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Abstract

This paper provides a game-theoretic model of network formation with a continuous effort choice. Efforts are strategic complements for direct neighbors in the network and display global substitution/congestion effects. We show that if the parameter governing local strategic complements is larger than the one governing global strategic substitutes, then all pairwise Nash equilibrium networks are nested split graphs. We also consider the problem of a planner, who can choose effort levels and place links according to a network cost function. Again all socially optimal configurations are such that the network is a nested split graph. However, the socially optimal network may be different from equilibrium networks and efficient effort levels do not coincide with Nash equilibrium effort levels. In the presence of strategic substitutes, Nash equilibrium effort levels may be too high or too low relative to efficient effort levels. The relevant applications are crime networks and R&D collaborations among firms, but also interbank lending and trade.

Key Words: Strategic network formation, local strategic complements, global strategic substitutes, congestion effects, positive externalities, negative externalities.
JEL Codes: D62, D85.

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1 Introduction

Many social and economic environments are characterized by local bilateral influences, inducing local strategic complementarities, and global competition/congestion effects, which generate global strategic substitutabilities. Frequent examples include the choice of criminal effort in a network of criminals and production choices of firms engaging in bilateral R&D agreements. Further applications are interbank lending and trade. We focus on the first two settings and briefly explain the mechanism leading to the type of strategic interaction considered.[1] In the case of crime networks, local strategic complementarities are due to a direct know-how transfer on how to commit a crime. That is, the criminal activities of a criminal's direct neighbors in the network translate into a lower probability of being caught and therefore increased incentives to commit crimes. Global strategic substitution effects stem from global competition/congestion effects for crime opportunities. For R&D collaborations, consider firms that compete in quantities in a common market. Production leads to learning-by-doing, while bilateral R&D agreements allow for a direct, cost reducing know-how transfer, which gives rise to local strategic complementarities. Global strategic substitution effects arise directly when the goods produced are substitutes and sold in a common market. More generally, introducing competition/congestion effect is relevant for many applications, as disregarding them often requires adopting strong assumptions. For example, in the case of crime networks, one would need to assume that there is no competition for crime opportunities, and for R&D networks that all firms operate in completely separate markets. Similarly, disregarding global substitution effects implies for interbank lending that consumers cannot substitute across loans, while in trade networks goods produced by different countries cannot be substitutable.[2]

Network structures are crucial in determining individual behavior and aggregate outcomes. It is therefore important to understand why certain network structures arise.[3] In this paper we endogenize the network in a simple game-theoretic setup, allowing for local strategic complements, as well as global strategic substitutes. In accordance with empirically observed networks, we obtain nested split graphs, which are a subset of core-periphery networks. The defining feature of nested split graphs is "nestedness" of neighborhoods in the following sense: agents with a higher number of links are connected to all agents to which an agent with fewer links is connected. Recently, these type of networks have drawn increased attention in the economics literature on networks.[4] The particular structure of crime networks depends on the type of criminal activity. Canter (2004), for example, finds that networks of hooligans are less structured than property crime and drug networks. However, the presence of a core group is described

[1]Formal derivations are provided for crime and R&D networks in the appendix. For a detailed discussion, including interbank lending and trade, see König et al. (2014).

[2]See also König et al. (2014).

[3]A related discussion is provided in Jackson et al. (2017).

[4]Goyal and Joshi (2003) is a very early paper that features nested split graphs (the authors call them interlinked stars). For a good discussion of nested split graphs, see König et al. (2014).

as the most recognized structural feature.[5] Nestedness and core-periphery structures have been frequently observed in R&D networks.[6]

In the following we provide a brief description of the model, together with a more detailed account of the main results. We propose a simple simultaneous move game, in which agents choose a non-negative, continuous effort level and announce with whom they wish to be linked. A bilateral link is created when the announcement is mutual. Gross payoffs are based on the linear quadratic payoff function first presented in Ballester et al. (2006). We assume that the parameter governing strategic complementarities is (weakly) larger than the one for global strategic substitutes. Note that this is assumption has empirical support for the main applications considered.[7] Links are assumed to be unweighted, undirected and to incur a linear cost. The equilibrium concept used is pairwise Nash equilibrium. Pairwise Nash equilibrium refines Nash equilibrium and allows for deviations in which agents simultaneously create a link (and best respond to each other's effort level). This rules out configurations in which pairs of agents are not connected, but both agents find it profitable to create a link among themselves. We find that all pairwise Nash equilibrium networks are nested split graphs and that a pairwise Nash equilibrium always exists. Finally, we analyze the problem of a planner, who can place links according to a network cost function. It is shown that all optimal networks are again nested split graphs. However, the optimal network may be different from equilibrium networks and efficient effort levels do not coincide with equilibrium effort levels.

Ballester et al. (2006) was the starting point for a rich body of theoretical and empirical research (see, for example, Calvó-Armengol et al., 2009, Ballester et al., 2010 and Helsley and Zenou, 2014). However, endogenizing the network proved to be challenging. Recent efforts have focused on models of network formation with stochastic elements and action choices in a dynamic setting with myopic agents (see, for example, König et al., 2014 and König et al., 2019). In these papers noise is introduced into the decision process and typically agents cannot revise their whole linking strategy (e.g., only create at most one one-sided link, often at zero cost, or only delete a single link) and/or the deletion of links is not strategic and occurs due to decay over time. It is then shown that the stochastically stable networks of the dynamic process are nested split graphs. One of the advantages of the aforementioned models is that they can be brought to the data. However, the stochastically stable states obtained need not be Nash equilibria and are therefore also not pairwise Nash equilibria (as illustrated in Example 1). In contrast we show in a simple game-theoretic model that if the parameter governing local complements is larger than the one governing global substitutes, then all pairwise Nash equilibria are nested split graphs.

Joshi and Mahmud (2016) assume the payoff function provided in Ballester et al. (2006) and present a two-stage game, in which agents create links in the first stage and play Nash equilibrium

[5]See also Dorn and South (1990), Dorn, Murji and South (1992), Ruggiero and South (1997) and Johnston (2000).

[6]See, for example, Tomasello et al. (2017), Kitsak et al. (2010), Rosenkopf and Schilling (2007).

[7]See König et al. (2019) for the case of R&D networks and, for example, Pattacchini and Zenou (2012) for peer effects in crime.

effort levels in the second stage. The authors show that with local complementarities and global substitutes all pairwise stable equilibria are nested split graphs. However, only limiting cases of parameters are considered and no existence results are provided for the network formation game. Hiller (2017) studies pairwise Nash equilibrium networks assuming a general class of payoff functions for which the linear-quadratic specification is a special case, but disregards the global substitution term.[8] Pairwise Nash equilibrium and socially optimal networks are shown to be nested split graphs. However, in the presence of substitution effects, socially optimal networks may display different type of structures within the class of nested split graphs. For linear network cost functions the bang-bang type solution of an empty or complete network, as presented in Hiller (2017), may not be socially optimal with global strategic substitutes (as illustrated in Example 3). Note that introducing congestion/competition effects significantly complicates the analysis. For the equilibrium characterization, we can follow Hiller (2017) for the outline of the argument, but each step requires a different proof. For the efficiency result we need to resort to a different proof strategy.

Belhaj et al. (2015) consider the linear quadratic payoff specification in Ballester et al. (2006), disregarding the global substitution term and solve the following two planner problems. In the first case the planner can place links according to a network cost function and agents play Nash equilibrium effort levels, given the network. In the second case, the planner can not only choose the network, but also agents' effort levels. The authors show that in both cases the optimal networks are nested split graphs. Note that in the absence of global substitution effects, the two problems are closely related and one can show that characterizing one also characterizes the respective other problem. This relationship breaks down when introducing global substitution effects. We show here that in the latter case, i.e. when the planner can choose both the network and effort levels, then again nested split graphs are socially optimal.[9] The former case, however, remains an open problem.

The paper is organized as follows. Section 2 provides the model description, while Section 3 shows that all pairwise Nash equilibria are nested split graphs and that a pairwise Nash equilibrium always exists. Section 4 solves the planner's problem. A formal derivation of the payoff function for crime and R&D networks is provided in the appendix and we also relate our payoff function in detail to Ballester et al. (2006). All proofs are relegated to the appendix.

[8]For further work on network formation games with simultaneous action and linking choices, see Bätz (2014), Galeotti and Goyal (2010) and Kinateder and Merlino (2017).

[9]König et al. (2018) show that if the cost function is linear, then the first best solution is a nested split graph. However, the proof relies on the linearity of what corresponds to the network cost function and therefore does not extend to the type of network design problem, as presented in Belhaj et al. (2015).

2 Model Description

Let $N = \{1, 2, ..., n\}$ be the set of players with $n \geq 3$. Each agent i chooses an effort level $x_i \in X$ and announces a set of agents to whom the agent wishes to be linked, which is represented by a row vector $\mathbf{g}_i = (g_{i,1}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n})$, with $g_{i,j} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. An entry $g_{i,j} = 1$ in $\mathbf{g}_i$ is interpreted as agent i announcing a link to agent j, while an entry $g_{i,j} = 0$ in $\mathbf{g}_i$ is taken to mean that agent i does not announce a link to agent j. Assume $X = [0, +\infty)$ and $\mathbf{g}_i \in G_i = \{0, 1\}^{n-1}$. The set of agent i's strategies is denoted by $S_i = X \times G_i$ and the set of strategies of all players by $S = S_1 \times S_2 \times ... \times S_n$. A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g}) \in S$ then specifies each player's individual effort level, $\mathbf{x} = (x_1, x_2, ..., x_n)$, and intended links, $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. A link between agents i and j, denoted by $\bar{g}_{i,j} = 1$, is created if and only if *both* agents i and j announce the link. That is, $\bar{g}_{i,j} = 1$ if and only if $g_{i,j} = g_{j,i} = 1$ (and $\bar{g}_{i,j} = 0$ otherwise) and therefore $\bar{g}_{i,j} = \bar{g}_{j,i}$. The undirected graph $\bar{\mathbf{g}}$ is defined as $\bar{\mathbf{g}} = \{\{i, j\} \in N : \bar{g}_{i,j} = 1\}$. That is, $\bar{\mathbf{g}}$ is a collection of links, which are listed as subsets of N of size 2. The presence of a link $\bar{g}_{i,j} = 1$ allows players to directly benefit from the effort level exerted by the respective other agent involved in the link. Denote the set of i's neighbors in $\bar{\mathbf{g}}$ with $N_i(\bar{\mathbf{g}}) = \{j \in N : \bar{g}_{i,j} = 1\}$ and the corresponding cardinality with $\eta_i(\bar{\mathbf{g}}) = |N_i(\bar{\mathbf{g}})|$.[10] The aggregate effort level of agent i's neighbors in $\bar{\mathbf{g}}$, which we sometimes call agent i's effort level "accessed", is written as $y_i(\bar{\mathbf{g}}) = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$. The aggregate effort level of all agents other than i is written as $z_i(\bar{\mathbf{g}}) = \sum_{j \in N \backslash \{i\}} x_j$. We write y_i for $y_i(\bar{\mathbf{g}})$ and z_i for $z_i(\bar{\mathbf{g}})$ when it is clear from the context. Given a network $\bar{\mathbf{g}}$, $\bar{\mathbf{g}} + \bar{g}_{i,j}$ and $\bar{\mathbf{g}} - \bar{g}_{i,j}$ have the following interpretation. When $\bar{g}_{i,j} = 0$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} + \bar{g}_{i,j}$ adds the link $\bar{g}_{i,j} = 1$, while if $\bar{g}_{i,j} = 1$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} + \bar{g}_{i,j} = \bar{\mathbf{g}}$. Similarly, if $\bar{g}_{i,j} = 1$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} - \bar{g}_{i,j}$ deletes the link $\bar{g}_{i,j}$, while if $\bar{g}_{i,j} = 0$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} - \bar{g}_{i,j} = \bar{\mathbf{g}}$. The network is called empty and denoted by $\bar{\mathbf{g}}^e$ if $\bar{g}_{i,j} = 0 \ \forall i, j \in N$, while it is called complete and denoted by $\bar{\mathbf{g}}^c$ if $\bar{g}_{i,j} = 1 \ \forall i, j \in N$ such that $i \neq j$.

Payoffs to player i under strategy profile $\mathbf{s} = (\mathbf{x},\mathbf{g})$ are given by

$$\Pi_i(\mathbf{s}) = \pi_i(\mathbf{x},\bar{\mathbf{g}}) - \eta_i(\bar{\mathbf{g}})\kappa,$$

where κ denotes linking cost with $\kappa > 0$. Gross payoffs, i.e. payoffs excluding linking cost, $\pi_i(\mathbf{x},\bar{\mathbf{g}})$, are given by the frequently employed linear-quadratic payoff function with local complementarities and global substitutes (Ballester et al., 2006). That is,

$$\pi_i(\mathbf{x},\bar{\mathbf{g}}) = \alpha x_i - \tfrac{1}{2}(\beta + \gamma)x_i^2 + \lambda x_i \textstyle\sum_{j \in N_i(\bar{\mathbf{g}})} x_j - \gamma x_i \sum_{j \in N \backslash \{i\}} x_j \ \forall i \in N.$$

Note that gross payoffs $\pi_i(\mathbf{x},\bar{\mathbf{g}})$ can be written as a function of own effort, x_i, the sum of effort levels of direct neighbors, $y_i(\bar{\mathbf{g}}) = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$, and the sum of effort levels of all agents different

[10] Note that agents are not linked to themselves and therefore not included in their own neighborhood.

from i, $z_i(\bar{\mathbf{g}}) = \sum_{j \in N \setminus \{i\}} x_j$. For ease of notation we sometimes write $\pi_i(x_i, y_i, z_i)$ and drop the subscripts when they are clear from the context.

We assume $\lambda > 0$ and $\lambda \geq \gamma \geq 0$. The first assumption guarantees local strategic complementarities and local positive externalities for changes in effort level accessed while keeping total effort levels constant, i.e. via the addition/deletion of links.[11] The second assumption ensures local strategic complementarities and positive externalities for changes in effort levels of connected agents.[12] Together these assumptions yield local strategic complementarities and positive externalities when a pair of agents creates a link and agents adjust their effort levels. Note also that if $\gamma > 0$, then an agent i's effort induces a negative externality on any agent that is not a direct neighbor of i in $\bar{\mathbf{g}}$. Moreover, effort levels are strategic substitutes for agents that are not direct neighbors in $\bar{\mathbf{g}}$. These assumptions are in accordance with our main applications.[13] Finally, to guarantee existence and uniqueness of a Nash equilibrium in effort levels for any fixed network $\bar{\mathbf{g}}$, we can resort to Ballester et al. (2006) and assume that $\beta > (n-1)\lambda$.

Below we present the best response and value function, which are useful for our equilibrium characterization.

Best response function. The unique best response of player i to the vector of effort levels $\mathbf{x}_{-i}$ in network $\bar{\mathbf{g}}$ is given by

$$\bar{x}_i(\mathbf{x}_{-i}, \bar{\mathbf{g}}) = \bar{x}_i(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) = \tfrac{1}{\beta+\gamma}\left(\alpha + \lambda \sum_{j \in N_i(\bar{\mathbf{g}})} x_j - \gamma \sum_{j \in N \setminus \{i\}} x_j\right).$$

Value function. The maximized gross payoff for $\mathbf{x}_{-i}$ in network $\bar{\mathbf{g}}$ is given by

$$\pi_i(\bar{x}_i, \mathbf{x}_{-i}, \bar{\mathbf{g}}) = v_i(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) = \tfrac{1}{2(\beta+\gamma)}\left(\alpha + \lambda \sum_{j \in N_i(\bar{\mathbf{g}})} x_j - \gamma \sum_{j \in N \setminus \{i\}} x_j\right)^2.$$

To simplify notation, we often write x_i, y_i, z_i, π_i, $\bar{x}_i$ and v_i, and drop the subscripts when it is clear from the context.

Next we define *pairwise Nash equilibrium* (PNE). When agents i and j deviate to create a link, then deviation effort levels are assumed to be mutual best responses (while the remaining agent's effort levels remain unchanged). The corresponding deviation effort levels are denoted by $x_i' = \bar{x}(y_i(\bar{\mathbf{g}}) + x_j', z_i(\bar{\mathbf{g}}) + x_j' - x_j)$. We sometimes use the notation $x_i'(\bar{\mathbf{g}} + \bar{g}_{i,j})$ to denote agent i's deviation effort level when creating a link with agent j.

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[11] Note that $\partial^2 \pi(x,y,z)/\partial x \partial y = \lambda > 0$ and $\partial \pi(x,y,z)/\partial y = \lambda x_i \geq 0$ holds $\forall x, y, z$.

[12] To see this, we can write $z_i = y_i + \sum_{j \notin \{N_i(\bar{\mathbf{g}}) \cup \{i\}\}} x_j$ and note that when an agent's neighbors change their effort levels, then not only y_i, but also z_i changes. The payoff function can then be written as $\pi_i(\mathbf{x}, \bar{\mathbf{g}}) = \alpha x_i - \frac{1}{2}(\beta + \gamma)x_i^2 + \lambda x_i y_i - \gamma x_i(y_i + \sum_{j \notin \{N_i(\bar{\mathbf{g}}) \cup \{i\}\}} x_j)$, so that $\partial^2 \pi(x,y,z)/\partial x \partial y = \lambda - \gamma \geq 0$ $\forall x, y, z$ and $\partial \pi(x,y,z)/\partial y = (\lambda - \gamma)x_i \geq 0$ $\forall x, y, z$.

[13] For example, for the case of R&D networks, König et al. (2019) provide estimates for λ and γ and show that $\lambda > \gamma > 0$. See Pattacchini and Zenou (2012) for peer effects in crime.

A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ is a pairwise Nash equilibrium *iff*

- for any $i \in N$ and every $\mathbf{s}_i \in S_i$, $\Pi_i(\mathbf{s}) \geq \Pi_i(\mathbf{s}_i, \mathbf{s}_{-i})$;
- for all $\bar{g}_{i,j} = 0$, if $\Pi_i(x'_i, x'_j, \mathbf{x}_{-i,-j}, \bar{\mathbf{g}} + \bar{g}_{i,j}) > \Pi_i(\mathbf{s})$,

  then $\Pi_j(x'_i, x'_j, \mathbf{x}_{-i,-j}, \bar{\mathbf{g}} + \bar{g}_{i,j}) < \Pi_j(\mathbf{s})$.

Note that a pairwise Nash equilibrium is both a Nash equilibrium and pairwise stable and therefore refines Nash equilibrium. Pairwise Nash equilibrium allows for deviations where a pair of agents creates a link (and deviating agents best respond to each other's effort level). Furthermore, pairwise Nash equilibrium allows for deviations in which an agent deletes any subset of existing links (and adjusts her effort level). However, deviations where a pair of agents creates a link and/or adjusts effort levels and *simultaneously* deletes any subset of existing links are not considered. We write $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ to denote a network $\bar{\mathbf{g}}$ and the corresponding vector of Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}})$. The configuration $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ is a pairwise Nash equilibrium if and only if the above conditions are satisfied for all agents/pairs of agents.

We sometimes write $\mathbf{g}'_i$ for an agent i's deviation linking strategy and denote the network after proposed deviation by $\bar{\mathbf{g}}'_i$. We drop the subscript when it is clear from the context. If $\bar{g}_{i,j} = 0$, then we write $\bar{g}'_{i,j} = 1$ to denote that agents i and j create a link.

3 Network Formation

Note first that for a configuration $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ to be a pairwise Nash equilibrium, we need that agents play Nash equilibrium effort levels given the network $\bar{\mathbf{g}}$. To show existence of a unique Nash equilibrium in effort levels for any fixed network, we can turn to Theorem 1 in Ballester et al. (2006). We then show that Nash equilibrium effort levels must be equal for all players in a complete component and that singleton agents display same effort levels. Both of the latter statements follow directly from the best response functions.

Proposition 1: *For any fixed network, $\bar{\mathbf{g}}$, there exists a unique NE in effort levels and the unique NE is interior. Furthermore, (i) NE effort levels are equal for all agents in a complete component, (ii) NE effort levels are equal for all singleton agents.*

Before presenting Proposition 2, which shows that a pairwise Nash equilibrium always exists, we describe two cost thresholds, $\underline{\kappa}$ and $\overline{\kappa}$, which are formally defined in the appendix (Definition 1). The lower threshold, $\underline{\kappa}$, is given by the gross marginal payoff when a pair of agents creates a link in the empty network, $\bar{\mathbf{g}}^e$. The higher threshold, $\overline{\kappa}$, is defined as the average gross marginal payoff of linking to $n-1$ agents in the complete network, $\bar{\mathbf{g}}^c$. We denote the unique Nash equilibrium effort level in the complete network, $\bar{\mathbf{g}}^c$, by $x(\bar{\mathbf{g}}^c)$ and the unique Nash equilibrium effort level in the empty network, $\bar{\mathbf{g}}^e$, by $x(\bar{\mathbf{g}}^e)$.

Since the lower of the two bounds, $\underline{\kappa}$, is defined as the marginal payoffs of a pair of agents creating a link in the empty network, the empty network is a pairwise Nash equilibrium for linking cost $\kappa \geq \underline{\kappa}$. In turn, the higher of the two bounds, $\overline{\kappa}$, is given by the average gross marginal payoff of linking to $n-1$ agents in the complete network. Since the value function is convex in effort level accessed (for any fixed value of z_i) and all agents display the same Nash equilibrium effort levels, an agent in the complete network either finds it profitable to delete all links or none. Therefore, if $\kappa < \underline{\kappa}$, then no agent finds it profitable to delete any links and the complete network is a pairwise Nash equilibrium. One can then show that $\underline{\kappa} < \overline{\kappa}$ always holds for $\lambda \geq \gamma$, which guarantees that a pairwise Nash equilibrium always exists, as summarized in the statement of Proposition 2.

Proposition 2: *$\underline{\kappa} < \overline{\kappa}$ holds. Furthermore, (i) if $\kappa < \underline{\kappa}$ then $(\mathbf{x}(\bar{\mathbf{g}}^c), \bar{\mathbf{g}}^c)$ is a PNE, (ii) if $\kappa > \overline{\kappa}$ then $(\mathbf{x}(\bar{\mathbf{g}}^e), \bar{\mathbf{g}}^e)$ is a PNE and (iii) if $\kappa \in [\underline{\kappa}, \overline{\kappa}]$ then $(\mathbf{x}(\bar{\mathbf{g}}^c), \bar{\mathbf{g}}^c)$ and $(\mathbf{x}(\bar{\mathbf{g}}^e), \bar{\mathbf{g}}^e)$ are PNE.*

We formally define nested split graphs below, which are a strict subset of core-periphery networks.[14,15] Note that the star, the complete and the empty network are nested split graphs.

Definition 2: A network $\bar{\mathbf{g}}$ is a *nested split graph* if and only if

$$[\bar{g}_{i,l} = 1 \text{ and } \eta_k(\bar{\mathbf{g}}) \geq \eta_l(\bar{\mathbf{g}})] \Rightarrow \bar{g}_{i,k} = 1.$$

Below we present our first main result, namely that all pairwise Nash equilibrium networks are nested split graphs. Moreover, the ranking of effort levels, number of links and gross payoffs coincides.

Theorem 1: *In any PNE, $(\mathbf{x}, \bar{\mathbf{g}})$, the network $\bar{\mathbf{g}}$ is a nested split graph such that $x_i < x_k \Leftrightarrow \eta_i(\bar{\mathbf{g}}) < \eta_k(\bar{\mathbf{g}}) \Leftrightarrow v(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) < v(y_k(\bar{\mathbf{g}}), z_k(\bar{\mathbf{g}}))$ holds.*

To build intuition for Theorem 1, it is instructive to first consider the case when $\gamma = 0$. The value function can then be thought of as independent of z_i and strictly convex in y_i. Due to the strict convexity of the value function, agents that access higher effort levels benefit more from linking to any particular agent. Conversely, agents gain relatively more from linking to agents with higher effort levels. Moreover, due to strategic complementarities, agents that access higher effort levels also exert higher effort levels. Then, in any pairwise Nash equilibrium, agents with higher effort levels (and therefore higher effort levels accessed) must be linked to all agents to which agents with lower effort levels (and therefore lower effort levels accessed) are connected. Neighborhoods are nested and agents with a higher number of links are connected to all agents to which agents with fewer links are connected. In other words, if an agent is connected to some

[14] A network $\bar{\mathbf{g}}$ is a core-periphery network if the set of agents N can be partitioned into two sets, $C(\bar{\mathbf{g}})$ (the core) and $P(\bar{\mathbf{g}})$ (the periphery), such that $\bar{g}_{i,j} = 1\ \forall i, j \in C(\bar{\mathbf{g}})$ and $\bar{g}_{i,j} = 0\ \forall i, j \in P(\bar{\mathbf{g}})$.

[15] For a formal proof that all nested split graphs are core-periphery networks see Chvátal and Hammer (1977).

agent with a given number of links, then the former agent is connected to any agent with a (weakly) higher number of links than the latter agent. This corresponds to the definition of a nested split graph. As the network is nested in any pairwise Nash equilibrium, a higher number of links also implies a higher effort level accessed and, since the value function is increasing in effort levels accessed, higher gross payoffs.

The case when $\gamma > 0$ is more involved. Note that $\gamma > 0$ introduces a negative externality in the sum of the respective other agents' effort levels. This implies a difference in payoffs whether an increase in effort levels accessed stems from the creation of a new link or an increase in (existing) neighbors' effort levels. Moreover, an agent's value function is strictly convex in the sum of direct neighbors' effort level, y_i, for any *fixed* aggregate effort level of the remaining agents, z_i. But, due to effort level adjustments in a link creation, the aggregate effort level of remaining agents changes when creating new link. One can show that agents always increase their effort levels when creating a new link and, since $\lambda \geq \gamma$, agents are no worse off when direct neighbors increase their effort levels. This allows us to establish that, if an agent does not find it profitable to delete a link with an agent exerting a given effort level, then it is profitable to create a link to any agent with a weakly higher effort level. Additionally, the aggregate effort levels of all remaining agents, z_i, is lower the higher own effort level and, since the cross derivative of the value function with respect to y_i and z_i is negative, agents with a higher effort level gain relatively more from creating a new link. From the first order conditions one can then show that agents displaying a higher effort level also access higher effort levels. That is, agents with higher effort levels (and therefore higher effort levels accessed) are linked to all agents to which agents with lower effort levels (and therefore lower effort levels accessed) are linked to and neighborhoods are nested. Again reinforcing incentives to create and sustain links yield nested split graphs as the only pairwise Nash equilibrium networks.[16]

Next we provide an example that highlights differences of our results relative to the literature using stochastic stability. König et al. (2018) employ the same gross payoff function as the one presented here, but in a dynamic setting with myopic agents. Agents receive opportunities at predetermined rates to either 1) adjust their effort level, 2) create a single link or 3) delete a single link.[17] Link formation is one-sided, but both agents involved in the link incur a linking cost. The latter assumption is justified by the fact that, since agents cannot revise their effort levels when creating a single link, the marginal benefit of i creating a link to j is always the same for both agents, i.e. $\lambda x_i x_j$. Note that this is not the case in our model, because agents can adjust their effort levels when creating links. The characterization of equilibrium in König et al. (2018) is then derived in the form of a Gibbs measure when the level of noise approaches zero. It is shown that in any stochastically stable state a link between a pair of agents i and j

[16] Conditions for the existence of a pairwise Nash equilibrium other than the complete or the empty network (for example a star network) can easily be obtained and are available from the author upon request.

[17] The underlying rationale is that there is some inertia in changing links or output levels, similar to Calvo pricing models with price stickiness. The advantage of this assumption is that it allows the authors to derive a likelihood function that can be conveniently estimated with real world data.

is present if and only if $\lambda x_i x_j > \kappa$. The example below shows that a configuration that is in the support of the stochastically stable states need not be a (pairwise) Nash equilibrium of the the static game. The reason is that under Nash equilibrium we allow for deviations in which agents adjust their effort levels when deleting a link and agents may delete any subset of links.

Example 1: Assume $n = 8$, $\alpha = 271/32$, $\beta = 8$, $\lambda = 1$, $\gamma = 1/2$, $\kappa = 1$ and $\bar{\mathbf{g}}$ is a star network. Denote by $x_c(\bar{\mathbf{g}})$ the Nash equilibrium effort level of the agent in the center and by $x_p(\bar{\mathbf{g}})$ the Nash equilibrium effort level of agents in the periphery. We round numerical values to the second decimal. Then $x_c(\bar{\mathbf{g}}) = 1.32$, $x_p(\bar{\mathbf{g}}) = 0.79$ and therefore $\lambda x_c x_p = 1.05 > \kappa$, while $\lambda x_c x_p = 0.63 < \kappa$. The above configuration is therefore part of the stochastically stable states, as shown in König et al. (2018). However, when accounting for adjustments in effort levels, the marginal payoff for an agent in the periphery from its link to the center is 0.95, while the average payoff for the central agent from her links to all peripheral players is 0.79. That is, all players have an incentive to delete all their links and the star network is not a Nash equilibrium (and therefore also not a pairwise Nash equilibrium).

4 A Planner's Problem

In this section we present the problem of a planner, who chooses effort levels and can place links at a cost. The total cost of the network (*the network cost*) is given by $\Phi(\eta(\bar{\mathbf{g}}))$, where $\eta(\bar{\mathbf{g}})$ is the total number of links in network $\bar{\mathbf{g}}$. Assume that $\Phi(0) = 0$ and $\Phi'(\cdot) > 0$. To simplify notation we sometimes write $\Phi(\bar{\mathbf{g}})$ for $\Phi(\eta(\bar{\mathbf{g}}))$ and η for $\eta(\bar{\mathbf{g}})$. The case of linear linking cost κ corresponds to a linear Φ and $\Phi(\eta) - \Phi(\eta - 1) = 2\kappa \ \forall \eta$. Note that then the solution of the planner's problem coincides with the first best solution of the corresponding network formation game. The planner aims to maximize total social welfare, which is defined as the sum of individual gross payoffs minus the network cost. For any strategy profile $\mathbf{s} = (\mathbf{x},\mathbf{g})$ and resulting network, $\bar{\mathbf{g}}$, social welfare is given by

$$W(\mathbf{x}, \mathbf{g}) = \textstyle\sum_{i \in N} \pi_i(\mathbf{x}, \bar{\mathbf{g}}) - \Phi(\bar{\mathbf{g}}).$$

A strategy profile $\hat{\mathbf{s}}$ is socially optimal if $W(\hat{\mathbf{s}}) \geq W(\mathbf{s}) \ \forall \mathbf{s} \in S$. Denote the vector of efficient effort levels for a given network $\bar{\mathbf{g}}$ with $\hat{\mathbf{x}}(\bar{\mathbf{g}})$ and agent i's entry in vector $\hat{\mathbf{x}}(\bar{\mathbf{g}})$ by $\hat{x}_i(\bar{\mathbf{g}})$. That is, $\hat{\mathbf{x}}(\bar{\mathbf{g}})$ yields the highest sum of payoffs for a given network, $\bar{\mathbf{g}}$, so that $W(\hat{\mathbf{x}}(\bar{\mathbf{g}}), \bar{\mathbf{g}}) \geq W(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ $\forall \mathbf{x} \in \mathbb{R}^n_+$. To simplify notation, we sometimes write $\hat{\mathbf{x}}$ for $\hat{\mathbf{x}}(\bar{\mathbf{g}})$ and $\hat{x}_i$ for $\hat{x}_i(\bar{\mathbf{g}})$. Denote the socially optimal network by $\hat{\mathbf{g}}$, so that $W(\hat{\mathbf{x}}(\hat{\mathbf{g}}), \hat{\mathbf{g}}) \geq W(\hat{\mathbf{x}}(\bar{\mathbf{g}}), \bar{\mathbf{g}}) \ \forall \bar{\mathbf{g}} \in G$. For a vector of parameters $\theta = (\alpha, \beta, \lambda, \gamma)$, we assume that $2\lambda/(\beta - \gamma) < 1/(n-1)$ and $\beta > \gamma$, which ensures that the problem is well defined. To see this, define $\tilde{\theta}(\theta) = (\tilde{\alpha}(\theta), \tilde{\beta}(\theta), \tilde{\lambda}(\theta), \tilde{\gamma}(\theta))$ with $\tilde{\alpha}(\theta) = \alpha$, $\tilde{\beta}(\theta) = \beta - \gamma$, $\tilde{\lambda}(\theta) = 2\lambda$, $\tilde{\gamma}(\theta) = 2\gamma$. The assumption $\beta > \gamma$ guarantees that $\tilde{\beta} > 0$. Note also that we allow for $\gamma > \lambda$. One can then show via the first order conditions that the vector of efficient effort levels, $\hat{\mathbf{x}}(\bar{\mathbf{g}},\theta)$, for parameter vector θ is equal to the vector of Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}},\tilde{\theta})$, for parameter vector $\tilde{\theta}$. Since then $\tilde{\alpha}/\tilde{\beta} < 1/(n-1)$ holds, we know from Ballester et al. (2006) that $\mathbf{x}(\bar{\mathbf{g}},\tilde{\theta})$ exists and is unique. Therefore, $\hat{\mathbf{x}}(\bar{\mathbf{g}},\theta)$ exists and is unique. For each network $\bar{\mathbf{g}}$ we can then calculate a finite maximum value associated with it, given by $W(\hat{\mathbf{x}}(\bar{\mathbf{g}}),\bar{\mathbf{g}})$. Since there is a finite number of networks for a given number of agents, the optimization problem is well defined.

Note, however, that Nash equilibrium effort levels are generically not the same for different parameter values, i.e. $\mathbf{x}(\bar{\mathbf{g}},\tilde{\theta}) \neq \mathbf{x}(\bar{\mathbf{g}},\theta)$. Therefore, Nash equilibrium effort levels and the socially optimal effort levels typically do not coincide for a given vector of parameters θ. Likewise, there are pairwise Nash equilibrium networks that are different from the socially optimal network. This will, for example, be the case when there is multiplicity of equilibria. Moreover, it is easy to construct examples such that the socially optimal network is not sustainable as a pairwise Nash equilibrium.

Next we define a neighborhood switch, as first presented in Belhaj et al. (2016), which will be useful for showing our result. Take agents j and k and let $N_{j\backslash k}(\bar{\mathbf{g}})$ be the set of agent that are neighbors of j but not of k. More formally, define $N_{j\backslash k}(\bar{\mathbf{g}})$ as $N_{j\backslash k}(\bar{\mathbf{g}}) = \{i \in N : \bar{g}_{i,j} = 1 \textit{ and } \bar{g}_{i,j} = 0\}$. A neighborhood switch from j to k, written as a $N_{(j,k)}$-switch, is a reallocation of all links between j and agents in $N_{j\backslash k}(\bar{\mathbf{g}})$ to links between k and agents in $N_{j\backslash k}(\bar{\mathbf{g}})$.

Definition 3 (Belhaj et al., 2016): ($N_{j\backslash k}$-switch). Consider a network $\bar{\mathbf{g}}$. A $N_{j\backslash k}$-switch is a reallocation of links leading to the network $\bar{\mathbf{g}}'$ where $\bar{\mathbf{g}}' = \bar{\mathbf{g}} + \sum_{l\in N_{j\backslash k}} \bar{g}_{k,l} - \sum_{l\in N_{j\backslash k}} \bar{g}_{j,l}$.

We are now in the position to present our second main result, which shows that any socially optimal configuration must be such that the network is a nested split graph.

Theorem 2: *In any optimal strategy profile,* $\hat{\mathbf{s}}$, $\hat{\mathbf{g}}$ *is a nested split graph. Moreover,* $\hat{x}_i(\hat{\mathbf{g}}) > \hat{x}_j(\hat{\mathbf{g}}) \Leftrightarrow \hat{y}_i(\hat{\mathbf{g}}) > \hat{y}_j(\hat{\mathbf{g}}) \Leftrightarrow \hat{\eta}_i(\hat{\mathbf{g}}) > \hat{\eta}_j(\hat{\mathbf{g}})$ *holds.*

In the following we outline the main arguments used. Belhaj et al. (2016) show for a payoff function which corresponds to the case when $\alpha = 1$, $\beta = 1$, $\lambda = \delta$ and $\gamma = 0$, that the sum of Nash equilibrium effort levels strictly increases after a $N_{j\backslash k}$-switch from an agent j with lower Bonacich centrality to an agent k with higher Bonacich centrality. Since for $\gamma = 0$ the sum of Nash equilibrium effort levels are proportional to the sum of Bonacich centralities, the sum of Bonacich centralities also increases strictly after a $N_{j\backslash k}$-switch. Note next that from Ballester et al. (2006) we know that the sum of Nash equilibrium effort levels is strictly increasing in the sum of Bonacich centralities. That is, we know that after a $N_{j\backslash k}$-switch the sum of Nash equilibrium effort levels also increases strictly when $\gamma > 0$ (where we set $\delta = \lambda/\beta$). One can then show via the first order conditions that for a given network $\bar{\mathbf{g}}$, the socially optimal payoffs are proportional to the sum of the socially optimal effort levels. Since these are equal to the corresponding Nash equilibrium effort levels for $\tilde{\theta}$, we know that a $N_{j\setminus k}$-switch also strictly increases socially optimal payoffs when $\gamma > 0$. Finally, Belhaj et al. (2016) show that the only networks where no $N_{j\setminus k}$-switch from an agent with lower Bonacich centrality to an agent with higher Bonacich centrality is possible are nested split graphs. Therefore, all socially optimal networks $\hat{\mathbf{g}}$ are nested split graphs. The second part of the statement in Theorem 2, regarding effort levels, effort levels accessed and agents' degrees follows directly from the fact that the efficient effort levels for θ coincide with the Nash equilibrium effort levels for $\tilde{\theta}$. Finally, note that since our proof involves rewiring of links, the network cost remains unchanged after rewiring, irrespective of the particular shape of the network cost function Φ. In other words, for any optimal number of links to be placed by the planner, the socially optimal network is a nested split graph.

Next we present an example with a network cost of 2κ for each link. The complete and the star networks are both pairwise Nash equilibrium networks, while the complete network is the socially optimal network. Note further that the center's pairwise Nash equilibrium effort level is too low, while periphery agents' pairwise Nash equilibrium effort levels are too high, when comparing with the efficient effort levels for a star. This is different from the case when disregarding congestion/substitution effects, where pairwise Nash equilibrium effort levels are always lower than what would be socially optimal. Finally, we provide a network cost function, such that the star network is optimal.[18]

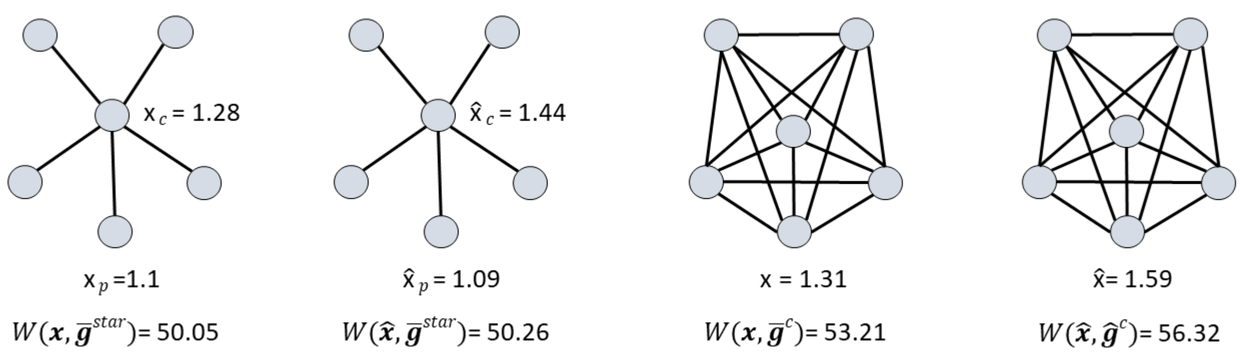

Figure 4.1: Example 2

Example 2: Assume $n = 6$, $\alpha = 17$, $\beta = 15$, $\lambda = 17/26$, $\gamma = 1/5$ and $\kappa = 5/6$. Assume that placing a link incurs a cost of 2κ. Then the socially optimal network is the complete network, while both the complete and the star network are pairwise Nash equilibria. The subscript c denotes the effort level of an agent in the center of a star, while the subscript p denotes the effort level of a peripheral agent. Effort levels and the total payoffs are rounded to the second decimal. For the star network we find that the Nash equilibrium effort level of the center of the star is lower than what would be efficient, i.e. $x_c(\bar{\mathbf{g}}^{star}) = 1.28 < \hat{x}_c(\bar{\mathbf{g}}^{star}) = 1.44$, while the converse

[18] Note that if Φ is strictly convex, then a sufficient condition for the socially optimal network $\hat{\mathbf{g}}$ to be different from the empty and complete network is given by $\Phi(1) \simeq 0$ and $\Phi(\frac{n(n-1)}{2}) > \sum_{i\in N} \pi_i(\hat{x}(\bar{\mathbf{g}}^c), \bar{\mathbf{g}}^c)$.

is true for periphery players, i.e. $x_p(\bar{\mathbf{g}}^{star}) = 1.1 > \hat{x}_p(\bar{\mathbf{g}}^{star}) = 1.09$. Note that for network cost function $\Phi(\eta) = (1/5)\eta^2$ the star socially optimal.

Next we highlight a further difference to the case when no congestion effects are present. Recall that if $\gamma = 0$, then for a linear network cost function the socially optimal network is either empty or complete (see Hiller, 2017). With $\gamma > 0$, however, even with a linear network cost function, networks that are neither complete nor empty may be socially optimal. We provide an example such that in the efficient configuration the network is a star.

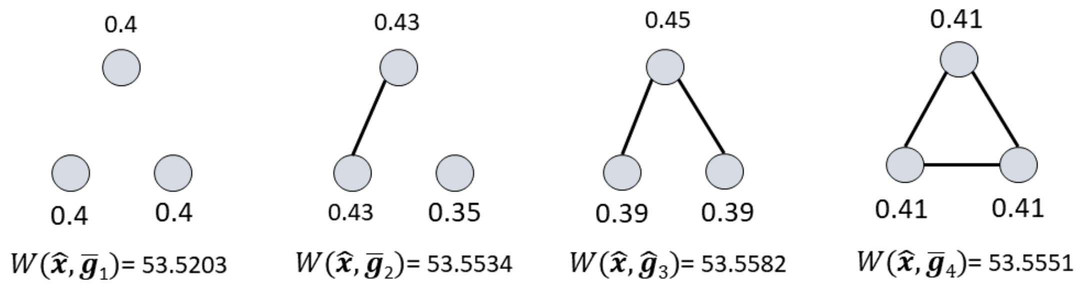

Figure 4.2: Example 3

Example 3: Assume $n = 3$, $\alpha = 89$, $\beta = 48$, $\lambda = 49/39$, $\gamma = 174/5$, $\kappa = 847/4218$. Assume that placing a link incurs a cost of 2κ. Then the socially optimal network is a star network and the one link network yields higher total payoffs than the empty and the complete network. Effort levels are stated below and are rounded to the second decimal, total payoffs to the fourth decimal.

5 Conclusion

This paper presents a simple model of network formation in the presence of local strategic complements and global strategic substitutes. We show that if, in accordance with empirical estimates, the parameter governing local strategic complements is larger than the one governing global strategic substitutes, then any pairwise Nash equilibrium displays a nested split graph. We then solve a planner's problem, in which the planner may choose effort levels and place links according to a network cost function. We show that the socially optimal networks are again nested split graph. However, socially optimal networks may differ from pairwise Nash equilibrium networks and Nash equilibrium effort levels do not coincide with socially optimal effort levels. The relevant applications are crime and R&D networks.

6 Appendix

The relationship with Ballester et al. (2006)

Ballester et al. (2006) start by assuming the utility function below

$$u_i(x_1, ..., x_n) = \alpha x_i + \tfrac{1}{2}\sigma x_i^2 + \textstyle\sum_{j \neq i} \sigma_{ij} x_i x_j,$$

with $\alpha > 0$ and $\sigma < 0$. By defining $\underline{\sigma} = min\{\sigma_{ij} \mid i \neq j\}$ and $\overline{\sigma} = max\{\sigma_{ij} \mid i \neq j\}$, the utility functions is rewritten as follows

$$u_i(\mathbf{x}) = \alpha x_i - \tfrac{1}{2}(\beta - \gamma)x_i^2 - \gamma x_i \textstyle\sum_{j \in N} x_j + \lambda \sum_{j \in N} g_{ij} x_i x_j,$$

where $\gamma = -min\{\underline{\sigma}, 0\} \geq 0$, $\lambda = \overline{\sigma} + \gamma \geq 0$. The authors assume $\lambda > 0$ (and thereby rule out the case when $\underline{\sigma} = \overline{\sigma}$) and interpret $g_{ij} = (\sigma_{ij} + \gamma)/\lambda$ as a directed link from i to j in network $\mathbf{g}$. Note that $u_i(\mathbf{x})$ can then be rearranged and rewritten as

$$u_i(\mathbf{x}) = \alpha x_i - \tfrac{1}{2}(\beta + \gamma)x_i^2 - \gamma x_i \textstyle\sum_{j \in N \setminus \{i\}} x_j + \lambda \sum_{j \in N} g_{ij} x_i x_j.$$

When $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$, then adjacency matrix corresponding to $\mathbf{g}$ is a symmetric $(0, 1)$-matrix and $\mathbf{g}$ is undirected and unweighted. We further assume that $\lambda \geq \gamma$, which yields strategic complementarities in effort levels for connected agents. In the context of Ballester et al. (2006) this corresponds to $\overline{\sigma} \geq 0$. Finally, note that when $\overline{\sigma} = 0$, then $\lambda = \gamma > 0$.

Derivation of payoff function: Crime

Before defining pairwise Nash equilibrium, the above payoff function is derived, based on Jackson and Zenou (2014) in the context of crime. Assume that expected gains of crime to agent i are given by

$$\pi_i(\mathbf{x}, \overline{\mathbf{g}}) = b_i(\mathbf{x}) - p_i(\mathbf{x}, \overline{\mathbf{g}})f,$$

with

$$\begin{cases} b_i(\mathbf{x}) = \alpha' x_i - \tfrac{1}{2}(\beta - \gamma)x_i^2 - \gamma x_i \sum_{j \in N} x_j \\ p_i(\mathbf{x}, \overline{\mathbf{g}}) = p_0 x_i (A - \lambda' \sum_{j \in N_i(\overline{\mathbf{g}})} x_j). \end{cases}$$

Expected cost of criminal activity, $p_i(\mathbf{x}, \overline{\mathbf{g}})f$, increases in own criminal activity, x_i, since being involved in more criminal activities increases the chance of being caught. Local strategic complementarities stem from a decrease in the apprehension probability in direct neighbors' involvement in crime, due to a direct know-how transfer. Note that A is assumed to be sufficiently large, so that the apprehension probability is always positive for all criminals.[19] Finally, global strategic substitutes are due to congestion effects for crime opportunities, captured by $\gamma x_i \sum_{j \in N} x_j$ in the expression for $b_i(\mathbf{x})$.[20]

Direct substitution yields

$$\pi_i(\mathbf{x}, \overline{\mathbf{g}}) = (\alpha' - p_0 f A)x_i - \tfrac{1}{2}\beta x_i^2 + p_0 f \lambda' x_i \sum_{j \in N_i(\overline{\mathbf{g}})} x_j - \gamma x_i \sum_{j \in N \setminus \{i\}} x_j.$$

For $\alpha = \alpha' - p_0 f A > 0$ and $\lambda = p_0 f \lambda'$ these payoffs are equivalent to the specification used in Ballester et al. (2006).

Derivation of payoff function: Research and Development

We present here the arguably simplest derivation of the payoff specification in Ballester et al. (2006) in the context of R&D collaborations. For alternative derivations which include R&D efforts and explicitly model consumers and multiple markets see, for example, König (2016). Firms may enter into R&D collaborations, which cause knowledge spillovers due to learning-by-doing effects. Cost reduction depends on a firm's own production level and on the production level of collaborating firms. Given production level q_i, marginal cost of firm i, c_i, are given by

$$c_i = \overline{c} - \mu q_i - \lambda \sum_{j \in N_i(\overline{\mathbf{g}})} q_j$$

Firm i's profits are given by

$$\pi_i = p_i q_i - c_i q_i.$$

Assume inverse demand for the good q_i is given by $p_i = a - bq_i - \gamma \sum_{j \neq i} q_j$. Substituting into firm i's profits we obtain

$$\pi_i = (a_i - bq_i - \gamma \sum_{j \neq i} q_j)q_i - (\overline{c}_i - gq_i - \lambda \sum_{j \in N_i(\overline{\mathbf{g}})} q_j)q_i.$$

Collecting terms yields

$$\pi_i = (a - \overline{c})q_i - (b - g)q_i^2 + \lambda q_i \sum_{j \in N_i(\overline{\mathbf{g}})} q_j - \gamma q_i \sum_{j \neq i} q_j.$$

Setting a, $\overline{c}$, b and g such that $(a - \overline{c}) = \alpha$, $(b - g) = \beta + \gamma$ then yields the specification used in Ballester et al. (2006).

[19] See König, Liu and Zenou (2014) for how to calculate an appropriate lower bound on A.

[20] One way to argue for as to why congestion effects should affect agents with higher criminal activity more, as reflected in the term $\gamma x_i \sum_{j \in N} x_j$, is that when aggregate crime levels are higher, the public may become more vigilant, which in turn has a higher impact on agents with high individual levels of criminal activity.

Proposition 1: *For any fixed network, $\bar{\mathbf{g}}$, there exists a unique NE in effort levels and the unique NE is interior. Furthermore, (i) NE effort levels are equal for all agents in a complete component, (ii) NE effort levels are equal for all singleton agents.*

Proof. Ballester et al. (2006) applies to the payoff function considered and we can therefore rely on Theorem 1 in Ballester et al. (2006). More specifically, a NE exists, is unique and interior if $\beta > \lambda\mu_1(\bar{\mathbf{g}})$, where $\mu_1(\bar{\mathbf{g}})$ is the largest eigenvalue of the adjacency matrix of $\bar{\mathbf{g}}$. Note that the largest eigenvalue of the adjacency matrix of $\bar{\mathbf{g}}$ lies between the following bounds $max\{d_{avg}(\bar{\mathbf{g}}), \sqrt{d_{max}(\bar{\mathbf{g}})}\} \leq \mu_1(\bar{\mathbf{g}}) \leq d_{max}(\bar{\mathbf{g}})$, where $d_{max}(\bar{\mathbf{g}})$ is the maximum degree and $d_{avg}(\bar{\mathbf{g}})$ the average degree in network $\bar{\mathbf{g}}$.[21] The largest eigenvalue of the adjacency matrix of $\bar{\mathbf{g}}$ is then maximal and equal to $n-1$ in the complete network, $\bar{\mathbf{g}}^c$. The existence of a unique NE is therefore guaranteed by the assumption that $\beta > \lambda(n-1)$.

Part (i): Assume to the contrary that there exists a NE for a fixed network $\bar{\mathbf{g}}$, $\mathbf{x}(\bar{\mathbf{g}})$, such that a pair of players k and l are in a complete component with $x_k \neq x_l$ and assume without loss of generality that $x_k > x_l$. Note that in a complete component $N_k(\bar{\mathbf{g}}) \setminus \{l\} = N_l(\bar{\mathbf{g}}) \setminus \{k\}$ holds and therefore $\sum_{j\in N_l(\bar{\mathbf{g}})} x_j = \sum_{j\in N_k(\bar{\mathbf{g}})} x_j + (x_k - x_l) > \sum_{j\in N_k(\bar{\mathbf{g}})} x_j$. Note further that $\sum_{j\in N\setminus\{l\}} x_j = \sum_{j\in N\setminus\{k\}} x_j + (x_k - x_l)$. Plugging the above into the best response functions for agent k and l, respectively, we obtain

$$\begin{aligned}\bar{x}_l(\mathbf{x}_{-l}, \bar{\mathbf{g}}) &= \tfrac{1}{\beta+\gamma}\left(\alpha + \lambda\sum_{j\in N_l(\bar{\mathbf{g}})} x_j - \gamma\sum_{j\in N\setminus\{l\}} x_j\right)\\ &= \tfrac{1}{\beta+\gamma}\left(\alpha + \lambda\left(\sum_{j\in N_k(\bar{\mathbf{g}})} x_j + (x_k - x_l)\right) - \gamma\left(\sum_{j\in N\setminus\{k\}} x_j + (x_k - x_l)\right)\right)\\ &= \tfrac{1}{\beta+\gamma}\left(\alpha + \lambda\sum_{j\in N_k(\bar{\mathbf{g}})} x_j - \gamma\sum_{j\in N\setminus\{k\}} x_j + (\lambda-\gamma)(x_k - x_l)\right)\\ &\geq \tfrac{1}{\beta+\gamma}\left(\alpha + \lambda\sum_{j\in N_k(\bar{\mathbf{g}})} x_j - \gamma\sum_{j\in N\setminus\{k\}} x_j\right) = \bar{x}_k(\mathbf{x}_{-k}, \bar{\mathbf{g}}),\end{aligned}$$

where the inequality follows from $x_k - x_l > 0$ and $\lambda - \gamma \geq 0$. We have thereby reached a contradiction.

Part (ii): The result follows from an analogous argument to the one provided in Part (i). *Q.E.D.*

Definition 1: $\underline{\kappa} = v\left(x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}^+), x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}^+) + (n-2)x(\bar{\mathbf{g}}^e)\right) - v\left(0, (n-1)x(\bar{\mathbf{g}}^e)\right)$

$$= \left(\alpha^2(\beta+\gamma)(2\beta+4\gamma-\lambda)\lambda\right) / \left(2(\beta+n\gamma)^2(\beta+2\gamma-\lambda)^2\right) \text{ and}$$

$$\begin{aligned}\bar{\kappa} &= \tfrac{1}{n-1}\left(v\left((n-1)x(\bar{\mathbf{g}}^c), (n-1)x(\bar{\mathbf{g}}^c)\right) - v(0, (n-1)x(\bar{\mathbf{g}}^c))\right)\\ &= \left(\alpha^2\lambda(2\beta - \lambda(n-1) + 2\gamma)\right) / \left(2(\beta+\gamma)(\beta - \lambda(n-1) + n\gamma)^2\right).\end{aligned}$$

[21] See, for example, L. Lovasz, Geometric Representations of Graphs (2009).

Lemma 1: *For any network* $\bar{\mathbf{g}}$ *and corresponding vector of NE effort levels,* $\mathbf{x}(\bar{\mathbf{g}})$*, if* $\bar{g}_{i,j} = 0$ *and* i *and* j *deviate by creating the link* $\bar{g}'_{i,j} = 1$*, then* $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) > x_i(\bar{\mathbf{g}})$ *and* $x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) > x_j(\bar{\mathbf{g}})$*.*

Proof. Note that from $X = [0,\infty)$ we know that $x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) \geq 0$ and $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) \geq 0$ holds. Assume first that $\gamma = 0$. For a fixed network $\bar{\mathbf{g}}$, Nash equilibrium effort levels for agent i and j are given by $x_i(\bar{\mathbf{g}}) = \frac{1}{\beta}\left(\alpha + \lambda y_i(\bar{\mathbf{g}})\right)$ and $x_j(\bar{\mathbf{g}}) = \frac{1}{\beta}\left(\alpha + \lambda y_j(\bar{\mathbf{g}})\right)$, while $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j})$ and $x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j})$ are given by

$$x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) = \tfrac{1}{\beta}\left(\alpha + \lambda\left(y_i(\bar{\mathbf{g}}) + x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j})\right)\right)$$

and

$$x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) = \tfrac{1}{\beta}\left(\alpha + \lambda\left(y_j(\bar{\mathbf{g}}) + x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j})\right)\right).$$

Assume to the contrary and without loss of generality that $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) \leq x_i(\bar{\mathbf{g}})$. From the best response function we know that then $x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) = 0$. However, from $y_j(\bar{\mathbf{g}}) \geq 0$ and $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) \geq 0$ we know that $x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) = \frac{1}{\beta}\left(\alpha + \lambda\left(y_j(\bar{\mathbf{g}}) + x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j})\right)\right) > 0$ and we have reached a contradiction. Assume next that $\gamma > 0$. Nash equilibrium effort levels for agent i and j in a fixed network $\bar{\mathbf{g}}$ are then given by $x_i(\bar{\mathbf{g}}) = \frac{1}{\beta+\gamma}\left(\alpha + \lambda y_i(\bar{\mathbf{g}}) - \gamma z_i(\bar{\mathbf{g}})\right)$ and $x_j(\bar{\mathbf{g}}) = \frac{1}{\beta+\gamma}\left(\alpha + \lambda y_j(\bar{\mathbf{g}}) - \gamma z_i(\bar{\mathbf{g}})\right)$, while $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j})$ and $x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j})$ are given by

$$x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) = \tfrac{1}{\beta+\gamma}\left(\alpha + \lambda\left(y_i(\bar{\mathbf{g}}) + x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j})\right) - \gamma\left(z_i(\bar{\mathbf{g}}) + \left(x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) - x_j(\bar{\mathbf{g}})\right)\right)\right)$$

and

$$x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) = \tfrac{1}{\beta+\gamma}\left(\alpha + \lambda\left(y_j(\bar{\mathbf{g}}) + x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j})\right) - \gamma\left(z_j(\bar{\mathbf{g}}) + \left(x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) - x_i(\bar{\mathbf{g}})\right)\right)\right).$$

We can now rewrite the latter expressions as

$$x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) = x_i(\bar{\mathbf{g}}) + \tfrac{1}{\beta+\gamma}\left((\lambda-\gamma)x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) + \gamma x_j(\bar{\mathbf{g}})\right)$$

and

$$x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) = x_j(\bar{\mathbf{g}}) + \tfrac{1}{\beta+\gamma}\left((\lambda-\gamma)x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) + \gamma x_i(\bar{\mathbf{g}})\right).$$

But then $(\lambda-\gamma)x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j})+\gamma x_j(\bar{\mathbf{g}}) > 0$ and $(\lambda-\gamma)x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j})+\gamma x_i(\bar{\mathbf{g}}) > 0$ hold. To see this, note that $x_i(\bar{\mathbf{g}}) > 0$ and $x_j(\bar{\mathbf{g}}) > 0$ since any Nash equilibrium is interior by Theorem 1 in Ballester et al. (2006). Moreover, by assumption $\lambda \geq \gamma$ and $\gamma > 0$, while from $X = [0,\infty)$ we know that $x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) \geq 0$ and $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) \geq 0$. Therefore, $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}) > x_i(\bar{\mathbf{g}})$ and $x'_j(\bar{\mathbf{g}}+\bar{g}_{i,j}) > x_j(\bar{\mathbf{g}})$ hold. *Q.E.D.*

Proposition 2: $\underline{\kappa} < \overline{\kappa}$ *holds. Furthermore, (i) if* $\kappa < \underline{\kappa}$ *then* $(\mathbf{x}(\bar{\mathbf{g}}^c), \bar{\mathbf{g}}^c)$ *is a PNE, (ii) if* $\kappa > \overline{\kappa}$ *then* $(\mathbf{x}(\bar{\mathbf{g}}^e), \bar{\mathbf{g}}^e)$ *is a PNE and (iii) if* $\kappa \in [\underline{\kappa}, \overline{\kappa}]$ *then* $(\mathbf{x}(\bar{\mathbf{g}}^c), \bar{\mathbf{g}}^c)$ *and* $(\mathbf{x}(\bar{\mathbf{g}}^e), \bar{\mathbf{g}}^e)$ *are PNE.*

Proof. We first derive the two bounds on linking cost, $\underline{\kappa}$ and $\overline{\kappa}$. $\overline{\kappa}$ is given by the average marginal payoff per link of an agent in the complete network, $\bar{\mathbf{g}}^c$. Note that the effort level of an agent in the complete network, $\bar{\mathbf{g}}^c$, is given by $x(\bar{\mathbf{g}}^c) = x_i(\bar{\mathbf{g}}^c) = \alpha/(\beta - \lambda(n-1) + n\gamma)$ $\forall i \in N$, while the deviation effort level of an agent deleting all links in the complete network, denoted by $x_i'(\bar{\mathbf{g}}^c - \sum_{j \in N\setminus\{i\}} \bar{g}_{i,j})$, is given by $x_i'(\bar{\mathbf{g}}^c - \sum_{j \in N\setminus\{i\}} \bar{g}_{i,j}) = \alpha(\beta - \lambda(n-1) + \gamma)/\left((\beta+\gamma)(\beta - \lambda(n-1) + n\gamma)\right)$. We then obtain $\overline{\kappa}$, i.e. the average gross marginal payoff of linking to $n-1$ agents in the complete network, $\bar{\mathbf{g}}^c$, by substituting $x(\bar{\mathbf{g}}^c)$ into the relevant expression for $\overline{\kappa}$. Both expressions are provided in Definition 1. In turn, $\underline{\kappa}$ is given by the marginal payoff of two agents creating a link in the empty network, $\bar{\mathbf{g}}^e$. The effort level in the empty network, $x(\bar{\mathbf{g}}^e)$, is given by $x(\bar{\mathbf{g}}^e) = x_i(\bar{\mathbf{g}}^e) = \alpha/(\beta + n\gamma)$ $\forall i \in N$ and the deviation effort level of a pair of agents i and j creating a link in the empty network, denoted by $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$, is given by $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) = \alpha(\beta + 2\gamma)/\left((\beta + n\gamma)(\beta - \lambda + 2\gamma)\right)$. By symmetry, $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) = x_j'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$. To obtain $\underline{\kappa}$ we substitute $x(\bar{\mathbf{g}}^e)$ and $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$ into the relevant expression for $\overline{\kappa}$. Again, both expressions are provided in Definition 1. Next we show that $\underline{k} > 0$. Note that the best response function in the empty network, $\bar{\mathbf{g}}^e$, is given by $\bar{x}_i(0, (n-1)x(\bar{\mathbf{g}}^e)) = \frac{1}{\beta+\gamma}\left(\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e)\right)$. We can then write $\underline{\kappa}$ in terms of $x(\bar{\mathbf{g}}^e)$ and $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$ as follows

$$\underline{\kappa} = \tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha + (\lambda-\gamma)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e)\right)^2 - \left(\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e)\right)^2\right).$$

From Theorem 1 in Ballester et al. (2006) we know that all Nash equilibria are interior for any $\bar{\mathbf{g}}$, and therefore $\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e) > 0$. Moreover, from Lemma 1 we know that $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) > x(\bar{\mathbf{g}}^e)$ holds. From $\lambda - \gamma \geq 0$ it then follows that $\alpha + (\lambda-\gamma)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e) > \alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e) > 0$ and therefore $\underline{\kappa} > 0$. Next we first show that $x(\bar{\mathbf{g}}^c) > x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$, which we then use to show that $\overline{\kappa} > \underline{\kappa}$ also holds. Recall first that the expression for $x(\bar{\mathbf{g}}^c)$ is given by $x(\bar{\mathbf{g}}^c) = x_i(\bar{\mathbf{g}}^c) = \alpha/(\beta - n(\lambda-\gamma) + \lambda)$. To see that $\beta - n(\lambda - \gamma) + \lambda > 0$ holds, we rewrite the expression as $\beta + n\gamma > \lambda(n-1)$, which follows directly from our assumption that $\beta > \lambda(n-1)$. Denote by $x(\bar{\mathbf{g}}^c_{n=2})$ the auxiliary expression obtained when setting $n = 2$ in $x(\bar{\mathbf{g}}^c)$, i.e. $x(\bar{\mathbf{g}}^c_{n=2}) = \alpha/(\beta + 2\gamma - \lambda)$. Note that $x(\bar{\mathbf{g}}^c) \geq x(\bar{\mathbf{g}}^c_{n=2})$ holds. To see this, rewrite $\beta - n(\lambda-\gamma) + \lambda$ as $\beta + 2\gamma - \lambda - (\lambda-\gamma)(n-2)$ and note that $x(\bar{\mathbf{g}}^c) = x(\bar{\mathbf{g}}^c_{n=2})$ for $\lambda = \gamma$, while $x(\bar{\mathbf{g}}^c) > x(\bar{\mathbf{g}}^c_{n=2})$ for $\lambda > \gamma$ and $n > 2$. Next we compare $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$ with the expression for $x(\bar{\mathbf{g}}^c_{n=2})$. Recall that $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) = \alpha(\beta + 2\gamma)/\left((\beta + n\gamma)(\beta - \lambda + 2\gamma)\right)$, which we can write as $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) = (\beta+\gamma)/(\beta + (n-1)\gamma)x(\bar{\mathbf{g}}^c_{n=2})$. Since $(\beta+\gamma)/(\beta+(n-1)\gamma) < 1$ (given that $n \geq 3$) we know that $x(\bar{\mathbf{g}}^c_{n=2}) > x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$ holds. From the above we know that $x(\bar{\mathbf{g}}^c) \geq x(\bar{\mathbf{g}}^c_{n=2})$ holds and we can therefore write $x(\bar{\mathbf{g}}^c) > x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$. Note that since $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) > x(\bar{\mathbf{g}}^e)$ (by Lemma 1) it then also follows that $x(\bar{\mathbf{g}}^c) > x(\bar{\mathbf{g}}^e)$. This allows us to show the following:

$$\begin{aligned}
\overline{\kappa} &= \tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha + (\lambda-\gamma)(n-1)x(\bar{\mathbf{g}}^c)\right)^2 - \left(\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^c)\right)^2\right)/(n-1)\\
&> \tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha + (\lambda-\gamma)(n-1)x(\bar{\mathbf{g}}^c)\right)^2 - \left(\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e)\right)^2\right)/(n-1)\\
&> \tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha + (\lambda-\gamma)(n-1)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})\right)^2 - \left(\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e)\right)^2\right)/(n-1)\\
&> \tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha + (\lambda-\gamma)(n-1)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e)\right)^2 - \left(\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e)\right)^2\right)/(n-1)\\
&> \tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha + (\lambda-\gamma)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e)\right)^2 - \left(\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e)\right)^2\right) = \underline{\kappa}.
\end{aligned}$$

The first inequality follows from $x(\bar{\mathbf{g}}^c) > x(\bar{\mathbf{g}}^e)$, while the second inequality follows from $x(\bar{\mathbf{g}}^c) > x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$. To see that the third inequality holds, note first that $\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e) > 0$ holds (since Nash equilibrium effort levels are interior in $\bar{\mathbf{g}}^e$), so that $\alpha + (\lambda - \gamma)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e) > 0$ holds, which follows from $\lambda - \gamma \geq 0$ and $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) > 0$. Therefore, $\alpha + (\lambda - \gamma)(n-1)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e) > 0$ holds and the inequality follows directly from $\gamma(n-2)x(\bar{\mathbf{g}}^e) > 0$. Finally, the last inequality follows immediately from the quadratic functional form. Therefore, $\overline{\kappa} > \underline{\kappa} > 0$. Note next that, if $\kappa \leq \overline{\kappa}$, then an agent in the complete network does not find it profitable to delete all her links. Since $v(y_i, z_i)$ is convex in y_i and $x(\bar{\mathbf{g}}^c) = x_i(\bar{\mathbf{g}}^c)$ $\forall i \in N$, deleting any subset of links is then also not profitable. Therefore, if $\kappa \leq \overline{\kappa}$, the complete network is a PNE. If $\kappa \geq \overline{\kappa}$, then no pair of agents finds it profitable to create a link in the empty network, and therefore the empty network is a PNE. $Q.E.D.$

Theorem 1: *In any PNE, $(\mathbf{x}, \bar{\mathbf{g}})$, the network $\bar{\mathbf{g}}$ is a nested split graph such that $x_i < x_k \Leftrightarrow \eta_i(\bar{\mathbf{g}}) < \eta_k(\bar{\mathbf{g}}) \Leftrightarrow v(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) < v(y_k(\bar{\mathbf{g}}), z_k(\bar{\mathbf{g}}))$ holds.*

Proof . We first provide four lemmas, which directly imply that any PNE network is a nested split graph.

Lemma 2: *In any PNE, $(\mathbf{x}, \bar{\mathbf{g}})$, if $\bar{g}_{i,l} = 1$, then $\bar{g}_{i,k} = 1$ for all agents k with $x_k \geq x_l$.*

Proof. Assume that $(\mathbf{x}, \bar{\mathbf{g}})$ is a PNE and, contrary to the above, that $\bar{g}_{i,l} = 1$ and $\bar{g}_{i,k} = 0$ for some agent k with $x_k \geq x_l$. Note first that for $\bar{g}_{i,l} = 1$ to be part of a PNE, it must be that $v(y_i, z_i) - v(y_i - x_l, z_i) \geq \kappa$ holds, as otherwise agent i can profitably deviate by deleting the link with agent l (and adjust her effort level). To simplify notation we write x_k' for the deviation effort level of agent k when linking to agent i and, analogously, x_i' for the deviation effort level of agent i when linking to agent k. Next we show that, if the latter condition holds, then agent i also finds it profitable to create the link $\bar{g}_{i,k}' = 1$. More formally, we show that $v(y_i + x_k', z_i + (x_k' - x_k)) - v(y_i, z_i) > v(y_i, z_i) - v(y_i - x_l, z_i) \geq \kappa$ holds. Recall that from Lemma 1 we know that $x_k' > x_k$. Since the value function is strictly convex in the first argument for fixed z_i, $v(y_i + x_l, z_i) - v(y_i, z_i) > v(y_i, z_i) - v(y_i - x_l, z_i)$ holds. Furthermore, since $x_k \geq x_l$, the following also holds $v(y_i + x_k, z_i) - v(y_i, z_i) \geq v(y_i + x_l, z_i) - v(y_i, z_i)$. Finally, to show that $v(y_i + x_k', z_i + (x_k' - x_k)) - v(y_i, z_i) \geq v(y_i + x_k, z_i) - v(y_i, z_i)$, we make us of the functional form of $v(y_i + x_k', z_i + (x_k' - x_k))$ and $v(y_i, z_i)$, respectively, given by

$$v(y_i + x_k', z_i + (x_k' - x_k)) = \tfrac{1}{2(\beta+\gamma)} \left(\alpha + \lambda y_i + \lambda x_k' - \gamma z_i - \gamma(x_k' - x_k)\right)^2$$

and

$$v(y_i + x_k, z_i) = \tfrac{1}{2(\beta+\gamma)} \left(\alpha + \lambda y_i + \lambda x_k - \gamma z_i\right)^2.$$

To show that $v(y_i + x'_k, z_i + (x'_k - x_k)) > v(y_i + x_k, z_i)$, it is then sufficient to show that $\lambda x'_k - \gamma(x'_k - x_k) > \lambda x_k$. To see this, note that $\lambda x'_k - \gamma(x'_k - x_k) = \lambda x_k + (\lambda - \gamma)(x'_k - x_k) \geq \lambda x_k$, where the inequality follows from $x'_k > x_k$ (by Lemma 1) and $\lambda \geq \gamma$. Therefore, $v(y_i + x'_k, z_i + (x'_k - x_k)) - v(y_i, z_i) > v(y_i, z_i) - v(y_i - x_l, z_i) \geq \kappa$ holds. That is, if agent i does not find it profitable to delete the link with agent l, then agent i finds it profitable to create the link with agent k. For $\bar{g}_{i,k} = 0$ to hold in a PNE, it would therefore have to be that agent k does not find it profitable to link to agent i. In the following we show that this cannot be the case. Note that for $\bar{g}_{i,l} = 1$ to hold, $v(y_l, z_l) - v(y_l - x_i, z_l) \geq \kappa$ must hold, as otherwise agent l can profitably deviate by deleting the link with agent i (and adjust her effort level). From the convexity of the value function in the first argument, we know that $v(y_l + x_i, z_l) - v(y_l, z_l) > v(y_l, z_l) - v(y_l - x_i, z_l)$ holds. Since $x_k \geq x_l$ holds, we know from the best response functions $\bar{x}_k(\bar{\mathbf{g}}) = \frac{1}{\beta+\gamma}(\alpha + \lambda y_k(\bar{\mathbf{g}}) - \gamma z_k(\bar{\mathbf{g}}))$ and $\bar{x}_l(\bar{\mathbf{g}}) = \frac{1}{\beta+\gamma}(\alpha + \lambda y_l(\bar{\mathbf{g}}) - \gamma z_l(\bar{\mathbf{g}}))$ that $\lambda y_k(\bar{\mathbf{g}}) - \gamma z_k(\bar{\mathbf{g}}) \geq \lambda y_l(\bar{\mathbf{g}}) - \gamma z_l(\bar{\mathbf{g}})$ also holds. From the value functions for $v(y_k, z_k) = \frac{1}{2(\beta+\gamma)}(\alpha + \lambda y_k(\bar{\mathbf{g}}) - \gamma z_k(\bar{\mathbf{g}}))^2$ and $v(y_l, z_l) = \frac{1}{2(\beta+\gamma)}(\alpha + \lambda y_l(\bar{\mathbf{g}}) - \gamma z_l(\bar{\mathbf{g}}))^2$ it then follows directly that $v(y_k + x_i, z_k) - v(y_k, z_k) \geq v(y_l + x_i, z_l) - v(y_l, z_l)$ holds. Finally, by an argument analogous to the one presented above, we then know that $v(y_k + x'_i, z_k + (x'_i - x_i)) - v(y_k, z_k) \geq v(y_k + x_i, z_k) - v(y_k, z_k) > \kappa$ holds. That is, agent k finds it profitable to link to agent i and proposed deviation is profitable. Therefore, in any PNE, if $\bar{g}_{i,l} = 1$, then $\bar{g}_{i,k} = 1$ for all agents k with $x_k \geq x_l$. *Q.E.D.*

Lemma 3: *In any PNE, $(\mathbf{x}, \bar{\mathbf{g}})$, $x_i = x_k \Leftrightarrow y_i = y_k$ and $x_i > x_k \Leftrightarrow y_i > y_k$.*

Proof. Define $z = \sum_{j \in N} x_j$ and write z_i and z_k as $z_i = z - x_i$ and $z_k = z - z_k$. Subtracting $x_k(\bar{\mathbf{g}})$ from $x_i(\bar{\mathbf{g}})$ then yields $x_i(\bar{\mathbf{g}}) - x_k(\bar{\mathbf{g}}) = \frac{\lambda}{\beta}(y_i(\bar{\mathbf{g}}) - y_k(\bar{\mathbf{g}}))$ and therefore $x_i = x_k \Leftrightarrow y_i = y_k$ and $x_i > x_k \Leftrightarrow y_i > y_k$.

Lemma 4: *In any PNE, $(\mathbf{x}, \bar{\mathbf{g}})$, $x_i = x_k \Leftrightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$.*

Proof. First we show that $x_i = x_k \Rightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Assume to the contrary that $x_i = x_k$ and $N_i(\bar{\mathbf{g}}) \setminus \{k\} \neq N_k(\bar{\mathbf{g}}) \setminus \{i\}$. There must then exist an agent l such that either $l \in N_i(\bar{\mathbf{g}}) \setminus \{k\}$ and $l \notin N_k(\bar{\mathbf{g}}) \setminus \{i\}$ or $l \notin N_i(\bar{\mathbf{g}}) \setminus \{k\}$ and $l \in N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Since $x_i = x_k$, this contradicts Lemma 2. Next we show that $N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\} \Rightarrow x_i = x_k$. Assume to the contrary that $N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$ and $x_i \neq x_k$. Without loss of generality assume that $x_i > x_k$. We consider two cases. Assume first $\bar{g}_{i,k} = 0$. Then $y_i = y_k$ and $x_i > x_k$ holds, which contradicts Lemma 3. Assume next that $\bar{g}_{i,k} = 1$. Then $y_k > y_i$ holds and $x_i > x_k$ again contradicts Lemma 3. *Q.E.D.*

Lemma 5: *In any PNE, $(\mathbf{x}, \bar{\mathbf{g}})$, $x_i < x_k \Leftrightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$.*

Proof. First we show that $x_i < x_k \Rightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Assume to the contrary that $x_i < x_k$, but $N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$ does not hold. We distinguish two subcases. Assume first that $x_i < x_k$ and $N_k(\bar{\mathbf{g}}) \setminus \{i\} = N_i(\bar{\mathbf{g}}) \setminus \{k\}$. This contradicts Lemma 4. Next, assume $x_i < x_k$

and $N_k(\bar{\mathbf{g}}) \setminus \{i\} \neq N_i(\bar{\mathbf{g}}) \setminus \{k\}$ holds, while $N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$ does not hold. There must then exist an agent l such that $l \in N_i(\bar{\mathbf{g}}) \setminus \{k\}$ and $l \notin N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Since $x_i < x_k$, this contradicts Lemma 2. Next we show that $N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\} \Rightarrow x_i < x_k$. Assume to the contrary $N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$ and $x_i \geq x_k$. Assume first $\bar{g}_{i,k} = 0$. Then $y_k > y_i$ holds and $x_i \geq x_k$ contradicts Lemma 3. Assume next that $\bar{g}_{i,k} = 1$. Then again $y_k > y_i$ holds and $x_i \geq x_k$ again contradicts Lemma 3. *Q.E.D.*

Lemma 6: *In any PNE,* $(\mathbf{x}, \bar{\mathbf{g}})$*,* $x_i < x_k \Leftrightarrow \eta_i(\bar{\mathbf{g}}) < \eta_k(\bar{\mathbf{g}})$*,* $x_i = x_k \Leftrightarrow \eta_i(\bar{\mathbf{g}}) = \eta_k(\bar{\mathbf{g}})$ *and* $x_i(\bar{\mathbf{g}}) < x_k(\bar{\mathbf{g}}) \Leftrightarrow v(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) < v(y_k(\bar{\mathbf{g}}), z_k(\bar{\mathbf{g}}))$*.*

Proof. The first two equivalence relationships follow directly from the lemmas above. The third equivalence relationship follows directly from the best response functions. To see this, note that $x_i < x_k$ if and only if $\alpha + \lambda y_i - \gamma z_i < \alpha + \lambda y_k - \gamma z_k$. *Q.E.D.*

In any PNE the network is a nested split graph.

In any PNE if $\bar{g}_{i,l} = 1$ and $\eta_k(\bar{\mathbf{g}}) \geq \eta_l(\bar{\mathbf{g}})$, then $x_k \geq x_l$ by Lemma 6 and $\bar{g}_{i,k} = 1$ by Lemma 2. That is, $\bar{\mathbf{g}}$ is a nested split graph. *Q.E.D.*

Theorem 2: *In any optimal strategy profile,* $\hat{\mathbf{s}}$*,* $\hat{\mathbf{g}}$ *is a nested split graph. Moreover,* $\hat{x}_i(\hat{\mathbf{g}}) > \hat{x}_j(\hat{\mathbf{g}}) \Leftrightarrow \hat{y}_i(\hat{\mathbf{g}}) > \hat{y}_j(\hat{\mathbf{g}}) \Leftrightarrow \hat{\eta}_i(\hat{\mathbf{g}}) > \hat{\eta}_j(\hat{\mathbf{g}})$ *holds.*

Proof. We show that $\hat{\mathbf{g}}$ is a nested split graph in four steps. We first show that the sum of gross payoffs is proportional to the sum of optimal effort levels.

Step 1: $\sum_{i \in N} \pi_i(\hat{\mathbf{x}}(\bar{\mathbf{g}}), \bar{\mathbf{g}}, \theta) = \frac{1}{2}\alpha \sum_{i \in N} \hat{x}_i(\bar{\mathbf{g}}, \theta)$

For a given vector of parameters $\theta = (\alpha, \beta, \lambda, \gamma)$ and network $\bar{\mathbf{g}}$, the planner maximizes gross welfare by choosing the vector of effort levels, $\mathbf{x}$. That is, we can write

$max_{\mathbf{x} \in \mathbb{R}^n_+} \sum_{i \in N} \pi_i(\bar{\mathbf{g}}, \mathbf{x}, \theta) =$

$$= \alpha \sum_{i \in N} x_i - \tfrac{1}{2}(\beta + \gamma) \sum_{i \in N} x_i^2 + \alpha \sum_{i \in N} (x_i \sum_{j \in N_i(\mathbf{g})} x_j) - \gamma \sum_{i \in N} (x_i \sum_{j \in N \setminus \{i\}} x_j).$$

The optimal effort levels $\hat{x}_i$ solve the following first order conditions conditions, so that

$$\alpha - (\beta + \gamma)\hat{x}_i + 2\lambda \sum_{j \in N_i(\mathbf{g})} \hat{x}_j - 2\gamma \sum_{j \neq i} \hat{x}_j = 0 \ \forall i \in N.$$

Multiplying each first order condition by $\hat{x}_i$ and summing over all agents' first order conditions yields

$$\tfrac{1}{2}\alpha \sum_{i \in N} \hat{x}_i = \tfrac{1}{2}(\beta + \gamma) \sum_{i \in N} \hat{x}_i^2 + \lambda \sum_{i \in N} (\hat{x}_i \sum_{j \in N_i(\mathbf{g})} \hat{x}_j) - \gamma \sum_{i \in N} (\hat{x}_i \sum_{j \neq i} \hat{x}_j).$$

Plugging back into the objective function yields

$$\sum_{i\in N} \pi_i(\bar{\mathbf{g}}, \hat{\mathbf{x}}(\bar{\mathbf{g}}), \theta) = \tfrac{1}{2}\alpha \sum_{i\in N} \hat{x}(\bar{\mathbf{g}}, \theta).$$

Next define $\theta = (\alpha, \beta, \lambda, \gamma)$ and $\tilde{\theta}(\theta) = (\tilde{\alpha}(\theta), \tilde{\beta}(\theta), \tilde{\lambda}(\theta), \tilde{\gamma}(\theta))$ with $\tilde{\alpha}(\theta) = \alpha,\ \tilde{\beta}(\theta) = \beta - \gamma,\ \tilde{\lambda}(\theta) = 2\lambda,\ \tilde{\gamma}(\theta) = 2\gamma$. To simplify notation, we simply write $\tilde{\theta}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\lambda}$ and $\tilde{\gamma}$ when it is clear from the context. Assume that θ is such that $\tilde{\lambda}/\tilde{\beta} < 1/(n-1)$ and that $\tilde{\beta}(\theta) > 0$.

Step 2: If θ is such that $\tilde{\lambda}/\tilde{\beta} < 1/(n-1)$ and that $\tilde{\beta}(\theta) > 0$, then $\mathbf{x}(\bar{\mathbf{g}}, \tilde{\theta})$ is well defined, unique and $\mathbf{x}(\bar{\mathbf{g}}, \tilde{\theta}) = \hat{\mathbf{x}}(\bar{\mathbf{g}}, \theta)$.

From Theorem 1 in Ballester et al. (2006) we know that if $\tilde{\lambda}/\tilde{\beta} < 1/(n-1)$, then the vector of Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}}, \tilde{\theta})$, is well defined an unique. Note next that in a Nash equilibrium for a fixed network $\bar{\mathbf{g}}$, agent i solves the following maximization problem

$$max_{x_i\in\mathbb{R}_+} u_i(\bar{\mathbf{g}}, x_i(\bar{\mathbf{g}}), \mathbf{x}_{-i}, \tilde{\theta}) = \tilde{\alpha}x_i - \tfrac{1}{2}(\tilde{\beta} + \tilde{\gamma})x_i^2 + \tilde{\lambda} \sum_{j\in N_i(\mathbf{g})} x_i x_j - \tilde{\gamma} \sum_{j\in N\setminus i} x_i x_j.$$

In a Nash equilibrium each agent i's first order condition holds, i.e. $\mathbf{x}(\bar{\mathbf{g}}, \tilde{\theta})$ is such that

$$\tilde{\alpha} - (\tilde{\beta} + \tilde{\gamma})x_i + \tilde{\lambda} \sum_{j\in N_i(\mathbf{g})} x_j - \tilde{\gamma}\sum_{j\neq i} x_j = 0 \ \forall i \in N.$$

We can now substitute for $\tilde{\theta}$ to obtain

$$\alpha - (\beta + \gamma)x_i + 2\lambda \sum_{j\in N_i(\mathbf{g})} x_j - 2\gamma\sum_{j\neq i} x_j = 0 \ \forall i \in N.$$

That is, the first order conditions coincide with the ones obtained in Step 1 and therefore $\mathbf{x}(\bar{\mathbf{g}}, \tilde{\theta}) = \hat{\mathbf{x}}(\bar{\mathbf{g}}, \theta)$.

Step 3: Corollary to Lemma 1 (Belhaj et al. 2016). Consider a network $\bar{\mathbf{g}}$ with agents j and k such that $b_i(\bar{\mathbf{g}}, \frac{\tilde{\lambda}}{\tilde{\beta}}) \leq b_i(\bar{\mathbf{g}}, \frac{\tilde{\lambda}}{\tilde{\beta}})$ and $N_{j\setminus k} \neq \emptyset$. Let $\bar{\mathbf{g}}' = \bar{\mathbf{g}} + A_k^{j\setminus k} - A_j^{j\setminus k}$. Then for any $\frac{1}{n-1} > \frac{\tilde{\lambda}}{\tilde{\beta}} > 0$, $\sum b_i(\bar{\mathbf{g}}', \frac{\tilde{\lambda}}{\tilde{\beta}}) > \sum b_i(\bar{\mathbf{g}}, \frac{\tilde{\lambda}}{\tilde{\beta}})$.

Belhaj et al. (2016) consider a payoff function in which, relative to the payoff function considered here, $\tilde{\alpha} = 1$, $\tilde{\beta} = 1$, $\tilde{\lambda} = \delta$ and $\tilde{\gamma} = 0$. From Ballester et al. (2006) we know that $x_i^*(\bar{\mathbf{g}}, \tilde{\theta}) = \tilde{\alpha} b_i(\bar{\mathbf{g}}, \frac{\tilde{\lambda}}{\tilde{\beta}})/(\tilde{\beta} + \tilde{\gamma} b(\bar{\mathbf{g}}, \frac{\tilde{\lambda}}{\tilde{\beta}}))$, where $b(\bar{\mathbf{g}}, \frac{\tilde{\lambda}}{\tilde{\beta}}) = \sum_{i\in N} b_i(\bar{\mathbf{g}}, \frac{\tilde{\lambda}}{\tilde{\beta}})$. For $\tilde{\theta}$ such that $\tilde{\alpha} = 1$, $\tilde{\beta} = 1$, $\tilde{\lambda} = \delta$ and $\tilde{\gamma} = 0$ we can therefore write $x_i^*(\bar{\mathbf{g}}, \tilde{\theta}) = b_i(\bar{\mathbf{g}}, \delta)$. In the proof of Lemma 1 (Belhaj et al., 2016), the authors show that a j,k-switch strictly increases $\sum_{i\in N} x_i(\bar{\mathbf{g}}, \tilde{\theta})$. Since $x_i(\bar{\mathbf{g}}, \tilde{\theta}) = b_i(\bar{\mathbf{g}}, \delta)$, we know that $\sum b_i(\bar{\mathbf{g}}, \delta)$ also strictly increases. Since for a given network, $\sum b_i(\bar{\mathbf{g}}, \delta)$ is only a function of δ and since this is true for any $\delta = \frac{\tilde{\lambda}}{\tilde{\beta}}$ such that $\frac{1}{n-1} > \delta > 0$, we know that $\sum b_i(\bar{\mathbf{g}}', \frac{\tilde{\lambda}}{\tilde{\beta}}) > \sum b_i(\bar{\mathbf{g}}, \frac{\tilde{\lambda}}{\tilde{\beta}})$ holds after a j,k-switch.

Step 4: Any socially optimal network is a nested split graph.

Note that since $\mathbf{x}(\bar{\mathbf{g}}, \tilde{\theta}) = \hat{\boldsymbol{x}}(\bar{\mathbf{g}}, \theta)$ we know that there is a unique vector or socially optimal effort levels for any $\bar{\mathbf{g}}$ and, plugging back into $\sum_{i \in N} \pi_i(\bar{\mathbf{g}}, \mathbf{x}, \theta)$, we know that $W(\hat{\mathbf{x}}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ is well defined. Note next that $\bar{\mathbf{g}}$ is finite, so that a maximum exists. If the number of links in the network, $\bar{\mathbf{g}}$, denoted with $\eta(\bar{\mathbf{g}})$ is such that $\eta(\bar{\mathbf{g}}) = \frac{n(n-1)}{2}$ ($\eta(\bar{\mathbf{g}}) = 0$), then the network is complete (empty) and therefore a nested split graph. Similarly, if $\eta(\bar{\mathbf{g}}) = 1$ or $\eta(\bar{\mathbf{g}}) = \frac{n(n-1)}{2} - 1$, then $\bar{\mathbf{g}}$ is again a nested split graph. Assume next that $\bar{\mathbf{g}}$ is not a nested split graph such that $2 \leq \eta(\bar{\mathbf{g}}) \leq \frac{n(n-1)}{2} - 2$. Using the expression in Theorem 1 of Ballester et al. (2006) and summing over Nash equilibrium effort levels for a fixed network, $\bar{\mathbf{g}}$, we know that $\sum x_i(\bar{\mathbf{g}}, \tilde{\theta}) = \tilde{\alpha} b(\bar{\mathbf{g}}, \tilde{\theta})/(\tilde{\beta} + \tilde{\gamma} b(\bar{\mathbf{g}}, \tilde{\theta}))$. The first derivative with respect to $b(\bar{\mathbf{g}}, \tilde{\theta})$ is given by $\alpha\beta/(\beta + \lambda\gamma)^2$, which is strictly positive for any $\tilde{\theta}$ given our parameter assumptions. Moreover, from our assumptions on parameters and Step 2 we know that for any θ we can find a $\tilde{\theta}$ such that $\sum x_i(\bar{\mathbf{g}}, \tilde{\theta}) = \sum \hat{x}_i(\bar{\mathbf{g}}, \theta)$. That is, $\sum \hat{x}_i(\bar{\mathbf{g}}, \theta)$ strictly increases after an j, k-switch. From Step 1 we know that $\sum_{i \in N} \pi_i(\bar{\mathbf{g}}, \hat{\mathbf{x}}(\bar{\mathbf{g}}), \theta) = \frac{1}{2}\alpha \sum_{i \in N} \hat{x}(\bar{\mathbf{g}}, \theta)$ and therefore a j, k-switch also increases the sum of payoffs. From Theorem 1 in Belhaj et al. (2016) we know that in any network that is not a nested split graph, there exists an N-switch, i.e. any socially optimal network is a nested split graph.

Finally, from $\mathbf{x}(\hat{\mathbf{g}}, \tilde{\theta}) = \hat{\mathbf{x}}(\hat{\mathbf{g}}, \theta)$ and since $\hat{\mathbf{g}}$ is a nested split graph, $\hat{x}_i(\hat{\mathbf{g}}) > \hat{x}_j(\hat{\mathbf{g}}) \Leftrightarrow \hat{y}_i(\hat{\mathbf{g}}) > \hat{y}_j(\hat{\mathbf{g}}) \Leftrightarrow \hat{\eta}_i(\hat{\mathbf{g}}) > \hat{\eta}_j(\hat{\mathbf{g}})$ follows directly from Theorem 1. *Q.E.D.*

7 References

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