strictly fewer reserves than every organization outside the periphery, it must link to every higher-reserve organization. Hence every peripheral organization is connected to every core organization. Moreover, if a peripheral organization links to a higher-reserve organization, every other organization must also link to that organization: otherwise an organization with a weaker link could profitably increase its link and reoptimize its own reserves. Thus the higher-reserve organizations are connected to one another as well, yielding a complete core-periphery partition.

Finally, a link from the core to the periphery would imply sufficiently extensive outgoing access that, because access to others' reserves substitutes for own reserves, it is incompatible with the higher reserve holdings that characterize core organizations. Hence no core organization links to the periphery under this partition. If the associated graph is complete, the all-core partition applies. The Appendix provides the formal proof.

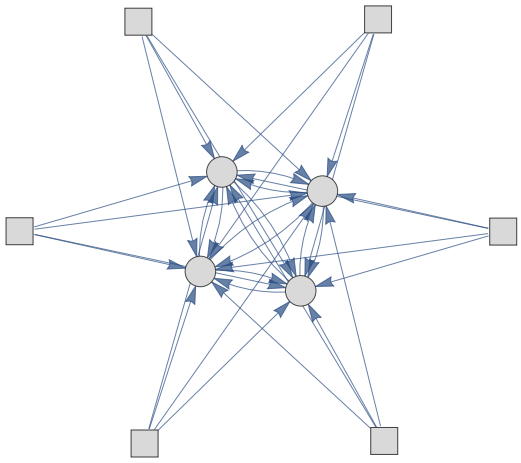
Example 1 presents two different equilibrium networks.

Figure 1: Equilibrium Networks and Emergency Reserves.



	Node	z^*
Core	○	0.17
Periphery	□	0.1

(a)



	Node	z^*
Core	○	0.2
Periphery	□	0.12

(b)

Reserve capacities are rounded to the second decimal. Panel (a) presents an equilibrium network with 5 agents in the core. Panel (b) presents an equilibrium network with 4 agents in the core. In both panels, all links are binary, i.e. $g_{i,j} \in \{0,1\}, \forall i,j \in N$.

Example 1: *Assume* $n = 10$, $\gamma = 12$, $\lambda = 9$, $\beta = 1$, $\psi = 0.5$, $\kappa = 3$, $\zeta = 5$. *The following are two equilibrium networks: the complete core-periphery graph with 5 agents in the core and binary links only, as depicted in Panel (a) of Figure 1, and the complete core-periphery graph with* 4 *agents in the core and binary links only, as depicted in Panel (b) of Figure 1. Holding parameters fixed, the sparser equilibrium with four core organizations has higher aggregate reserves and higher reserves per organization in both the core and the periphery.*

Existence of a Strict Equilibrium in the Network Formation Game

In Proposition 3 we provide a sufficient condition such that for any linking cost κ, the network formation game has a strict Nash equilibrium.

Define

$$\bar{\beta} \equiv \frac{\gamma\lambda^2\psi(\gamma + \lambda\psi)}{2\gamma^2(\gamma - \lambda) + \gamma\lambda^2\psi + \lambda^3\psi^2}.$$

Since $\gamma > \lambda > 0$, we have $\bar{\beta} > 0$.

Proposition 3. *If $\beta < \bar{\beta}$, then for any linking cost $\kappa > 0$, there exists a strict Nash equilibrium $\mathbf{s} = (\mathbf{z}, \mathbf{g})$. Moreover, if $\beta > 0$, then $\mathbf{z} > \mathbf{0}$.*[6]

Proposition 3 ensures that a strict equilibrium exists for every positive relationship cost. At high relationship costs κ the empty network is a strict Nash equilibrium. As relationships become less costly, weighted-star and denser complete core-periphery configurations can be sustained. The condition $\beta < \bar{\beta}$ guarantees that the parameter regions supporting these different configurations overlap, so that no positive value of κ is left without a strict equilibrium. These regions may overlap and Proposition 3 does therefore not imply uniqueness.

Recall that $\beta = \gamma - nr^x$. Thus, the condition is satisfied when the return to the active resource, r^x, is sufficiently close to γ/n, the marginal expected benefit of reserves at zero own and accessible reserve capacity. In this sense, when the incentive to hold reserves in isolation is not too strong, adjustments in network structure are sufficient to sustain a strict equilibrium across all positive relationship costs. This existence result provides the background for the next subsection, where we focus on weighted-star networks to obtain sharper equilibrium and comparative-statics results.

The Weighted Star

We first provide a formal definition of a star and a weighted star. Recall that we obtain the undirected network $\overline{\mathbf{g}}$ from $\mathbf{g}$ by taking $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\} = \overline{g}_{j,i}$.

Definition 2. *A network $\mathbf{g}$ is a star if its associated undirected network $\overline{\mathbf{g}}$ admits a complete core-periphery partition (C, P) with $|C| = 1$ and $g_{i,j} > 0$ for every $i \in P$ and $j \in C$. A weighted star is a star network $\mathbf{g}$ such that $g_{i,j} \in (0, 1)$ for at least one pair of distinct organizations $i, j \in N$.*

We refer to the sole core organization as the center and denote the set of weighted-star networks by $\mathbf{g}^s$. This configuration concentrates access on a single organization and allows network adjustment to be studied continuously through one link weight. Recall that $0 < \beta < \lambda$ is equivalent to $\gamma/n > r^x > c/n$. An organization without access to others' reserves holds $z_C = \zeta\beta/\gamma$. Define $\kappa_0 \equiv v'(0)\psi z_C$ and $\kappa_1 \equiv v'(\psi z_C)\psi z_C$, with $\kappa_1 < \kappa_0$. These thresholds measure the marginal benefit of increasing a link to such an organization at weights zero and one, respectively. Proposition 4 characterizes weighted-star equilibria and gives a sufficient condition for uniqueness among strict equilibria, providing the basis for the comparative statics and policy analysis below.

[6] If $\beta \leq 0$, then a strict Nash equilibrium trivially exists for every $\kappa > 0$ since each agent optimally chooses $z_i = 0$. Therefore, the unique equilibrium network for every κ is the empty network.

Proposition 4. *Suppose $\gamma/n > r^x > c/n$. A strict Nash equilibrium $\mathbf{s} = (\mathbf{z}, \mathbf{g})$ with $\mathbf{g} \in \mathbf{g}^s$ exists if and only if $\kappa \in (\kappa_1, \kappa_0)$. When this condition holds, every such equilibrium has the following properties:*

(i) Every peripheral organization extends a link of the same weight $g_s \in (0,1)$ to the center. The center extends no links.

(ii) The common link weight satisfies $\kappa = v'(\psi g_s z_C)\,\psi z_C$ and $z_C = \frac{\zeta(\gamma - n r^x)}{\gamma}$, where z_C is the reserve capacity of the center. Holding all other primitives fixed, g_s is differentiable with respect to γ and r^x in the open parameter region in which a strict weighted-star equilibrium exists.

(iii) If, in addition, $r^x > \gamma/(2n)$, then $\mathbf{s}$ is the unique strict Nash equilibrium of the network formation game, up to relabeling of organizations.

The weighted star reflects the substitution between own reserves and accessible reserve capacity. The center has no outgoing access and therefore holds z_C, whereas peripheral organizations reduce their own reserves in response to access to the center. The center thus provides more accessible reserve capacity per unit of relationship cost than any peripheral organization. Peripheral organizations prefer to concentrate their access on the center, while their lower reserves make links to them unattractive to the center under the proposition's assumptions.

All peripheral organizations face the same trade-off. After optimally adjusting own reserves, the marginal benefit of strengthening a link decreases with its weight. A cost between κ_1 and κ_0 therefore makes some access worthwhile but full access excessive, yielding a common interior weight. The smooth, strictly decreasing marginal benefit also makes this weight vary continuously in r^x and γ within the weighted-star region.

The additional condition $r^x > \gamma/2n$ rules out full-weight links in any strict Nash equilibrium. Under this condition, the marginal benefit of a full-weight link, evaluated without access through other links, increases with the reserves of the organization receiving it. Access through additional links reduces this marginal benefit, since v is strictly concave. It is therefore bounded above by $v'(\psi z_C)\psi z_C = \kappa_1$, obtained when the organization receiving the link holds the maximum equilibrium reserves z_C and no reserve capacity is accessible through other links. This comparison allows own reserves to be chosen optimally, as incorporated in v. Since the marginal relationship cost satisfies $\kappa > \kappa_1$, reducing a full-weight link is profitable.

Consequently, every positive link in a strict Nash equilibrium must have weight strictly between zero and one. If an organization extended two such links to organizations with

different reserves, it could reallocate their weights toward the organization holding more reserves, preserving accessible reserve capacity while reducing relationship costs. If their reserves were equal, the same reallocation would leave both accessible reserve capacity and costs unchanged. Holding own reserves fixed, the first deviation would increase its payoff, while the second would give a distinct strategy with the same payoff. Both contradict strict Nash equilibrium, so each organization extends at most one positive outgoing link. Each organization can therefore extend at most one positive link. Together with the complete core-periphery characterization and the preference for linking to organizations with greater reserves, this leaves a single center as the only possible nonempty strict Nash equilibrium configuration. Since $\kappa < \kappa_0$ also excludes the empty network, uniqueness follows among strict equilibria, up to the identity of the center. The argument does not exclude non-strict equilibria.

3.2 Comparative Statics

In this section, we analyze how changes in the return on the active resource r^x and the penalty for unmet demand γ affect equilibrium reserve capacity. We first focus on weighted-star equilibria. Our first result shows that, locally within the weighted-star region, a decrease in r^x can, counterintuitively, lead peripheral organizations to reduce their reserves when r^x is sufficiently high. An increase in γ also reduces peripheral reserves throughout this region. We explain these responses through changes in accessible reserve capacity and compare them with a benchmark in which link weights are fixed. The reversal with respect to r^x depends on link adjustment, whereas the reduction in reserves following an increase in γ can also occur with fixed link weights.

We express equilibrium outcomes in terms of r^x and γ, holding all other primitive parameters fixed and suppressing them from the notation.

Equilibrium Network and Reserve Capacity: Recall that $\beta \equiv \gamma - nr^x$ and $\lambda \equiv \gamma - c$. For given r^x and γ, let $\mathbf{g}(r^x, \gamma)$ denote the network of a selected strict Nash equilibrium, and write organization i's corresponding reserve capacity as $z_i(r^x, \gamma, \mathbf{g}(r^x, \gamma))$. Keeping the network as a separate argument allows us to distinguish parameter changes with the network fixed from changes that also involve network adjustment.

We first study weighted-star equilibria and then compare binary complete core-periphery equilibria with multiple organizations in the core.

Weighted Star Network: For the weighted-star comparisons, select a strict weighted-star equilibrium, fix the identity of the center, and vary parameters within an open region in which this equilibrium exists. Denote the common weight of peripheral organizations' links

to the center by $g_s(r^x, \gamma)$. By Proposition 4, this weight is differentiable with respect to r^x and γ within this region, and the center's reserve capacity is

$$z_C(r^x, \gamma) \equiv \frac{\zeta(\gamma - nr^x)}{\gamma}.$$

Using the best response (4), each of the $n-1$ peripheral organizations holds

$$z_P(r^x, \gamma, g_s(r^x, \gamma)) \equiv \frac{\zeta(\gamma - nr^x)}{\gamma}\left(1 - \frac{\gamma - c}{\gamma}\psi g_s(r^x, \gamma)\right).$$

We retain the link weight as a separate argument to facilitate the comparison with fixed link weights below.

Proposition 5 presents the comparative statics for weighted-star equilibria. The center's reserve capacity is strictly decreasing in the return on the active resource r^x and strictly increasing in the penalty for unmet demand γ. For peripheral organizations, reserve capacity is strictly increasing in r^x when the return exceeds the threshold identified in the proposition, and strictly decreasing in γ.

Proposition 5. *Let r^x and γ be such that the equilibrium network $\mathbf{g}(r^x, \gamma)$ is a weighted star network with corresponding link $g_s(r^x, \gamma)$. Then:*

(i) $\partial z_C(r^x, \gamma)/\partial r^x \leq 0$ and $\partial z_C(r^x, \gamma)/\partial \gamma \geq 0$, i.e. the reserve capacity of the center is decreasing in r^x and increasing in γ.

(ii) There exists $\overline{r^x} = \frac{1}{n}[\gamma - \sqrt{\gamma\kappa/(\zeta^2\psi)}] \in (c/n, \gamma/n)$ such that, wherever a weighted-star equilibrium exists, $\partial z_P(r^x, \gamma, g_s(r^x, \gamma))/\partial r^x \geq 0$ if and only if $r^x \geq \overline{r^x}$, i.e. peripheral organizations' reserve capacity is nondecreasing in r^x when $r^x \geq \overline{r^x}$.[7]

(iii) $\partial z_P(r^x, \gamma, g_s(r^x, \gamma))/\partial \gamma \leq 0$, i.e. peripheral organizations' reserve capacity is decreasing in γ.

We briefly comment on Proposition 5. The reserve-capacity best response can be written as

$$z_i(y_i) = \zeta\left(1 - \frac{nr^x}{\gamma}\right) - \frac{\lambda}{\gamma}y_i.$$

Thus, holding accessible reserve capacity fixed, a higher return to the active resource reduces reserve capacity. For the center, $y_C = 0$, so this directly implies that its reserve capacity decreases with r^x. For a peripheral organization, the effect is less immediate. A higher r^x

[7]The threshold $\overline{r^x}$ need not itself lie in a weighted-star interval. The derivative comparison applies only where a weighted-star equilibrium exists.

reduces the center's reserve capacity and hence, for a given link weight, reduces the capacity accessible from the center. Because own and accessible capacity are substitutes, this offsets the direct incentive to reduce reserves. With a fixed link weight, however, the direct effect dominates and peripheral reserve capacity still decreases with r^x.

Endogenous link formation introduces an additional adjustment margin. Wherever a strict weighted-star equilibrium exists, peripheral reserve capacity is nondecreasing in r^x if and only if $r^x \geq \overline{r^x}$. Since it would decrease with r^x at a fixed link weight, this reversal requires the equilibrium link weight to fall sufficiently, reducing accessible capacity and inducing peripheral organizations to hold more of their own reserves. The threshold $\overline{r^x}$ need not itself lie at a parameter value supporting a strict weighted-star equilibrium.

The comparative static with respect to γ is different. A higher penalty for unmet emergency needs increases the center's reserve capacity but reduces peripheral organizations' own reserve capacity. The latter effect is not specific to endogenous network formation: as shown in the Appendix, peripheral reserve capacity can also decrease with γ when the link weight is fixed. What matters for shortage exposure is instead the resulting change in total available capacity.

Let

$$y_P(r^x, \gamma, g_s(r^x, \gamma)) = \psi g_s(r^x, \gamma) z_C(r^x, \gamma)$$

denote the accessible reserve capacity of a peripheral organization, and let $y_C(r^x, \gamma) = 0$. The next proposition shows that an increase in γ raises $z_i + y_i$ for every organization, despite the reduction in peripheral organizations' own reserve capacity. Conversely, an increase in r^x reduces total available reserve capacity. Thus, a higher penalty for unmet emergency needs, or a lower return to the active resource, reduces organizations' exposure to shortages.

Proposition 6. *Let γ and r^x be such that the equilibrium network $\mathbf{g}(r^x, \gamma)$ is a weighted star network with corresponding link weight $g_s(r^x, \gamma)$. Then total available reserve capacity $z_i + y_i$ is strictly increasing in γ and strictly decreasing in r^x for every $i \in N$.*

Proposition 6 distinguishes own reserve capacity from the capacity available through the network. When γ increases, peripheral organizations reduce their own reserve capacity, but their total available reserve capacity $z_P + y_P$ increases. Accessible capacity therefore rises by more than own reserve capacity falls, reducing their exposure to shortages.

The comparative static with respect to r^x illustrates a different margin. For sufficiently large r^x, peripheral organizations can increase their own reserve capacity even though their total available reserve capacity falls. In this case, the reduction in accessible capacity more than offsets the increase in own reserves: the center becomes a weaker source of reserve

capacity, and peripheral organizations substitute toward their own reserves while becoming more exposed to shortages.

Corollary 1. *Let*

$$Z = z_C + (n-1)z_P$$

denote aggregate reserve capacity. At any parameter value supporting a strict weighted-star equilibrium,

$$\frac{\partial Z}{\partial r^x} \geq 0 \quad \Longleftrightarrow \quad r^x \geq r_Z^x,$$

where

$$r_Z^x \equiv \frac{\gamma}{n}\left[1 - \sqrt{\frac{(n-1)\kappa}{\zeta^2\psi[\gamma(n-1)+c]}}\right].$$

At the same time, $\partial(z_i + y_i)/\partial r^x < 0$ for every $i \in N$. Hence, whenever $r^x > r_Z^x$ and a strict weighted-star equilibrium exists, aggregate reserve capacity increases with r^x even though every organization's total available reserve capacity falls.

Corollary 1 shows that every agent can become more exposed to shortages in equilibrium, even as aggregate reserves increase. When $r^x > r_Z^x$ and a strict weighted-star equilibrium exists, a marginal increase in r^x reduces the center's own reserves, while peripheral agents lose more in accessible capacity than they gain in own reserves. The combined increase in peripheral reserves nevertheless exceeds the center's reduction, raising Z. Thus, the endogenous adjustment of reserve holdings and access relationships can leave every agent more exposed to a shock, despite an increase in aggregate reserves: what matters is not only how much reserve capacity is held, but also where it is held and who can access it.

Multiple Agents in the Core

We next examine how the forces identified in the weighted-star case extend to equilibria with multiple core organizations. We focus on strict binary complete core-periphery equilibria with at least one core organization and at least one peripheral organization. The main analysis examines changes in the return to the active resource, r^x.[8]

For a given r^x, several such strict Nash equilibria may exist and differ in the number of core organizations. Let $\mathbf{g}(r^x, m)$ denote a binary complete core-periphery network with m core organizations, and define the set of core sizes sustained in strict Nash equilibrium by

$$N_c(r^x) = \{m \in \{1, \ldots, n-1\} : \text{there exists a strict Nash equilibrium with network } \mathbf{g}(r^x, m)\}.$$

[8]The Appendix also reports reserve capacity responses to γ while holding a binary configuration fixed.

The set $N_c(r^x)$ may be empty. By construction, it excludes the empty network, the complete network, and weighted configurations. Since strict Nash equilibria need not be unique, the comparative static in r^x compares the sets of sustainable core sizes. When both sets being compared are nonempty, we order them using the strong set order defined below.

Definition 3. *Let S and S' be nonempty subsets of $\mathbb{N}$. We say that S' dominates S in the strong set order if, for every $s' \in S'$ and $s \in S$, $\max\{s', s\} \in S'$ and $\min\{s', s\} \in S$.*

Intuitively, S' lies weakly above S: values belonging only to S' exceed every value in S, while values belonging only to S lie below every value in S'. The sets may overlap, so this does not require every element of S' to exceed every element of S.

Proposition 7 characterizes reserve holdings and network structure when the core contains multiple organizations. As r^x increases while the equilibrium network remains unchanged, all organizations reduce their reserves. When two equilibrium networks with different core sizes coexist at a given r^x, reserve holdings are higher in the sparser network in both the core and the periphery. Finally, for sufficiently large r^x, the equilibrium network becomes sparser as r^x increases.

Proposition 7. *For the binary configurations defining N_c, let $c/n < \tilde{r}^x \leq r^x < \gamma/n$. Then:*

(i) If $\mathbf{g}$ is an equilibrium network under r^x and $\tilde{r}^x$, then $z_i(r^x, \mathbf{g}) \leq z_i(\tilde{r}^x, \mathbf{g})$ for all $i \in N$.

(ii) If $n_c, \tilde{n}_c \in N_c(r^x)$ with $n_c < \tilde{n}_c$, then $z_i(r^x, \mathbf{g}(r^x, n_c)) \geq z_i(r^x, \mathbf{g}(r^x, \tilde{n}_c))$ for every organization i that belongs to the core in both networks or to the periphery in both networks.

(iii) If $\tilde{r}^x > (\gamma + c)/(2n)$ and both $N_c(\tilde{r}^x)$ and $N_c(r^x)$ are nonempty, then $N_c(\tilde{r}^x)$ dominates $N_c(r^x)$ in the strong set order.

Proposition 7 is presented in three parts. Part (i) considers increases in r^x for which the equilibrium network, including its weights, remains unchanged. All organizations then reduce their reserve capacity. Part (ii) holds r^x fixed and compares different equilibrium networks. Organizations remaining in the same role hold more reserve capacity in the equilibrium with fewer core organizations. Although each core organization holds more reserves, the smaller number of core organizations implies less accessible reserve capacity for organizations retaining their roles, inducing them to hold more reserves themselves. Finally, part (iii) shows that, for sufficiently large r^x, the set of sustainable core sizes shifts weakly downward in the strong set order as r^x increases, provided both sets are nonempty. If the equilibrium moves to a smaller core, the associated reduction in accessible reserve capacity for organizations retaining their roles works against the direct incentive to reduce their own reserves.

4 Optimal Subsidy of Reserve Capacity

In this section, we study the problem of a public authority that can provide a uniform per-unit subsidy to organizations' capacity reserves. The authority's objective incorporates the opportunity cost of allocating resources to reserves rather than to ordinary activity, the expected cost of unmet emergency needs, and the fiscal cost of the subsidy.[9] To keep the policy analysis tractable, we focus on weighted-star equilibria. This allows us to study adjustment through link intensity while abstracting from changes in core size that can arise when equilibria have multiple core organizations.

Subsidy for Reserve Capacity The authority chooses a per-unit amount of subsidy $s \geq 0$ to agents' reserve capacities. In this case, the expected utility for each i is given by:

$$\hat{u}_i(z_i, \mathbf{g}_i; s) = r^x(w_i - z_i) + sz_i - \frac{1}{n}\widehat{\Pi}_i(z_i, y_i) - \tilde{\kappa}\sum_{j \in N} g_{i,j}. \tag{6}$$

A subsidy s reduces the opportunity cost of reserves to $r^x - s$. For $0 \leq s < r^x$, Lemma 10 establishes that the original subsidized game and the auxiliary game with payoff $u_i + n\zeta s z_i$ have identical joint best responses, Nash equilibria, and strict Nash equilibria.

With links fixed, this equivalence requires additional conditions. To cover the fixed-network benchmark, we use the reserve best response derived from the original payoff. For $0 \leq s < r^x$, it is

$$z_i(y_i; s) = \max\left\{0,\, \zeta\left(1 - \frac{n(r^x - s)}{c}\right),\, \frac{\zeta[\gamma - n(r^x - s)] - (\gamma - c)y_i}{\gamma}\right\}. \tag{7}$$

At $s = r^x$, every reserve level in $[\zeta, w_i]$ is privately optimal, and we select $z_i = \zeta$, the minimum optimal reserve level and the authority's preferred choice. For $s > r^x$, each organization uniquely chooses $z_i = w_i$. The selection at $s = r^x$ does not affect the optimal subsidy, since Lemma 11 establishes that every equilibrium at $s \geq r^x$ is dominated by zero subsidy.

Denote by $z_i(s, \mathbf{g}(s))$ and $y_i(s, \mathbf{g}(s))$, respectively, the equilibrium reserves and accessible reserve capacity for agent i as a function of the subsidy s and the equilibrium network $\mathbf{g}(s)$ given s.

The Public Authority's Optimization Problem: For notational convenience, write $z_i(s) \equiv z_i(s, \mathbf{g}(s))$ and $y_i(s) \equiv y_i(s, \mathbf{g}(s))$.

[9] FEMA's SAFER program, for example, evaluates funding applications according to their effects on daily operations, financial need, and benefits to fire departments and their communities, and reviews proposed costs and feasibility (FEMA, 2025, pp. 22, 25). These criteria motivate attention to service outcomes and public expenditure.

Both quantities incorporate the response of the equilibrium network to the subsidy. The authority chooses $s \geq 0$ to maximize

$$W(s) = \sum_{i=1}^{n} \left[r^x(w_i - z_i(s)) - s z_i(s) - \frac{\gamma}{2n\zeta} \big(\zeta - z_i(s) - y_i(s)\big)_+^2 \right]. \tag{8}$$

The authority weighs the benefits of ordinary activity and reductions in expected unmet emergency needs against the fiscal cost of the subsidy. We interpret $r^x(w_i - z_i)$ as the value of services delivered through ordinary activity and the shortage term as the expected loss from unmet emergency needs. We assume that the authority shares the organizations' valuations of these outcomes, r^x and γ. Subsidy expenditure reduces the funds available for other uses, to which the authority assigns unit marginal value.

Relationship and access costs are borne by the organizations and receive no direct weight in the public authority's objective. They nevertheless affect the authority's payoff through organizations' reserve capacity and linking decisions and the resulting expected level of unmet emergency needs. Their exclusion reflects the assumption that the authority evaluates service outcomes and expenditure from its own budget, without separately valuing organizations' operating surplus or subsidy receipts. Accordingly, $W(s)$ represents the funding authority's objective rather than aggregate social welfare.[10]

4.1 Analysis

Lemma 11 shows that every equilibrium at a subsidy $s \geq r^x$ yields a lower value of the authority's objective than every equilibrium at zero subsidy. We can therefore restrict the characterization of an optimal subsidy to $0 \leq s < r^x$.

Linking Cost: We maintain $c < nr^x < \gamma$. For each s, let $\kappa_0(s)$ and $\kappa_1(s)$ denote the formal threshold expressions obtained by replacing r^x with $r^x - s$ in the algebraic expressions for κ_0 and κ_1, respectively. Their explicit formulas are given in the Appendix. We impose the linking-cost conditions

$$\kappa > \kappa_1(s) \text{ for all } s \geq 0 \quad \text{and} \quad \kappa > \kappa_0(s) \text{ for all } s \geq r^x - c/n. \tag{9}$$

For $0 \leq s < r^x$, we restrict attention to strict Nash equilibria. Under (9), Lemma 12 shows that the weighted star is the unique strict equilibrium up to relabeling when $\kappa < \kappa_0(s)$; otherwise, the empty configuration is the unique pure Nash equilibrium and is strict.

[10] Including both relationship and expected access costs can change the optimal subsidy, even with fixed links, because reserve capacity holdings and expected transfers respond to the subsidy. With endogenous links, the modified objective need not remain concave under the maintained assumptions.

If $\kappa < \max_{s\geq 0} \kappa_0(s)$, let $\underline{s} < \overline{s}$ denote the two real solutions to $\kappa_0(s) = \kappa$. Under (9), these thresholds satisfy $0 < \overline{s} < r^x - c/n$, while $\underline{s}$ may be negative. Within $0 \leq s < r^x$, the selected network is a weighted star precisely for $s \in (\underline{s}, \overline{s}) \cap [0, \infty)$ and is empty otherwise, including at threshold equality. If instead $\kappa \geq \max_{s\geq 0} \kappa_0(s)$, the selected network is empty throughout $0 \leq s < r^x$.

To complete the equilibrium specification outside the range relevant for optimal policy, consider $s \geq r^x$. At $s = r^x$, each organization is indifferent among reserve capacity levels in $[\zeta, w_i]$; we select the minimum optimal reserve level, $z_i = \zeta$, for every i. For $s > r^x$, each organization uniquely chooses $z_i = w_i$. In both cases, the equilibrium network is empty. Thus, the selected equilibrium network $\mathbf{g}(s)$ is either a weighted star or the empty network for every $s \geq 0$.

The selected equilibrium link weight, defined to be zero when the network is empty, is continuous in the subsidy s, but may have a kink at thresholds where the selected network switches between a weighted star and the empty network.

4.2 The Optimal Subsidy

In this subsection, we characterize the planner's optimal subsidy for reserve capacity. We focus on environments in which the resulting equilibrium network is either a weighted star or the empty network. We first establish existence of the optimal subsidy and provide conditions under which it is unique. We then compare the optimal subsidy to the benchmark in which the network is fixed, and derive conditions under which the optimal subsidy is higher or lower in the endogenous case. Finally, we present comparative statics and show that the optimal subsidy is decreasing in the penalty for unmet emergency needs, γ.

Proposition 8 presents our first result on the optimal subsidy. In particular, we show that an optimal subsidy exists and provide sufficient conditions under which it is unique.

Proposition 8. *There exists $s^* \geq 0$ such that $s^* \in \arg\max_{s\geq 0} W(s)$. Moreover, if $\kappa < \kappa_0(0)$ and $r^x > \gamma/2n$, then s^* is unique.*

To build intuition for Proposition 8, recall the equilibrium characterization in Lemma 12. Equilibrium reserve levels and link weights vary continuously within each network configuration and converge to the empty-network values at each transition. Hence $W(s)$ is continuous for $0 \leq s < r^x$. The proof shows that $W(s)$ is strictly decreasing on subsidy intervals where the selected network is empty and $s < r^x$. Separately, Lemma 11 shows that every subsidy $s \geq r^x$ yields a lower objective value than zero subsidy. Thus, if $\kappa \geq \max_{s\geq 0} \kappa_0(s)$, the equilibrium network is empty at every subsidy and the optimal subsidy is $s^* = 0$. Otherwise, the equilibrium network is empty for $\overline{s} \leq s < r^x$. Since $W(s)$ is strictly decreasing on this

interval, $W(s) < W(\overline{s})$ for every $s \in (\overline{s}, r^x)$. Together with the exclusion of $s \geq r^x$, this reduces the problem to maximizing the continuous objective on the compact interval $[0, \overline{s}]$, establishing existence.

For uniqueness, the condition $r^x > \gamma/(2n)$ ensures strict concavity of $W(s)$ on the weighted-star interval. The additional condition $\kappa < \kappa_0(0)$ ensures that the equilibrium network is a weighted star throughout $[0, \overline{s})$. Continuity at $\overline{s}$ then extends strict concavity to $[0, \overline{s}]$, yielding a unique optimal subsidy. In what follows, we maintain these conditions for uniqueness of s^*.

Next, we compare the optimal subsidy when organizations choose the weights of their links to the center with the optimal subsidy when those weights are fixed at some $g \in [0, 1]$. This comparison highlights how network formation affects the optimal policy.

Write the authority's objective as $W(s, g_s(s))$ to make its dependence on the equilibrium link weight explicit. The optimal subsidy is $s^* = \arg\max_{s \geq 0} W(s, g_s(s))$. For a fixed network, define $s_F(g) = \arg\max_{s \geq 0} W(s, g)$. Using the exact reserve responses, the Appendix shows that the fixed-network objective is strictly concave until peripheral organizations have enough own and accessible reserves to meet every possible shock. For higher subsidies below r^x, the objective is strictly decreasing. Subsidies $s \geq r^x$ are again excluded by Lemma 11. The Appendix therefore establishes that $s_F(g)$ exists and is unique for every $g \in [0, 1]$.

Proposition 9. *Under the maintained assumptions, suppose that $\kappa < \kappa_0(0)$ and $r^x > \gamma/(2n)$. For every $g \in [0, 1]$:*

(i) If $s_F(g) \geq \overline{s}$, then $s^* \leq s_F(g)$. The same conclusion holds if $0 \leq s_F(g) < \overline{s}$, $g_s'(s_F(g)) \leq 0$, and $g_s(s_F(g)) \leq g$, where $g_s'(0)$ denotes the right derivative.

(ii) If $0 < s_F(g) < \overline{s}$, $g_s'(s_F(g)) \geq 0$, and $g_s(s_F(g)) \geq g$, then $s^* \geq s_F(g)$.

For $s_F(g) < \overline{s}$, part (i) reflects two mechanisms. First, at a given subsidy, stronger links allow peripheral organizations to hold less reserve capacity themselves while increasing their total available reserve capacity, $z_P + y_P$. This reduces the combined opportunity and fiscal costs of reserves and expected unmet emergency needs, so $W_g > 0$. If a marginal increase in the subsidy weakly reduces equilibrium link weights at $s_F(g)$, this adjustment weakly lowers the authority's marginal return to the subsidy.

Second, stronger links increase the marginal return to a subsidy, as captured by $W_{sg} > 0$. Hence, if $g_s(s_F(g)) \leq g$, the marginal return evaluated at the endogenous link weight is weakly lower than at the fixed benchmark weight. Together, these effects imply $s^* \leq s_F(g)$. For $0 < s_F(g) < \overline{s}$, the reverse conditions yield $s^* \geq s_F(g)$, as in part (ii).

The fixed-network benchmark can represent a short run in which organizations adjust reserves given predetermined assistance arrangements, while the endogenous-network case

allows these arrangements to be revised. Under the conditions in part (i), the optimal subsidy is therefore weakly lower when assistance arrangements can adjust.

The Appendix also provides a simpler sufficient condition for part (i). Define

$$\bar{\kappa} = \frac{\zeta^2(\gamma - c)\psi(\gamma - nr^x)}{2\gamma} < \kappa_0(0).$$

If both $\kappa < \bar{\kappa}$ and (9) hold, g_s is strictly decreasing on $[0, \overline{s}]$. Consequently, $g_s(0) \leq g$ is sufficient for $s^* \leq s_F(g)$.

Our final result on the optimal subsidy establishes its comparative statics with respect to the penalty for unmet emergency needs, γ. To make this dependence explicit, we write the authority's objective as $W(s, \gamma)$ and denote the corresponding optimal subsidy by $s^*(\gamma) = \arg\max_{s \geq 0} W(s, \gamma)$. The next proposition shows that, under its stated assumptions, $s^*(\gamma)$ is locally weakly decreasing in γ whenever $0 < s^*(\gamma) < \overline{s}$, so that the optimum is interior and the equilibrium network is a weighted star.

Proposition 10. *Suppose that $\kappa < \kappa_0(0)$ and $r^x > \gamma/2n$. If $s^*(\gamma) \in (0, \overline{s})$, then $\frac{\partial s^*(\gamma)}{\partial \gamma} \leq 0$.*

The conditions $\kappa < \kappa_0(0)$ and $r^x > \gamma/(2n)$ imply that $s^*(\gamma)$ maximizes $W(s, \gamma)$ over $[0, \overline{s}]$. The objective is strictly concave on this interval and twice continuously differentiable for $0 < s < \overline{s}$. Under the assumption that the optimum is interior, the proof follows immediately from the implicit function theorem, by showing that at $s^*(\gamma)$ the cross-partial derivative $W_{s\gamma}(s^*(\gamma), \gamma)$ is non-positive.

5 Conclusion

Joint choices of reserve capacity and access relationships determine both where reserves are held and who can use them. Organizations with greater capacity attract links, while access to others' reserves reduces incentives to hold reserves themselves, producing complete core-periphery structures in strict Nash equilibria. Within weighted-star equilibria, a marginal increase in the return to ordinary activity can raise aggregate reserves even as every organization becomes more exposed to unmet emergency needs. For a public authority weighing ordinary activity, expected unmet emergency needs, and subsidy expenditure, network adjustment also changes the return to subsidizing reserves. The weighted-star policy analysis identifies sufficient conditions for a weakly lower or higher optimal subsidy than under fixed links, depending on the response and relative level of equilibrium link weights. The effects of reserve subsidies thus depend on how organizations adjust access to others' reserves as well as their own holdings.

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Appendix

Exact payoffs and the auxiliary game

Maintain the uniform specialization $\epsilon \sim \mathcal{U}[0,\zeta]$, $0 < \zeta \leq \min_i w_i$, $r^x > 0$, $\gamma > c > 0$, $0 < \psi \leq 1$, and $\tilde{\kappa} > 0$, with

$$\beta = \gamma - nr^x, \qquad \lambda = \gamma - c, \qquad \kappa = n\zeta\tilde{\kappa}.$$

Write $(a)_+ = \max\{a, 0\}$. The stated three cases of realized loss are equivalently $c(\epsilon - z_i)_+ + \lambda(\epsilon - z_i - y_i)_+$. For $a \geq 0$, evaluating the integrals under the uniform distribution gives

$$E[(\epsilon - a)_+] = \frac{1}{\zeta}\int_0^\zeta (\epsilon - a)_+ \, d\epsilon = \frac{(\zeta - a)_+^2}{2\zeta}.$$

Hence, the conditional expected loss and the original payoff are exactly

$$\widehat{\Pi}_i(z_i, y_i) = \frac{c(\zeta - z_i)_+^2 + \lambda(\zeta - z_i - y_i)_+^2}{2\zeta}, \tag{10}$$

$$\hat{u}_i = r^x(w_i - z_i) - \frac{\widehat{\Pi}_i(z_i, y_i)}{n} - \tilde{\kappa}\sum_j g_{i,j}. \tag{11}$$

The factor $1/n$ is the probability that i is the single organization hit. Let $\widehat{\Gamma}$ be this original game and Γ the auxiliary game with payoff (3) on the same entire strategy space. Define the choice-independent constant $C_i = n\zeta r^x w_i - \gamma\zeta^2/2$ and $U_i = n\zeta\hat{u}_i - C_i$. Expanding the squares, using $(\zeta - a)_+^2 = (\zeta - a)^2 - (a - \zeta)_+^2$, gives the global identity

$$U_i = u_i + \frac{c}{2}(z_i - \zeta)_+^2 + \frac{\lambda}{2}(z_i + y_i - \zeta)_+^2. \tag{12}$$

Thus $U_i \geq u_i$ everywhere, with equality exactly when $z_i + y_i \leq \zeta$. Because the shock distribution has support only on $[0, \zeta]$, the quadratic expression derived for $z_i + y_i \leq \zeta$ does not in general equal the original expected payoff when $z_i + y_i > \zeta$. Beyond that region, we use the same quadratic expression only as an auxiliary extension.

Lemma 1. *Under the maintained assumptions above, the original game $\widehat{\Gamma}$ and auxiliary game Γ have identical joint best-response sets over reserve capacity and links for every profile of opponents' strategies. Every best response satisfies $z_i + y_i < \zeta$. Consequently, the two games have the same Nash equilibria and strict Nash equilibria.*

Proof. Fix i and the opponents' strategies. All deviations below respect $0 \leq z_i \leq w_i$, $0 \leq g_{i,j} \leq 1$, and $g_{i,i} = 0$. If $z_i > \zeta$, replace its strategy by $z_i' = \zeta$ and $\mathbf{g}_i' = \mathbf{0}$. The loss from the shock remains zero, ordinary-use return rises strictly, and relationship cost weakly falls. The deviation is feasible since $w_i \geq \zeta$.

If instead $z_i \leq \zeta$ and $z_i + y_i > \zeta$, keep z_i fixed and multiply every outgoing weight by $a = (\zeta - z_i)/y_i \in [0, 1)$. Then $y_i' = \zeta - z_i$. In (10), the first square is unchanged and the second is zero before and after the deviation. Relationship cost falls strictly, since $y_i > 0$ implies $\sum_j g_{i,j} > 0$ and $\tilde{\kappa} > 0$.

It follows that every maximizer of the original payoff belongs to

$$D_i = \{(z_i, \mathbf{g}_i) \in S_i : z_i + y_i \leq \zeta\}.$$

Since U_i is a positive affine transformation of the original payoff, it has the same maximizers. Continuity and compactness ensure that U_i attains a maximum M_i, and every maximizer lies in D_i, where $u_i = U_i$. Since $u_i \leq U_i \leq M_i$ everywhere by (12), it follows that the maximum of u_i is also M_i. Moreover, $u_i = M_i$ implies $U_i = M_i$, while every maximizer of U_i also maximizes u_i. Thus the two games have identical best-response sets, including at the zero-reserve boundary.

It remains to rule out $z_i + y_i = \zeta$. Any positive link to an organization with zero reserves can be removed while strictly reducing relationship costs. At a remaining strategy with $z_i + y_i = \zeta$ and $y_i > 0$, multiply all outgoing weights by $1 - t$. For sufficiently small $t > 0$, the change in U_i is

$$t\kappa \sum_j g_{i,j} - \frac{\lambda}{2} t^2 y_i^2 > 0.$$

If $y_i = 0$, after removing any links to organizations with zero reserves, equality requires $z_i = \zeta$. Reducing reserves by a sufficiently small $h > 0$ increases U_i by

$$n\zeta r^x h - \frac{\gamma}{2} h^2 > 0.$$

Hence every best response in either game satisfies $z_i + y_i < \zeta$.

Identical best-response sets imply identical Nash equilibria. A pure-strategy Nash equilibrium is strict precisely when each player's strategy is its unique best response to the opponents' strategies. Hence strict Nash equilibria are also preserved. □

For arbitrary accessible capacity y_i, maximizing the auxiliary payoff over own reserve capacity gives

$$z_i(y_i) = \max\left\{0, \frac{\zeta\beta - \lambda y_i}{\gamma}\right\}, \tag{13}$$

$$v(y) = \zeta\lambda y - \frac{\lambda}{2}y^2 + \frac{(\zeta\beta - \lambda y)_+^2}{2\gamma}, \tag{14}$$

where $v(y)$ is the optimized auxiliary payoff before relationship costs. The upper bound w_i does not bind, since $r^x > 0$ implies $\max\{\zeta\beta/\gamma, 0\} < \zeta \leq w_i$. The function v is continuously differentiable and strictly concave. We use (14) to evaluate deviations in access relationships in the auxiliary game. Lemma 1 implies that the same best responses apply in the original game.

Fixed networks: Lemma 2 and Proposition 1

Lemma 2. *For any fixed network, if $nr^x \geq c$ (equivalently $\beta \leq \lambda$), the original and auxiliary reserve games have identical best responses for every profile of opponents' reserves, and identical Nash equilibria and strict Nash equilibria. If $\beta \leq 0$, their unique equilibrium is $\mathbf{z} = \mathbf{0}$. If $0 < nr^x < c$, the two reserve responses to a given y_i coincide if and only if $y_i \leq \zeta nr^x/c$.*

Proof. For fixed y_i, the derivative of the normalized exact payoff is

$$\frac{\partial U_i}{\partial z_i} = -n\zeta r^x + c(\zeta - z_i)_+ + \lambda(\zeta - z_i - y_i)_+.$$

It is strictly decreasing on $[0, \zeta]$ and strictly negative for $z_i \geq \zeta$. On $[0, \zeta]$ it is the maximum of the two decreasing expressions $c(\zeta - z_i) - n\zeta r^x$ and $\gamma(\zeta - z_i) - \lambda y_i - n\zeta r^x$. The larger of their two roots, truncated at zero, therefore gives the unique exact reserve-capacity best response

$$\max\left\{0,\ \zeta\left(1 - \frac{nr^x}{c}\right),\ \frac{\zeta\beta - \lambda y_i}{\gamma}\right\}. \tag{15}$$

Each term in the maximum is strictly below ζ, so $w_i \geq \zeta$ is nonbinding. If $nr^x \geq c$, then $\zeta(1 - nr^x/c) \leq 0$, and (15) coincides with the auxiliary best response (13) for every $y_i \geq 0$, including $y_i > \zeta$. The original and auxiliary reserve games therefore have identical best-response sets. If $\beta \leq 0$, both reserve-capacity best-response functions are identically zero, so the unique equilibrium is $\mathbf{z} = \mathbf{0}$.

If $0 < nr^x < c$, then $\zeta(1 - nr^x/c) > 0$. Moreover,

$$\frac{\zeta\beta - \lambda y_i}{\gamma} - \zeta\left(1 - \frac{nr^x}{c}\right) = \frac{\lambda}{\gamma}\left(\frac{\zeta n r^x}{c} - y_i\right).$$

Hence the original and auxiliary reserve best responses coincide if and only if

$$y_i \leq \frac{\zeta n r^x}{c}.$$

For larger y_i, the original best response is $\zeta(1 - nr^x/c) > 0$, while the auxiliary best response is smaller. □

Proof of Proposition 1. Let $\alpha = \zeta\beta/\gamma$ and $\omega = \lambda\psi/\gamma$. The simultaneous constrained best-response map is

$$T_i(\mathbf{z}) = \max\left\{0, \alpha - \omega \sum_j g_{i,j} z_j\right\}.$$

For any $\mathbf{z} \in [0, \alpha]^n$, we have $0 \leq T_i(\mathbf{z}) \leq \alpha$, so T maps $[0, \alpha]^n$ into itself. Moreover, using

$$|\max\{0, a\} - \max\{0, b\}| \leq |a - b|,$$

we obtain

$$|T_i(\mathbf{z}) - T_i(\mathbf{z}')| \leq \omega \sum_j g_{i,j}|z_j - z'_j| \leq \omega(n-1)\|\mathbf{z} - \mathbf{z}'\|_\infty,$$

where the last inequality uses $g_{i,i} = 0$ and $0 \leq g_{i,j} \leq 1$. Taking the maximum over i gives

$$\|T(\mathbf{z}) - T(\mathbf{z}')\|_\infty \leq \omega(n-1)\|\mathbf{z} - \mathbf{z}'\|_\infty.$$

Since $\omega = \lambda\psi/\gamma$, the assumption $\psi < \gamma/[\lambda(n-1)]$ implies $\omega(n-1) < 1$. Thus T is a contraction on $[0, \alpha]^n$, and the contraction theorem gives a unique fixed point.

Every equilibrium lies in $[0, \alpha]^n$. At the fixed point,

$$z_i \geq \alpha - \omega \sum_j g_{i,j}\alpha \geq \alpha[1 - \omega(n-1)] > 0, \qquad z_i \leq \alpha < \zeta \leq w_i.$$

Thus the unique constrained equilibrium is interior and solves

$$\left[I + \frac{\lambda\psi}{\gamma} G\right]\mathbf{z} = \frac{\zeta\beta}{\gamma}\mathbf{1}. \tag{16}$$

By Lemma 2, the original game has the same unique equilibrium. □

Core-periphery structure: Proposition 2

All lemmas in this subsection are stated for strict Nash equilibria of the auxiliary game with strictly positive relationship costs. By Lemma 1, these results also hold in the original game. No organization links to a zero-reserve capacity orgainzation in equilibrium, since deleting that link saves cost without changing accessible capacity.

Lemma 3. *An agent extends at most one link with weight in* $(0, 1)$*. If such a link is directed to* k*, every full-weight link is directed to an agent with reserves strictly greater than* z_k*.*

Proof. Suppose two weights are interior, to l and k, with $z_k \geq z_l > 0$. Reduce $g_{i,l}$ by $\nu > 0$ and increase $g_{i,k}$ by $\nu z_l/z_k$, for ν sufficiently small. Accessible reserve capacity changes by $-\psi\nu z_l + \psi\nu(z_l/z_k)z_k = 0$, whereas cost weakly decreases. This contradicts the assumption that the strategy profile is a strict Nash equilibrium. The identical transfer from a full link to an interior link proves the second assertion if the organization receiving the full link has weakly smaller reserves. □

Lemma 4. *If* $g_{i,l} > 0$*, then* $g_{i,k} = 1$ *for every* $k \in N \setminus \{i, l\}$ *with* $z_k \geq z_l$*.*

Proof. If $g_{i,k} < 1$, decrease $g_{i,l}$ by $\nu > 0$ and increase $g_{i,k}$ by $\nu z_l/z_k$, for ν sufficiently small, exactly as in the preceding proof. This leaves all reserves unchanged and preserves i's total accessible reserve capacity y_i, while weakly reducing relationship cost, contradicting strictness. □

Lemma 5. *If* $z_i < z_l$*, then* $g_{i,l} > 0$*.*

Proof. Equation (13) implies $y_i > y_l$, even if $z_i = 0$. Suppose $g_{i,l} = 0$. If also $g_{l,i} = 0$, both agents face the same optimization over links to third agents and can imitate each other's optimized strategies. The reserve bounds do not prevent this imitation, since every optimal reserve choice lies below $\zeta \leq w_i, w_l$. Strictness and $z_i \neq z_l$ make this impossible. Thus $g_{l,i} > 0$, implying $z_i > 0$. By Lemma 4, l fully links to every third agent with reserves greater than z_i. The same lemma and $g_{i,l} = 0$ imply that i links only to agents with reserves strictly greater than z_l. Consequently

$$y_l \geq \psi g_{l,i} z_i + \psi \sum_{\substack{j \neq l \\ z_j > z_i}} z_j > \psi \sum_{z_j > z_l} z_j \geq y_i,$$

contradicting $y_i > y_l$. □

Lemma 6. *If* $g_{i,l} > 0$ *and* $z_i \leq z_l$*, then* $g_{k,l} \geq g_{i,l}$ *for every* $k \neq l$*.*

Proof. Suppose $g_{k,l} < g_{i,l}$, and set $\Delta = \psi(g_{i,l} - g_{k,l})z_l > 0$. Since $g_{k,l} < 1$, agent k has no link to any other organization with reserves weakly below z_l. Agent i fully links to every organization with reserves strictly above z_l, and i itself is not such an organization. It follows, taking account of the exclusion of self-links, that $y_k + \Delta \leq y_i$. Strict optimality for i gives

$$v(y_i) - v(y_i - \Delta) > \kappa(g_{i,l} - g_{k,l}).$$

Concavity of the globally constrained value (14) then implies

$$v(y_k + \Delta) - v(y_k) \geq v(y_i) - v(y_i - \Delta) > \kappa(g_{i,l} - g_{k,l}).$$

Agent k can profitably raise its weight to $g_{i,l}$ and reoptimize its reserves, a contradiction. □

Lemma 7. *If $\beta > 0$, all reserve capacities in a strict equilibrium are positive. If $\beta \leq 0$, the unique equilibrium is the empty network with zero reserves.*

Proof. The second assertion follows from (13) and strictly positive relationship costs. For the first, some agent must hold positive reserves, since at the profile with zero reserves and no links any agent can profitably increase its reserve capacity: the marginal payoff from reserves at zero is $\zeta\beta > 0$. Suppose $z_i = 0$. By the preceding two lemmas, for every positive-reserve organization j, $g_{i,j} > 0$ and $g_{k,j} \geq g_{i,j}$ for $k \neq j$. For a positive-reserve agent k, $y_k \geq y_i - \psi g_{i,k} z_k$. Set $\alpha = \zeta\beta/\gamma$ and $b = \lambda/\gamma$. Since $\alpha - by_i \leq 0$,

$$z_k = \alpha - by_k \leq \alpha - by_i + b\psi g_{i,k} z_k \leq b\psi z_k < z_k,$$

a contradiction. □

Proof of Proposition 2. For $\beta \leq 0$ take an empty core. Otherwise, all reserves are positive by Lemma 7. If the associated undirected graph is complete, take $C = N$ and $P = \emptyset$. Henceforth suppose that it is not complete, and let $\underline{Z}$ be the set of organizations with minimum reserves. If one organization in $\underline{Z}$ links to another organization in $\underline{Z}$, cutoff linking implies that it links to every other agent. Each of these organizations then receives a link from every remaining agent by Lemma 6. This would make the associated undirected graph complete, contrary to our supposition. Hence there are no links within $\underline{Z}$. Take $P = \underline{Z}$ and $C = N \setminus P$. Every peripheral agent links to every core agent by Lemma 5, and every core agent receives links from all other agents by Lemma 6. Thus, the associated undirected graph has a complete core-periphery structure. All peripheral organizations hold the same reserve capacity, and every core organization holds strictly more. It remains to verify the direction of core-periphery links.

Since every core organization receives a link from every other organization, $|P| = 1$ would make the associated undirected graph complete. Thus $|P| \geq 2$. If a core organization extends an interior-weight link to a peripheral organization, Lemma 4 implies that it fully links to every other peripheral organization, since peripheral reserves are equal. Consequently, any core-to-periphery link entails a full core-to-periphery link. It therefore suffices to exclude the latter.

If $i \in C$ fully links to $j \in P$, cutoff linking makes all of i's outgoing weights one. Since both organizations hold positive reserves, their reserve best-response conditions are

$$\gamma z_i = \zeta\beta - \lambda\psi\left(z_j + \sum_{k\neq i,j} z_k\right)$$

and

$$\gamma z_j = \zeta\beta - \lambda\psi\left(g_{j,i}z_i + \sum_{k\neq i,j} g_{j,k}z_k\right).$$

Subtracting and rearranging gives

$$(\gamma - \lambda\psi)(z_i - z_j) = \lambda\psi\left[(g_{j,i} - 1)z_i + \sum_{k\neq i,j}(g_{j,k} - 1)z_k\right] \leq \lambda\psi\sum_{k\neq i,j}(g_{j,k} - 1)z_k \leq 0.$$

Since $\gamma > \lambda\psi$, this contradicts $z_i > z_j$. □

Strict existence: Proposition 3

Throughout this subsection first take $0 < \beta < \lambda$. The empty network has $z_i = \zeta\beta/\gamma$ and is a strict equilibrium if and only if

$$\kappa \geq \kappa_0 = \frac{\zeta^2\beta\lambda\psi(\gamma - \beta)}{\gamma^2}. \tag{17}$$

Equality is included because, at $\kappa = \kappa_0$, the marginal gain from adding a link at zero is zero, but strict concavity of v implies that every strictly positive link deviation yields a strictly lower payoff.

Lemma 8. *A strict weighted-star equilibrium exists if and only if $\kappa_1 < \kappa < \kappa_0$, where*

$$\kappa_1 = \frac{\zeta^2\beta\lambda\psi[\gamma(\gamma - \beta) - \psi c\beta]}{\gamma^3}. \tag{18}$$

Its center holds $z_C = \zeta\beta/\gamma$, all peripheral weights equal $g_s \in (0, 1)$, and $\kappa = v'(\psi g_s z_C)\psi z_C$.

Proof. By Proposition 2, the center has no outgoing links. Each peripheral agent's joint choice of its own reserves and the weight of its single link to the center reduces to the same strictly concave problem

$$\max_{g\in[0,1]}\{v(\psi g z_C)-\kappa g\}.$$

Hence they choose the same weight, and an interior solution satisfies $\kappa = v'(\psi g_s z_C)\psi z_C$. Its marginal benefit decreases strictly from κ_0 at weight zero to κ_1 at weight one. At the candidate solution, $z_P = z_C(1-\lambda\psi g_s/\gamma) > 0$. A peripheral agent's unused link to another peripheral agent has strictly lower marginal benefit than its link to the center. For the center,

$$\kappa - v'(0)\psi z_P = \frac{\zeta^2 g_s \beta\lambda(\lambda-\beta)\psi^2}{\gamma^2} > 0.$$

Thus it strictly prefers not to link. Strict concavity of (14) makes these first-order link conditions sufficient. Moreover, since $z_C > z_P$, a peripheral agent obtains any given amount of accessible capacity at strictly lower relationship cost from the center than from another peripheral agent; together with the strict no-link condition for the center and the unique reserve best response, this gives a unique best response for every agent. □

We next characterize strict equilibria with binary complete core-periphery networks. These results are used in the proof of Proposition 3 and in the multiple-core comparison in Proposition 7.

Lemma 9. *Suppose $0 < \beta < \lambda$ and let $m \in \{1,\dots,n-1\}$. Consider a binary complete core-periphery configuration with m core organizations, in which every organization links with weight one to every core organization other than itself and all other links have weight zero. The reserve and accessible-capacity levels are*

$$z_C(m) = \frac{\zeta\beta}{\gamma+(m-1)\lambda\psi}, \qquad z_P(m) = z_C(m)\left(1-\frac{\lambda\psi}{\gamma}\right), \tag{19}$$
$$y_C(m) = \psi(m-1)z_C(m), \qquad y_P(m) = \psi m z_C(m).$$

Both reserve levels are positive and total available reserve capacities are below ζ.

The configuration is a strict Nash equilibrium precisely when

$$\underline{\kappa}(m) \le \kappa \le \bar{\kappa}(m),$$

where

$$\begin{aligned}\underline{\kappa}(m) &\equiv v'(y_C(m))\psi z_P(m) \\ &= \frac{\zeta^2\beta\lambda\psi(\gamma-\lambda\psi)[\gamma-\beta+(m-1)(\lambda-\beta)\psi]}{\gamma[\gamma+(m-1)\lambda\psi]^2}, \qquad (20)\\ \bar{\kappa}(m) &\equiv v'(y_P(m))\psi z_C(m) \\ &= \frac{\zeta^2\beta\lambda\psi[\gamma(\gamma+(m-1)\lambda\psi)-\beta(\gamma+\gamma m\psi-\lambda\psi)]}{\gamma[\gamma+(m-1)\lambda\psi]^2}. \qquad (21)\end{aligned}$$

The supporting interval has positive width:

$$\bar{\kappa}(m)-\underline{\kappa}(m)=\frac{\zeta^2\beta\lambda\psi^2(\lambda-\beta)}{\gamma[\gamma+(m-1)\lambda\psi]}>0. \qquad (22)$$

Both endpoints decrease with m.

Proof. At any strict equilibrium, Lemma 7 implies that all organizations hold positive reserves. Since every core organization fully links to every other core organization, subtracting the reserve capacity best-response conditions of any two core organizations gives $(\gamma-\lambda\psi)(z_i-z_j)=0$. Hence all core organizations hold the same reserves, z_C. All peripheral organizations access the same core reserves and therefore also hold the same reserve capacities, z_P. Their reserve capacity best-response conditions are

$$\gamma z_C=\zeta\beta-\lambda\psi(m-1)z_C, \qquad \gamma z_P=\zeta\beta-\lambda\psi m z_C.$$

Solving gives (19) and the stated expressions for y_C and y_P. Under the maintained parameter assumptions, both reserve levels are positive, and $z_C+y_C<\zeta$ and $z_P+y_P<\zeta$.

For fixed opponents' reserves, after maximizing over own reserves the marginal payoff from increasing $g_{i,j}$ is

$$v'(y_i)\psi z_j-\kappa.$$

Thus an absent link requires $v'(y_i)\psi z_j\le\kappa$, whereas a link at weight one requires $v'(y_i)\psi z_j\ge\kappa$. All absent links point to peripheral organizations and all full links point to core organizations. Since $y_P>y_C$ and v' is decreasing, the tightest absent-link condition is the core-periphery condition

$$\kappa\ge v'(y_C)\psi z_P=\underline{\kappa}(m),$$

while the tightest full-link condition is the periphery-to-core condition

$$\kappa\le v'(y_P)\psi z_C=\bar{\kappa}(m).$$

Substitution of (19) into (14) gives the closed-form expressions (20) and (21). If either bound is violated, a sufficiently small increase in an absent core-to-periphery link or decrease in a full periphery-to-core link is profitable. Hence the bounds are necessary.

Now suppose $\underline{\kappa}(m) \leq \kappa \leq \bar{\kappa}(m)$. For fixed opponents' reserve capacities, maximizing over own reserves gives the reduced payoff

$$v(y_i) - \kappa \sum_{j \neq i} g_{i,j}.$$

This payoff is jointly concave in outgoing link weights because v is concave and y_i is linear in those weights. At the candidate vector, the marginal payoff from increasing a link weight is nonpositive for absent links and nonnegative for full links. Since absent links can only be increased and full links can only be reduced, concavity implies that no feasible change in outgoing link weights can increase the payoff. Since v already incorporates the optimal reserve choice, this also rules out any profitable joint change in reserves and links.

It remains to show that each organization's best response is unique. Consider first a deviation to $\mathbf{g}_i'$ that changes accessible capacity from y_i to $y_i' \neq y_i$. Strict concavity of v and the boundary first-order conditions imply

$$v(y_i') - v(y_i) - \kappa \sum_{j \neq i} (g_{i,j}' - g_{i,j}) < v'(y_i)(y_i' - y_i) - \kappa \sum_{j \neq i} (g_{i,j}' - g_{i,j}) \leq 0.$$

Hence every such deviation is strictly worse, including when a relevant marginal-payoff inequality holds with equality.

If instead a nonzero change in link weights preserves y_i, it must reduce weights on links to core organizations and increase weights on links to peripheral organizations, since the former are all at one and the latter at zero. Since $z_C > z_P$, preserving y_i requires the total increase in peripheral-link weights to exceed the total reduction in core-link weights. Total relationship cost therefore rises strictly, while $v(y_i)$ is unchanged. Finally, with the link vector unchanged, (13) gives a unique optimal reserve choice, so any distinct reserve choice is strictly worse. Thus each organization has a unique best response, so the Nash equilibrium is strict, including when $\kappa = \underline{\kappa}(m)$ or $\kappa = \bar{\kappa}(m)$.

Subtracting (20) from (21) gives (22). Moreover, z_C and z_P decrease with m, while y_C and y_P increase with m. Since total available capacity is below ζ, v' is positive at y_C and y_P, and strict concavity makes v' decreasing. The representations $\underline{\kappa}(m) = v'(y_C)\psi z_P$ and $\bar{\kappa}(m) = v'(y_P)\psi z_C$ therefore imply that both endpoints decrease with m. □

Proof of Proposition 3. For $\beta \leq 0$, Lemma 7 proves the result for every $\kappa > 0$. For $\beta > 0$, the stated bound $\beta < \bar{\beta}$ implies $\beta < \lambda$. Substituting the definition of $\bar{\beta}$ and factoring gives

$$\lambda - \bar{\beta} = \frac{\lambda(\gamma - \lambda)(\gamma - \lambda\psi)(2\gamma + \lambda\psi)}{2\gamma^2(\gamma - \lambda) + \gamma\lambda^2\psi + \lambda^3\psi^2} > 0,$$

where the inequality follows from $\gamma > \lambda > 0$ and $0 < \psi \leq 1$, which make every factor in the numerator and every term in the denominator strictly positive.

Empty and weighted-star equilibria cover $[\kappa_0, \infty)$ and (κ_1, κ_0), respectively. Binary configurations cover the remaining costs, as follows.

For integers $m \geq 2$, define

$$B(m) = \frac{\lambda^2\psi(m-1)[\gamma + (m-2)\lambda\psi][\gamma + (m-1)\lambda\psi]}{B_1(m)},$$

where

$$\begin{aligned} B_1(m) = {} & 2\gamma^3 + \gamma^2\lambda[3(m-2)\psi - 2] \\ & + \lambda^3\psi^2[2m - 3 + (m-2)(m-1)^2\psi] \\ & + \gamma\lambda^2\psi[5 - 2m + (m-2)(2m-3)\psi]. \end{aligned}$$

To establish $B_1(m) > 0$, write $t = m - 2 \geq 0$. Then

$$\begin{aligned} B_1(m) = {} & 2\gamma^2(\gamma - \lambda) + \gamma\lambda^2\psi + \lambda^3\psi^2 \\ & + t\gamma\lambda\psi(3\gamma - 2\lambda) + \gamma\lambda^2\psi^2 t(2t+1) \\ & + \lambda^3\psi^2[2t + t(t+1)^2\psi] > 0. \end{aligned}$$

All terms are nonnegative, and the first is strictly positive.

For $2 \leq m \leq n-1$, Lemma 9 shows that both endpoints decrease with m. Hence the intervals supporting configurations with $m-1$ and m core organizations overlap if and only if $\underline{\kappa}(m-1) \leq \bar{\kappa}(m)$. Using (20) and (21) gives

$$\bar{\kappa}(m) - \underline{\kappa}(m-1) = \frac{\zeta^2\beta\lambda\psi^2 B_1(m)[B(m) - \beta]}{\gamma[\gamma + (m-2)\lambda\psi]^2[\gamma + (m-1)\lambda\psi]^2}.$$

Since all factors outside $B(m) - \beta$ are strictly positive, the overlap condition is equivalent to $\beta \leq B(m)$. The same identity holds at $m = n$ when $\bar{\kappa}(n)$ denotes the formal expression (21) evaluated at n.

Moreover, putting $B(m)$ and $B(2)$ over a common denominator and factoring the numerator gives

$$B(m) - B(2) = \frac{\lambda^2\psi(\gamma - \lambda\psi)(\gamma - \lambda)(m-2)\mathcal{P}(m)}{B_1(2)B_1(m)} \geq 0,$$

where

$$\mathcal{P}(m) = 2\gamma^3 + \gamma^2\lambda\psi(4m-3) + \gamma\lambda^2\psi^2(2m^2 - 2m - 1) + \lambda^3\psi^3(m-1)^2 > 0.$$

The sign follows from $m \geq 2$, $\gamma > \lambda > 0$, and $0 < \psi \leq 1$; in particular, $2m^2 - 2m - 1 \geq 3$. Since $B(2) = \bar{\beta}$, the assumed bound $\beta < \bar{\beta}$ guarantees all adjacent overlaps, as well as $\bar{\kappa}(n) > \underline{\kappa}(n-1)$. Because $\bar{\kappa}(1) = \kappa_1$, the binary intervals for $1 \leq m < n$ therefore cover $[\underline{\kappa}(n-1), \kappa_1]$.

For the complete binary network, all reserves equal $z_C(n)$, and each organization accesses $y_C(n) = \psi(n-1)z_C(n)$. Define

$$\kappa^c \equiv v'(y_C(n))\psi z_C(n) = \frac{\zeta^2\beta\lambda\psi[\gamma - \beta + (n-1)(\lambda - \beta)\psi]}{[\gamma + (n-1)\lambda\psi]^2}.$$

Every outgoing link has weight one and marginal payoff $\kappa^c - \kappa$. Thus, for $0 < \kappa \leq \kappa^c$, concavity of the reduced payoff rules out profitable link deviations. Any nonzero feasible change in link weights strictly reduces accessible capacity, since every outgoing link has weight one at the candidate configuration and all other organizations hold positive reserves. Strict concavity of v, together with the unique reserve best response, therefore makes this equilibrium strict, including at $\kappa = \kappa^c$.

The formal expression (21) evaluated at n satisfies

$$\kappa^c - \bar{\kappa}(n) = \frac{(\zeta\beta\psi)^2\lambda c}{\gamma[\gamma + (n-1)\lambda\psi]^2} > 0.$$

Consequently, $\kappa^c > \bar{\kappa}(n) > \underline{\kappa}(n-1)$, so the complete-network interval overlaps the binary intervals. Together with the empty and weighted-star equilibria, these configurations cover every $\kappa > 0$.

□

Weighted-star uniqueness: Proposition 4

Proof. Parts (i)–(ii) follow from Lemma 8. Solving its linear marginal condition gives

$$g_s = \frac{\gamma}{c\psi^2\beta^2}\left[nr^x\beta\psi - \frac{\gamma^2\kappa}{\zeta^2(\gamma - c)}\right]. \tag{23}$$

This is differentiable wherever $0 < g_s < 1$ and $0 < \beta < \lambda$.

For part (iii), $nr^x > \gamma/2$ means $\beta < \gamma/2$. At every strict equilibrium, reserves are in $(0, \zeta\beta/\gamma]$. On that interval

$$\frac{d}{dz}\{v'(\psi z)\psi z\} = \frac{\lambda\psi}{\gamma}[\zeta(\gamma - \beta) - 2\psi cz] > 0.$$

For the last sign, the bracket at the largest feasible z is strictly greater than $\zeta(\gamma - \beta)(1 - 2\beta/\gamma) > 0$, using $c = \gamma - \lambda < \gamma - \beta$ and $\psi \leq 1$. Therefore any full link in a strict equilibrium would satisfy

$$\kappa \leq v'(y_i)\psi z_j \leq v'(\psi z_j)\psi z_j \leq v'(\psi z_C)\psi z_C = \kappa_1,$$

contradicting $\kappa > \kappa_1$. Without full links, Lemma 3 allows at most one positive outgoing link per organization. By Proposition 2, every strict equilibrium admits a complete core-periphery partition with no links from the core to the periphery. If the associated undirected graph is not complete and the core contains at least two organizations, each peripheral organization must therefore link to at least two core organizations, contradicting the outgoing-link restriction.

A complete associated undirected graph requires at least $n(n-1)/2$ positive directed links. For $n \geq 4$, this exceeds the maximum of n permitted by the outgoing-link restriction. For $n = 3$, completeness requires exactly three directed links, one connecting each pair, with one outgoing link per organization. Up to relabeling, these form the cycle $1 \to 2$, $2 \to 3$, and $3 \to 1$. Since the reverse links are absent, Lemma 4 requires, respectively, $z_2 > z_3, z_3 > z_1, z_1 > z_2$, a contradiction.

The empty network is also excluded because $\kappa < \kappa_0$. The remaining configuration therefore has one core organization, with links from every peripheral organization to the center and no other links. Since full links are excluded, this is a weighted star. Lemma 8 gives a common peripheral link weight $g_s \in (0, 1)$, uniquely determined by (23); the reserve best responses then uniquely determine reserve capacities. Any organization may be the center, so the weighted star is the unique strict Nash equilibrium up to relabeling. □

Reserve comparative statics: Propositions 5–6

Proof of Proposition 5. All derivatives are evaluated at parameter values for which a strict weighted-star equilibrium exists, holding the remaining primitive parameters fixed. Substi-

tuting (23) into the expressions for reserve capacity gives

$$z_C = \frac{\zeta\beta}{\gamma}, \qquad z_P = \frac{\zeta(c - nr^x)}{c} + \frac{\gamma\kappa}{c\zeta\psi\beta}. \tag{24}$$

Consequently

$$z_{C,r^x} = -\frac{\zeta n}{\gamma} < 0, \qquad z_{C,\gamma} = \frac{\zeta n r^x}{\gamma^2} > 0,$$
$$z_{P,r^x} = \frac{n}{c}\left[\frac{\gamma\kappa}{\zeta\psi\beta^2} - \zeta\right], \qquad z_{P,r^x r^x} = \frac{2n^2\gamma\kappa}{c\zeta\psi\beta^3} > 0,$$
$$z_{P,\gamma} = -\frac{nr^x\kappa}{c\zeta\psi\beta^2} < 0.$$

Define the threshold

$$\overline{r^x} = \frac{1}{n}\left[\gamma - \sqrt{\frac{\gamma\kappa}{\zeta^2\psi}}\right]. \tag{25}$$

Since $\beta = \gamma - nr^x > 0$, the expression for z_{P,r^x} shows that

$$z_{P,r^x} \geq 0 \iff \beta^2 \leq \frac{\gamma\kappa}{\zeta^2\psi} \iff r^x \geq \overline{r^x}.$$

It remains to locate this threshold. A weighted-star equilibrium requires $\kappa < \kappa_0$. Moreover, for $c < nr^x < \gamma$,

$$\frac{\zeta^2\psi(\gamma - c)^2}{\gamma} - \kappa_0 = \frac{\zeta^2\lambda\psi}{\gamma^2}\left[\gamma\lambda - nr^x(\gamma - nr^x)\right] = \frac{\zeta^2\lambda\psi}{\gamma^2}\left[\lambda(\gamma - nr^x) + nr^x(nr^x - c)\right] > 0.$$

Hence $\kappa < \zeta^2\psi(\gamma - c)^2/\gamma$. Together with $\kappa > 0$, this implies

$$0 < \sqrt{\frac{\gamma\kappa}{\zeta^2\psi}} < \gamma - c,$$

and therefore $c/n < \overline{r^x} < \gamma/n$. The threshold $\overline{r^x}$ need not itself correspond to a parameter value at which a weighted-star equilibrium exists; the derivative comparison applies wherever such an equilibrium exists. □

Proof of Proposition 6. For the center, $y_C = 0$, so the claim follows from the strict inequalities established in the proof of Proposition 5. For a peripheral organization,

$$z_P + y_P = \zeta - \frac{\gamma\kappa}{\zeta\psi(\gamma - c)\beta}. \tag{26}$$

Its derivatives are

$$\frac{\partial(z_P + y_P)}{\partial r^x} = -\frac{n\gamma\kappa}{\zeta\psi(\gamma - c)\beta^2} < 0, \qquad \frac{\partial(z_P + y_P)}{\partial \gamma} = \frac{(\gamma^2 - cnr^x)\kappa}{\zeta\psi(\gamma - c)^2\beta^2} > 0.$$

□

We record two additional comparative statics that help interpret the preceding results. Proposition 6 concerns total available reserve capacity, $z_i + y_i$. Physical aggregate reserve capacity instead equals

$$Z = z_C + (n-1)z_P.$$

Since Z measures the physical stock of reserve capacity, it counts each organization's reserves only once; accessible capacity represents access to reserves already included in another organization's z_j. The derivatives of Z are

$$\begin{aligned} Z_{r^x} &= -\frac{\zeta n}{\gamma} + \frac{n(n-1)}{c}\left[\frac{\gamma\kappa}{\zeta\psi\beta^2} - \zeta\right], \\ Z_\gamma &= \frac{\zeta n r^x}{\gamma^2} - \frac{(n-1)nr^x\kappa}{c\zeta\psi\beta^2}. \end{aligned} \tag{27}$$

The first expression yields the threshold characterized in Corollary 1. The sign of Z_γ is ambiguous: the center's reserve capacity increases with γ, while peripheral reserve capacity decreases, so the aggregate effect depends on their relative magnitudes.

Proof of Corollary 1. From (27),

$$Z_{r^x} \geq 0 \quad \Longleftrightarrow \quad (\gamma - nr^x)^2 \leq \frac{(n-1)\gamma^2\kappa}{\zeta^2\psi[\gamma(n-1) + c]}.$$

Since $\beta = \gamma - nr^x > 0$ along a strict weighted-star equilibrium, this is equivalent to

$$r^x \geq \frac{\gamma}{n}\left[1 - \sqrt{\frac{(n-1)\kappa}{\zeta^2\psi[\gamma(n-1) + c]}}\right] = r^x_Z.$$

Proposition 6 gives $\partial(z_i + y_i)/\partial r^x < 0$ for every $i \in N$, which proves the remaining claim. □

The negative response of peripheral reserve capacity to γ is not specific to endogenous network adjustment. Holding the star weight fixed at g gives

$$z_{P,\gamma}|_g = \frac{\zeta}{\gamma^3}[nr^x\gamma(1 - \psi g) + c\psi g(2nr^x - \gamma)],$$

which can be negative.

Binary multiple-core comparison: Proposition 7

Here $N_c(r^x)$ denotes the core sizes $1 \leq m \leq n-1$ sustained by the strict binary complete core-periphery configurations characterized in Lemma 9. This correspondence may be empty and, by definition, excludes the empty network, the complete network, and weighted configurations.

Proof of Proposition 7. For part (i), hold the binary configuration $\mathbf{g}$ fixed. The expressions in (19) show that both core and peripheral reserve capacities are proportional to $\beta = \gamma - nr^x$. Hence increasing r^x reduces every organization's reserve capacity while its role and the network remain unchanged. For part (ii), both reserve-capacity expressions in (19) strictly decrease in m at fixed $\beta > 0$. The comparison therefore applies only to organizations that retain the same role across the two configurations.

For part (iii), assume

$$\frac{\gamma + c}{2n} < \tilde{r}^x \leq r^x < \frac{\gamma}{n}$$

and that both $N_c(\tilde{r}^x)$ and $N_c(r^x)$ are nonempty. Since $\tilde{r}^x \leq r^x < \gamma/n$ and $\tilde{r}^x > (\gamma + c)/(2n)$,

$$0 < \gamma - nr^x \leq \gamma - n\tilde{r}^x < \gamma - \frac{\gamma + c}{2} = \frac{\gamma - c}{2} = \frac{\lambda}{2}.$$

Thus, the corresponding values of $\beta = \gamma - nr^x$ lie in $(0, \lambda/2)$. Making the dependence explicit and differentiating (21) and (20) gives

$$\frac{\partial \bar{\kappa}(m,\beta)}{\partial \beta} = \frac{\zeta^2 \lambda\psi}{\gamma[\gamma + (m-1)\lambda\psi]^2} \left[\gamma[\gamma + (m-1)\lambda\psi] - 2\beta(\gamma + \gamma m\psi - \lambda\psi)\right],$$
$$\frac{\partial \underline{\kappa}(m,\beta)}{\partial \beta} = \frac{\zeta^2 \lambda\psi(\gamma - \lambda\psi)}{\gamma[\gamma + (m-1)\lambda\psi]^2} \left[\gamma + (m-1)\lambda\psi - 2\beta(1 + (m-1)\psi)\right].$$

For $\beta < \lambda/2$,

$$\gamma[\gamma+(m-1)\lambda\psi]-2\beta(\gamma+\gamma m\psi-\lambda\psi) > \gamma[\gamma+(m-1)\lambda\psi]-\lambda(\gamma+\gamma m\psi-\lambda\psi) = (\gamma-\lambda)(\gamma-\lambda\psi) > 0,$$

and

$$\gamma + (m-1)\lambda\psi - 2\beta(1 + (m-1)\psi) > \gamma + (m-1)\lambda\psi - \lambda(1 + (m-1)\psi) = \gamma - \lambda > 0.$$

Hence both thresholds are strictly increasing in β on $(0, \lambda/2)$. Both thresholds decrease with m, as already proved.

Write

$$\beta_L = \gamma - nr^x \leq \beta_H = \gamma - n\tilde{r}^x,$$

and make the dependence of the thresholds on β explicit below. Take arbitrary $a \in N_c(r^x)$ and $b \in N_c(\tilde{r}^x)$. If $a \leq b$, then $\max\{a, b\} = b \in N_c(\tilde{r}^x)$ and $\min\{a, b\} = a \in N_c(r^x)$. If $a > b$, monotonicity in m and β gives

$$\underline{\kappa}(a, \beta_H) \leq \underline{\kappa}(b, \beta_H) \leq \kappa, \qquad \bar{\kappa}(a, \beta_H) \geq \bar{\kappa}(a, \beta_L) \geq \kappa,$$

so $a \in N_c(\tilde{r}^x)$. Likewise,

$$\underline{\kappa}(b, \beta_L) \leq \underline{\kappa}(b, \beta_H) \leq \kappa, \qquad \bar{\kappa}(b, \beta_L) \geq \bar{\kappa}(a, \beta_L) \geq \kappa,$$

so $b \in N_c(r^x)$. Since a and b were arbitrary, this proves the stated strong set order, including threshold equalities. □

For completeness, consider the response to γ while holding a binary configuration with m core organizations fixed. We have

$$\frac{\partial z_C(m)}{\partial \gamma} = \frac{\zeta[nr^x(1 + (m-1)\psi) - (m-1)c\psi]}{[\gamma + (m-1)(\gamma - c)\psi]^2} > 0,$$
$$\frac{\partial z_P(m)}{\partial \gamma} = \left(1 - \psi + \frac{c\psi}{\gamma}\right) \frac{\partial z_C(m)}{\partial \gamma} - \frac{c\psi z_C(m)}{\gamma^2}.$$

The numerator of the first derivative can be written as $nr^x + (m-1)\psi(nr^x - c) > 0$, where the inequality follows from $nr^x > c$, $m \geq 1$, and $\psi > 0$. By contrast, the maintained assumptions do not determine the sign of $\partial z_P(m)/\partial \gamma$.

Equilibrium under subsidies

The planner's objective is the stated fiscal and shortage-loss criterion, without adding relationship costs or access costs:

$$W(s) = \sum_{i=1}^{n} \left[r^x(w_i - z_i) - sz_i - \frac{\gamma}{2n\zeta}(\zeta - z_i - y_i)_+^2 \right]. \tag{28}$$

Lemma 10. *Maintain the assumptions of Lemma 1. For $s \geq 0$, write*

$$\hat{u}_i(s) = \hat{u}_i + sz_i, \qquad u_i(s) = u_i + n\zeta sz_i.$$

Then:

(i) For $0 \leq s < r^x$, the original subsidized game and the auxiliary game have identical joint best responses, Nash equilibria, and strict Nash equilibria. Every joint best response satisfies $z_i + y_i < \zeta$.

(ii) For $0 \leq s < r^x$ and fixed links, the original reserve best response is (7). It coincides with the auxiliary reserve response for every accessible capacity when $n(r^x - s) \geq c$. If $0 < n(r^x - s) < c$, the two responses coincide if and only if

$$y_i \leq \frac{\zeta n(r^x - s)}{c}.$$

(iii) In the original game, at $s = r^x$ the joint best responses are $(z_i, \mathbf{0})$ with $z_i \in [\zeta, w_i]$. For $s > r^x$, the unique joint best response is $(w_i, \mathbf{0})$. With links fixed, the corresponding capacity reserve best-response sets are $[\zeta, w_i]$ and $\{w_i\}$, respectively.

Proof. For $0 \leq s < r^x$,

$$r^x(w_i - z_i) + sz_i = (r^x - s)(w_i - z_i) + sw_i.$$

The term sw_i is independent of the organization's choices. Since $r^x - s > 0$ and all other assumptions are unchanged, Lemma 1 applies with r^x replaced by $r^x - s$. The corresponding auxiliary payoff is $u_i(s)$: the substitution replaces β by $\gamma - n(r^x - s)$ while leaving λ and κ unchanged. This proves (i).

Applying Lemma 2 with the same substitution proves the equivalence conditions in (ii). Substitution in (15) gives (7). Every entry in that maximum is strictly below $\zeta \leq w_i$, so the upper resource bound does not bind.

At $s = r^x$, ordinary-use return plus subsidy income equals $r^x w_i$, independently of reserve holdings. Because $c > 0$, expected emergency loss is zero when $z_i \geq \zeta$. If links are chosen endogenously, positive relationship costs additionally require $\mathbf{g}_i = \mathbf{0}$ at a best response. Thus every $z_i \in [\zeta, w_i]$ is optimal. At $s > r^x$, ordinary-use return plus subsidy income is strictly increasing in z_i; choosing $z_i = w_i$ maximizes this return and eliminates emergency loss. Endogenous outgoing links are again zero. The same reserve choices apply when links are fixed. This proves (iii). □

Lemma 11. *Suppose $0 < nr^x < \gamma$. In the original subsidized game, every pure Nash equilibrium at zero subsidy yields a strictly higher value of the public authority's objective, $W(s)$, than every pure Nash equilibrium at any subsidy $s \geq r^x$. The result holds both when*

the network is endogenous, without requiring the linking-cost conditions in (9)*, and also when any given network is held fixed across subsidies.*

Proof. We first obtain a lower bound on $W(0)$ that holds at every pure equilibrium. Since $0 < nr^x < \gamma$ and $w_i \geq \zeta$, each organization can choose reserves $\zeta(1 - nr^x/\gamma) \in (0, \zeta)$ while keeping its outgoing links unchanged. This deviation is feasible whether the network is endogenous or fixed. By (10), access to others weakly reduces the expected emergency loss relative to having no access at the same reserve level. Equilibrium optimality therefore implies

$$\hat{u}_i(0) + \tilde{\kappa} \sum_j g_{i,j} \geq r^x(w_i - \zeta(1 - nr^x/\gamma)) - \frac{\gamma}{2n\zeta}(\zeta - \zeta(1 - nr^x/\gamma))^2.$$

At zero subsidy, $W(0)$ and the sum of organizations' original payoffs contain the same ordinary activity returns and expected shortage losses. The original payoffs additionally deduct relationship and expected access costs, which the authority excludes. Relationship costs have already been added back on the left-hand side of the preceding inequality. Summing over organizations and adding back the remaining nonnegative expected access costs therefore gives

$$W(0) \geq r^x \sum_i w_i - \zeta n r^x + \frac{\zeta(nr^x)^2}{2\gamma}. \tag{29}$$

We next obtain an upper bound on the authority's objective at subsidies $s \geq r^x$. The exact best responses established above imply $z_i \in [\zeta, w_i]$ at $s = r^x$ and $z_i = w_i$ at $s > r^x$, whether the network is endogenous or fixed. Thus every organization holds at least ζ, eliminating shortages because emergency needs are bounded above by ζ. Since $\sum_i z_i \geq n\zeta$ and $r^x + s \geq 2r^x$,

$$W(s) = r^x \sum_i w_i - (r^x + s) \sum_i z_i \leq r^x \sum_i w_i - 2\zeta n r^x.$$

The lower bound in (29) exceeds this upper bound by

$$\zeta n r^x + \frac{\zeta(nr^x)^2}{2\gamma} > 0.$$

Hence every pure equilibrium at zero subsidy yields a strictly higher value of the authority's objective than every pure equilibrium at any $s \geq r^x$. The argument applies both to endogenous networks and to any network held fixed across subsidies, without using (9) or imposing an equilibrium selection. □

Throughout the policy analysis, maintain $c < nr^x < \gamma$ and the linking-cost conditions (9). Define the threshold expressions for every s by

$$\kappa_0(s) = \frac{\zeta^2\psi(\gamma - c)n(r^x - s)[\gamma - n(r^x - s)]}{\gamma^2}, \tag{30}$$

$$\kappa_1(s) = \frac{\zeta^2\psi(\gamma - c)[\gamma - n(r^x - s)]\big[\gamma n(r^x - s) - \psi c[\gamma - n(r^x - s)]\big]}{\gamma^3}. \tag{31}$$

Both expressions are strictly concave in s. We define these for all s to state the linking-cost conditions. This does not imply, however, that a weighted-star equilibrium exists at every subsidy. The linking-cost conditions (9) require $\kappa > \kappa_1(s)$ for all $s \geq 0$ and $\kappa > \kappa_0(s)$ for all $s \geq r^x - c/n$.

Lemma 12. *Under the maintained assumptions and the linking-cost conditions* (9)*, for every* $0 \leq s < r^x$*:*

(i) No pure Nash equilibrium contains a full-weight link.

(ii) If $\kappa \geq \kappa_0(s)$*, the empty configuration, with* $z_i = \zeta[\gamma - n(r^x - s)]/\gamma$ *for every* i*, is the unique pure Nash equilibrium and it is strict.*

(iii) If $\kappa < \kappa_0(s)$*, necessarily* $s < r^x - c/n$*, and the weighted star of Lemma 8 is the unique strict Nash equilibrium up to relabeling.*

Proof. Fix $0 \leq s < r^x$. By Lemma 10, we may consider the auxiliary game. Let $\alpha = \zeta[\gamma - n(r^x - s)]/\gamma$. Since $0 < n(r^x - s) < \gamma$, we have $0 < \alpha < \zeta$. The reserve best response $z_i = \max\{0, \alpha - \lambda y_i/\gamma\}$, together with $y_i \geq 0$ and $\zeta \leq w_i$, therefore implies $0 \leq z_i \leq \alpha < \zeta \leq w_i$ at every equilibrium. Write v for the value function (14) with β replaced by $\gamma - n(r^x - s)$. Suppose a Nash equilibrium contains a full-weight link $g_{i,j} = 1$. Holding the other organizations' choices fixed, organization i can reduce this weight and adjust its reserves. Such a deviation cannot increase its payoff at equilibrium. Therefore, $v'(y_i)\psi z_j - \kappa \geq 0$. Moreover, $y_i \geq \psi z_j$ and v' is decreasing, so $v'(y_i) \leq v'(\psi z_j)$. Since $\psi z_j \geq 0$, it follows that

$$\kappa \leq v'(y_i)\psi z_j \leq v'(\psi z_j)\psi z_j. \tag{32}$$

This requires $z_j > 0$. At accessible capacity ψz_j, the reserve optimum is positive:

$$\alpha - \frac{\lambda\psi}{\gamma} z_j \geq \alpha\left(1 - \frac{\lambda\psi}{\gamma}\right) > 0.$$

Since $z_j \leq \alpha$ and $\lambda\psi < \gamma$, the auxiliary reserve best response at accessible capacity ψz_j is at least $\alpha(1 - \lambda\psi/\gamma) > 0$. We can therefore compute $v'(\psi z_j)$ using the formula derived for positive reserve holdings.

We now show that $v'(\psi z_j)\psi z_j < \kappa$, contradicting (32). The maintained assumptions and $z_j \leq \alpha$ imply

$$\begin{aligned} 0 < \frac{\gamma - nr^x}{\gamma} &< 1 - \frac{c}{\gamma}, \\ \frac{\gamma - nr^x}{\gamma} \leq \frac{\gamma - n(r^x - s)}{\gamma} &< 1, \\ 0 \leq \frac{z_j}{\zeta} \leq \frac{\gamma - n(r^x - s)}{\gamma}&, \end{aligned}$$

and

$$\frac{v'(\psi z_j)\psi z_j}{\zeta^2 \lambda\psi} = \frac{z_j}{\zeta}\left(1 - \frac{\gamma - n(r^x - s)}{\gamma} - \frac{c}{\gamma}\psi\frac{z_j}{\zeta}\right).$$

The two parts of (9) imply

$$\begin{aligned} \frac{\kappa}{\zeta^2\lambda\psi} &> q\left[1 - \left(1 + \frac{c}{\gamma}\psi\right)q\right] && \text{for every } q \geq \frac{\gamma - nr^x}{\gamma}, \\ \frac{\kappa}{\zeta^2\lambda\psi} &> q(1-q) && \text{for every } q \geq 1 - \frac{c}{\gamma}. \end{aligned}$$

If $\frac{z_j}{\zeta} \geq \frac{\gamma - nr^x}{\gamma}$, then $\frac{\gamma - n(r^x - s)}{\gamma} \geq \frac{z_j}{\zeta}$ gives

$$\begin{aligned} \frac{z_j}{\zeta}\left(1 - \frac{\gamma - n(r^x - s)}{\gamma} - \frac{c}{\gamma}\psi\frac{z_j}{\zeta}\right) &\leq \frac{z_j}{\zeta}\left[1 - \left(1 + \frac{c}{\gamma}\psi\right)\frac{z_j}{\zeta}\right] \\ &< \frac{\kappa}{\zeta^2\lambda\psi}. \end{aligned}$$

If $\frac{z_j}{\zeta} < \frac{\gamma - nr^x}{\gamma}$, the expression is at most

$$\frac{z_j}{\zeta}\left(1 - \frac{\gamma - nr^x}{\gamma} - \frac{c}{\gamma}\psi\frac{z_j}{\zeta}\right).$$

When $\frac{\gamma - nr^x}{\gamma} \leq \left(1 + 2\frac{c}{\gamma}\psi\right)^{-1}$, the preceding expression is nondecreasing in z_j/ζ on $\left[0, \frac{\gamma - nr^x}{\gamma}\right]$, so its value is at most

$$\frac{\gamma - nr^x}{\gamma}\left[1 - \left(1 + \frac{c}{\gamma}\psi\right)\frac{\gamma - nr^x}{\gamma}\right] < \frac{\kappa}{\zeta^2\lambda\psi}.$$

Otherwise, its unrestricted maximum gives

$$\begin{aligned}\frac{z_j}{\zeta}\left(1-\frac{\gamma-nr^x}{\gamma}-\frac{c}{\gamma}\psi\frac{z_j}{\zeta}\right) &\leq \frac{\left(1-\frac{\gamma-nr^x}{\gamma}\right)^2}{4\frac{c}{\gamma}\psi} \\ &< \frac{\frac{c}{\gamma}\psi}{\left(1+2\frac{c}{\gamma}\psi\right)^2} \\ &< \frac{c}{\gamma}\left(1-\frac{c}{\gamma}\right) < \frac{\kappa}{\zeta^2\lambda\psi}.\end{aligned}$$

The penultimate inequality follows from

$$1-\frac{c}{\gamma} > \frac{\gamma-nr^x}{\gamma} > \left(1+2\frac{c}{\gamma}\psi\right)^{-1}$$

and $\psi \leq 1$. The final inequality follows by substituting $q = 1-c/\gamma$ into $\kappa/(\zeta^2\lambda\psi) > q(1-q)$. Every case contradicts (32), proving (i).

For (ii), suppose $\kappa \geq \kappa_0(s)$ and fix opponents' capacity reserves bounded above by α. For any outgoing weights with total $L = \sum_j g_{i,j} > 0$, accessible capacity satisfies $y \leq \psi\alpha L$. Since $v'(0) > 0$ and $v'(0)\psi\alpha = \kappa_0(s)$, strict concavity gives, whenever $y > 0$,

$$v(y) - \kappa L < v(0) + v'(0)y - \kappa L \leq v(0).$$

If $y = 0$, positive relationship cost also makes no links strictly preferable. The unique best response is therefore no links and reserves α. Since every equilibrium reserve is bounded above by α, the empty configuration is the unique equilibrium and is strict, including at $\kappa = \kappa_0(s)$.

For (iii), $\kappa < \kappa_0(s)$ and the second part of (9) imply $s < r^x - c/n$, hence $0 < \gamma - n(r^x - s) < \lambda$. The first part gives $\kappa_1(s) < \kappa < \kappa_0(s)$, so Lemma 8, with r^x replaced by $r^x - s$, supplies a strict weighted-star equilibrium.

Part (i) excludes full-weight links, so Lemma 3 implies that each organization has at most one outgoing link in a strict equilibrium. The remaining argument in the proof of Proposition 4, using Proposition 2 and Lemma 4, then leaves only the empty network or a weighted star.

Since $\kappa < \kappa_0(s)$, the empty network is not an equilibrium. Lemma 8 uniquely determines the star's weight and reserve capacities, establishing uniqueness among strict Nash equilibria up to relabeling. □

Let $g_s(s)$ denote the common weight of peripheral links to the center when the selected equilibrium is a weighted star, and set $g_s(s) = 0$ when the network is empty. In the weighted-star case, $g_s(s)$ is given by (23), with r^x replaced by $r^x - s$ and β replaced by $\gamma - n(r^x - s)$.

If $\kappa < \max_{s \geq 0} \kappa_0(s)$, let $\underline{s} < \overline{s}$ be the two roots of $\kappa_0(s) = \kappa$. By Lemma 12, strict concavity of κ_0, and the linking-cost conditions (9), the weighted star is the unique strict Nash equilibrium up to relabeling for

$$s \in (\underline{s}, \overline{s}) \cap [0, \infty), \qquad 0 < \overline{s} < r^x - c/n.$$

Otherwise, the equilibrium network is empty at every subsidy. At each transition to the empty network, the star's link weight tends to zero and its reserve capacity holdings converge to those of the empty equilibrium. Hence $W(s)$ is continuous for $0 \leq s < r^x$.

The optimal subsidy: Proposition 8

For a star with fixed weight $g \in [0, 1]$ and $0 \leq s < r^x$, the following expression applies whenever the reserve equilibrium satisfies $z_i + y_i \leq \zeta$ for every organization. Substituting the reserve choices into (28) and suppressing the constant $r^x \sum_i w_i$ gives

$$\begin{aligned} W(s, g) = \frac{1}{2\gamma^3} \Big[& n\gamma^2 \{\zeta n (r^x - s)(r^x + 3s) - 2\zeta (r^x + s)\gamma\} \\ & + 2g\zeta (n-1)\gamma [\gamma - n(r^x - s)]\{(r^x + s)\gamma - 2cs\}\psi \\ & - \frac{c^2 g^2 \zeta (n-1)}{n} [\gamma - n(r^x - s)]^2 \psi^2 \Big]. \end{aligned} \tag{33}$$

This expression applies throughout the selected weighted-star interval, since Lemma 10 ensures $z_i + y_i < \zeta$ at every such equilibrium.

Proof of Proposition 8. Lemma 11 shows that every $s \geq r^x$ gives a strictly lower value than a zero subsidy, independently of equilibrium selection at those subsidies. It therefore suffices to consider $0 \leq s < r^x$.

Whenever the selected network is empty,

$$W_s(s, 0) = -\frac{\zeta n}{\gamma} [3ns + \gamma - nr^x] < 0, \tag{34}$$

where the inequality follows from $nr^x < \gamma$. If no weighted star is selected, $W(s)$ is therefore strictly decreasing on $[0, r^x)$, and the optimal subsidy is $s^* = 0$. Otherwise, the equilibrium network is empty for $\overline{s} \leq s < r^x$. Since $W(s)$ is strictly decreasing on this interval, $W(s) < W(\overline{s})$ for every $s \in (\overline{s}, r^x)$. Together with the exclusion of $s \geq r^x$, this reduces

the optimization problem to $[0, \overline{s}]$. Continuity of $W(s)$ on this compact interval establishes existence of an optimal subsidy.

If additionally $\kappa < \kappa_0(0)$, the selected network is a weighted star for every $s \in [0, \overline{s})$. Substituting $g_s(s)$ into (33) and differentiating gives, for $0 < s < \overline{s}$,

$$\begin{aligned}\frac{d^2}{ds^2}W\big(s, g_s(s)\big) = & -n\zeta\left[\frac{3}{\gamma} + \frac{2(n-1)}{c}\right] \\ & - \frac{2n(n-1)\gamma(2nr^x - \gamma)\kappa}{c\zeta\psi[\gamma - n(r^x - s)]^3} \\ & - \frac{3n(n-1)\gamma^3\kappa^2}{\zeta^3(\gamma - c)^2\psi^2[\gamma - n(r^x - s)]^4} < 0.\end{aligned} \tag{35}$$

when $nr^x > \gamma/2$. Strict concavity on $(0, \overline{s})$ and continuity at the endpoints imply strict concavity on $[0, \overline{s}]$, so the maximizer on this interval is unique. Every subsidy outside this interval has already been shown to yield a strictly lower value than some subsidy within it. The maximizer is therefore unique over all $s \geq 0$. □

Fixed weights and the subsidy comparison: Proposition 9

Proof of Proposition 9. For fixed $g \in [0, 1]$ and $0 \leq s < r^x$, the reserve capacity best-responses in (7) give

$$z_C = \frac{\zeta[\gamma - n(r^x - s)]}{\gamma}, \qquad y_P = \frac{\psi g\zeta[\gamma - n(r^x - s)]}{\gamma}.$$

Peripheral organizations have enough own and accessible reserves to meet every possible shock once the subsidy reaches

$$s_H(g) = r^x - \frac{\gamma c\psi g}{n(\gamma + c\psi g)}, \tag{36}$$

where $s_H(0) = r^x$ and $s_H(g) > r^x - c/n > 0$. Equation (33) applies for $0 \leq s < s_H(g)$ and, when $g > 0$, also at $s = s_H(g)$. For $0 < s < s_H(g)$, differentiating twice with respect to s, holding g fixed, gives

$$W_{ss}(s, g) = -\frac{n\zeta}{\gamma}\left[3 + (n-1)\left\{2\left(1 - \frac{(\gamma - c)\psi g}{\gamma}\right) + \left(1 + \frac{c\psi g}{\gamma}\right)^2\right\}\right] < 0.$$

As the subsidy approaches $s_H(g)$ from below, expected unmet emergency needs among peripheral organizations tend to zero. Using $z_P \geq 0$ and $\partial z_P/\partial s > 0$ from below gives

$$W_s(s_H(g)^-, g) \leq -\zeta + \frac{n\zeta}{\gamma}[r^x - 3s_H(g)] < 0.$$

For $s_H(g) < s < r^x$, the peripheral reserve response is $z_P = \zeta[1 - n(r^x - s)/c]$. Peripheral organizations can meet every possible shock, while the center still faces a positive probability of unmet emergency needs. Differentiating (28) gives

$$W_s(s, g) = -z_C - (n-1)z_P - \frac{2n\zeta s}{\gamma} - (r^x + s)\frac{n\zeta(n-1)}{c} < 0.$$

For $g > 0$, reserve holdings vary continuously through $s_H(g)$, so $W(s, g)$ is continuous at this threshold. Together with Lemma 11, these observations establish a unique $s_F(g) \in [0, s_H(g))$ for every $g \in [0, 1]$. In particular, (34) implies $s_F(0) = 0$.

Since $s_H(g) > r^x - c/n$ for every $g \in [0, 1]$, (33) applies throughout $0 \leq s \leq r^x - c/n$, $0 \leq g \leq 1$. Differentiation yields

$$\begin{aligned} W_g(s, g) = \frac{\zeta(n-1)\psi[\gamma - n(r^x - s)]}{n\gamma^3}\Big[&n\gamma\{(r^x + s)\gamma - 2cs\} \\ &- c^2 g\psi[\gamma - n(r^x - s)]\Big] > 0, \end{aligned} \tag{37}$$

$$\begin{aligned} W_{sg}(s, g) = \frac{\zeta(n-1)\psi}{\gamma^3}\Big[&\gamma\{2c(nr^x - 2ns - \gamma) + \gamma(2ns + \gamma)\} \\ &- 2c^2 g\psi[\gamma - n(r^x - s)]\Big] > 0. \end{aligned} \tag{38}$$

The expressions inside square brackets decrease with $g\psi$ and have constant slopes in s. Their minima over $0 \leq s \leq r^x - c/n$ therefore occur at an endpoint. For (37) and (38), setting $g\psi = 1$ and checking $s = 0, r^x - c/n$ gives the respective strict lower bounds $c(\gamma^2 - c\gamma + c^2) > 0$ and $\gamma^3 - 2\gamma^2 c + 2c^3$. The latter is positive because

$$\gamma^3 - 2\gamma^2 c + 2c^3 = (\gamma - c)^3 + c\left(\gamma - \frac{3}{2}c\right)^2 + \frac{3}{4}c^3 > 0.$$

Proposition 8 gives a unique $s^* \in [0, \overline{s}]$, and its proof establishes strict concavity of $W(s, g_s(s))$ on this interval. If $s_F(g) \geq \overline{s}$, then $s^* \leq s_F(g)$ directly. Otherwise, g_s is differentiable at $s_F(g)$, using the right derivative when $s_F(g) = 0$.

For part (i), optimality at fixed g gives $W_s(s_F(g), g) \leq 0$, with equality if $s_F(g) > 0$. By (38) and $g_s(s_F(g)) \leq g$, $W_s(s_F(g), g_s(s_F(g))) \leq 0$. Equation (37) and $g_s'(s_F(g)) \leq 0$ then

imply that the total derivative of $W(s, g_s(s))$ at $s_F(g)$ is nonpositive. Strict concavity yields $s^* \leq s_F(g)$, including $s^* = 0$ when $s_F(g) = 0$.

For part (ii), $s_F(g) = 0$ gives the conclusion immediately. If $0 < s_F(g) < \overline{s}$, then $W_s(s_F(g), g) = 0$. The reverse weight inequality and $g'_s(s_F(g)) \geq 0$, together with (37) and (38), make the total derivative nonnegative. Strict concavity therefore gives $s^* \geq s_F(g)$. □

The sufficient condition discussed after Proposition 9 follows by differentiating the equilibrium link weight:

$$\operatorname{sgn} g'_s(s) = \operatorname{sgn} \left\{ \frac{2\gamma^2\kappa}{\zeta^2(\gamma - c)} - \gamma\psi[\gamma - n(r^x - s)] \right\}.$$

The expression in braces is decreasing in s. Hence,

$$\kappa < \bar{\kappa} \equiv \frac{\zeta^2(\gamma - c)\psi(\gamma - nr^x)}{2\gamma}$$

ensures $g'_s(s) < 0$ throughout $0 \leq s < \overline{s}$, with the derivative at zero taken from the right. Moreover, $nr^x > \gamma/2$ implies $\bar{\kappa} < \kappa_0(0)$. The restriction $\kappa < \bar{\kappa}$ must hold together with the maintained linking-cost conditions (9).

The shortage-cost comparative static: Proposition 10

Proof. Consider an optimum satisfying $0 < s^*(\gamma) < \overline{s}$. The strict inequalities defining the weighted-star equilibrium hold in a neighborhood of this optimum. The explicit equilibrium formulas therefore make $W(s, \gamma)$ twice continuously differentiable in this region. Its derivatives incorporate the adjustment of both reserve holdings and link weights.

Holding the remaining primitives fixed, substitute the subsidized link weight from (23) into (33) and differentiate to obtain

$$\begin{aligned} W_s(s, \gamma) = & -n\zeta \left[1 - \frac{r^x - 3s}{\gamma} + \frac{2(n-1)s}{c} \right] \\ & + \frac{(n-1)\gamma(2nr^x - \gamma)\kappa}{c\zeta\psi[\gamma - n(r^x - s)]^2} \\ & + \frac{(n-1)\gamma^3\kappa^2}{\zeta^3\psi^2(\gamma - c)^2[\gamma - n(r^x - s)]^3}. \end{aligned} \tag{39}$$

Differentiating with respect to γ and collecting terms gives

$$\begin{aligned} W_{s\gamma}(s,\gamma) + \frac{W_s(s,\gamma)}{\gamma} = & -\frac{n\zeta[c+2(n-1)s]}{c\gamma} \\ & - \frac{(n-1)\kappa}{c\zeta\psi[\gamma - n(r^x - s)]^3} \\ & \times \Big\{\gamma[\gamma - n(r^x - s)] + 2n(r^x - s)(2nr^x - \gamma)\Big\} \\ & - \frac{(n-1)\gamma^2\kappa^2}{\zeta^3\psi^2(\gamma - c)^3[\gamma - n(r^x - s)]^4} \\ & \times \Big\{(\gamma + c)[\gamma - n(r^x - s)] + 3n(r^x - s)(\gamma - c)\Big\}. \end{aligned} \tag{40}$$

For $0 < s < \overline{s}$, we have $0 < n(r^x - s) < nr^x < \gamma$. Together with $2nr^x > \gamma$ and $\gamma > c$, this makes every term on the right-hand side strictly negative. At the optimum, $W_s(s^*(\gamma), \gamma) = 0$, so (40) implies $W_{s\gamma}(s^*(\gamma), \gamma) < 0$. Moreover, (35) gives $W_{ss}(s^*(\gamma), \gamma) < 0$. The implicit function theorem therefore yields

$$\frac{ds^*(\gamma)}{d\gamma} = -\frac{W_{s\gamma}(s^*(\gamma), \gamma)}{W_{ss}(s^*(\gamma), \gamma)} < 0.$$

Strict concavity ensures that the resulting local stationary point remains the unique optimum. □