TEXTO PARA DISCUSSÃO

No.704

Rational Sentiments and Financial Frictions

Paymon Khorrami

Fernando Mendo

DEPARTAMENTO DE ECONOMIA

PUC RIO

Rational Sentiments and Financial Frictions*

Paymon Khorrami
Duke Fuqua

Fernando Mendo
PUC-Rio

June 19, 2024

Abstract

We discover sentiment-driven equilibria in popular models of imperfect risk sharing. In these equilibria, sentiment dynamics behave like uncertainty shocks, in the sense that self-fulfilled beliefs about volatility drive aggregate fluctuations. Because such fluctuations can decouple from the wealth distribution, rational sentiment helps resolve two puzzles plaguing models emphasizing balance sheets: (i) financial crises emerge suddenly, featuring large volatility spikes and asset-price declines; (ii) asset-price booms, with below-average risk premia, predict busts and financial crises. Methodologically, our contribution is using stochastic stability theory to establish existence of sunspot equilibria.

JEL Codes: E00, E44, G01.

Keywords: financial frictions, sunspot equilibria, self-fulfilling beliefs, sentiment, financial crises, uncertainty shocks.

*Khorrami: paymon.khorrami@duke.edu; Mendo: fmendolopez@gmail.com. We are grateful to Jungsuk Han, Stavros Panageas, Alan Moreira and to seminar participants at Imperial College, the Central Bank of Chile, USC, Yale SOM, Duke Fuqua, HKU, NUS, the Macro-Finance Society, the EFA, LACEA-LAMES meetings, and in the virtual Junior European Finance series for helpful feedback. A previous version of this paper was titled "Where there is amplification, there are sunspots".

It has by now become commonplace, especially after the 2008 global financial crisis, for macroeconomic models to prominently feature banks, limited participation, imperfect risk-sharing, and other such "financial frictions." Incorporating these features allows macroeconomists to speak meaningfully about financial crises and desirable policy responses. Despite the dramatic growth in this literature, there remain two major sources of disconnect between these models and actual data. For one, standard models have difficulty reproducing the observed severity and suddenness of economic downturns and asset-price dislocations. Secondly, standard models struggle to generate booms that are inherently fragile and prone to bust. To address these shortcomings, some recent contributions add large and sudden bank runs[1] while others deviate from rational expectations to model booms as episodes of over-optimism.[2]

We embrace *rational sentiment* as a complementary approach. This paper makes two main contributions. First, we uncover a wide variety of novel sentiment-driven sunspot equilibria supported by standard financial friction models. The fluctuations in these equilibria can be self-fulfilling: they only occur because agents coordinate on them. Second, we demonstrate how sentiment fluctuations alleviate some of the shortcomings for this class of models. Rational sentiment can generate both (i) large and sudden fluctuations, similar to bank runs (footnote 1), and (ii) booms that breed fragility, similar to the "behavioral sentiment" adopted by some recent papers (footnote 2).

Model and mechanism. We study a simple stripped-down model with financial frictions, similar to Kiyotaki and Moore (1997), Brunnermeier and Sannikov (2014), and many others.[3] There are two types of agents ("experts" and "households") with identical preferences but different levels of productivity when managing capital. Heterogeneous productivity means the identity of capital holders matters for aggregate output. Ideally, in a world with complete financial markets, experts would manage all capital and issue sufficient equity to perfectly share with households any risks associated to capital. But in our model, incomplete markets prevent agents from sharing those risks, so optimal capital holdings depend to some degree on risk and not only on productivities. There are no

[1] For example, Gertler and Kiyotaki (2015) and Gertler et al. (2020) attempt to integrate bank runs into a conventional financial accelerator model, in order to capture additional amplification and non-linearity. Without runs or panic-like behavior, financial accelerator models have a difficult time inducing the financial intermediary leverage needed to generate large amounts of amplification.

[2] For example, Krishnamurthy and Li (2020) and Maxted (2023) build an extrapolative sentiment process on top of a relatively standard financial accelerator model. Agents' excessive optimism in booms lowers risk premia, erodes bank balance sheets, and creates fragility.

[3] We work in continuous time, contributing to a burgeoning literature (He and Krishnamurthy, 2012, 2013, 2019; Moreira and Savov, 2017; Di Tella, 2017, 2019; Klimenko et al., 2017; Silva, 2017; Drechsler et al., 2018; Caballero and Simsek, 2020).

other features: no ad-hoc collateral constraints, no default externalities, and no irrational beliefs. And yet, this basic model can feature a tremendous amount of multiplicity that has been overlooked in the literature.

Indeterminacy in this model comes from the combination of incomplete financial markets and heterogeneous productivities. With these features, asset prices today are not pinned down by "fundamentals"— namely the minimal set of state variables—and can also depend on agents' beliefs about the distribution of asset prices tomorrow. Different beliefs deliver different equilibria. Of particular importance in our specific model is the perceived dispersion in future asset prices, or price volatility.

The following story clarifies the mechanics. Suppose agents are *fearful*, anticipating high asset-price volatility. Despite their productivity advantage, experts will only manage a fraction of aggregate capital, as capital price risk cannot be fully shared through markets. Perceived volatility thus causes an inefficient capital allocation, hence low asset prices. On the other hand, if low asset-price volatility is anticipated, experts will hold a large share of capital, and asset prices will be high. Are both of these coordinated volatility perceptions justified? In many models, only one perception of volatility could be consistent with equilibrium, because future paths would otherwise be explosive.

But in our paper, many coordinated beliefs about volatility can satisfy equilibrium conditions and remain non-explosive, mirroring the conventional idea that dynamic stability of equilibrium supports indeterminacy. Here, stability means that asset prices eventually mean-revert, or "bounce back" from extreme values. Supposing the future distribution of asset prices q is characterized by a first and second moment (μ_q, σ_q^2), then a rise in σ_q (fear)—which depresses q—must be accompanied by an eventual rise in μ_q (bounce-back beliefs). In our continuous-time setup, bounce-back beliefs are just boundary conditions on μ_q at extreme states. Such boundary restrictions are both analytically-convenient and mild; rich dynamics are admissible away from extreme states.

If volatility is dynamically stable, we can use sunspot shocks to govern agents' beliefs about volatility and create sentiment dynamics. In other words, our model can feature a surprise increase in *fear* leading to a *fire sale*, which temporarily depresses asset prices and output. Conversely, sunspot bravery (decline in fear) raises asset prices, through coordinated purchases. These fear-driven dynamics are sustainable so long as they are expected to eventually subside. A distinctive feature is that sentiment dynamics are always characterized by time-varying endogenous uncertainty.

Overview of paper. While explaining our model above, we abstracted from the wealth distribution between experts and households. Typically in the financial frictions literature, this wealth distribution is the key state variable modulating the dynamics. In our analysis, the wealth distribution remains a state variable, but additional "sentiment" state variables naturally arise as potential drivers of equilibrium. Mathematically, we dispense with the assumption that equilibria be Markovian in the wealth distribution, which removes an ad-hoc restriction on agents' beliefs.[4]

Our main results provide an explicit construction and characterization of a broad class of such sentiment-driven equilibria (Section 2). As one might expect from deterministic models, the existence of sunspot equilibria is tied directly to the stability properties of the equilibrium dynamical system. For many models, such stability questions are settled via linearized spectral analysis near steady state. What is the analog in our stochastic nonlinear environment with multiple state variables? To tackle this problem, we leverage tools from the "stochastic stability" literature (the stochastic analog of Lyapunov stability for ODE systems). Conveniently, all of our stability analysis boils down to boundary conditions on our dynamical system.

Our sentiment-driven equilibria engender several new insights, related to the shortcomings in existing models (Section 3). First, fundamentals-based recessions are primarily about expert balance sheet impairment in our model, so they feature small volatility increases and very slow recoveries; sentiment-driven crises can feature far larger volatility spikes and fast recoveries. In fact, we prove that arbitrary capital price volatility and recovery speeds can be justified by sunspot equilibria. Second, whereas fundamentals-based booms always reduce the prospect of crisis, sentiment-driven booms can actually increase crisis probabilities. Relatedly, in the years before large busts, an economy with sentiment tends to feature asset-price and output booms, low volatility, and below-average risk premia. We argue all of these properties of sentiment-driven fluctuations better resemble real-world financial cycles.

Related literature. The theoretical focus on financial frictions and sunspots is not new to this paper. Several studies show how multiplicity emerges through the interaction between asset valuations and borrowing constraints.[5] Relative to these papers, we study different and more primitive financial frictions (equity-issuance constraints) that do not

[4] In a companion paper (Khorrami and Mendo, 2024), we study the possibility of multiple equilibria which are Markovian in the wealth distribution. While interesting, we show in that paper how the resulting dynamics of these self-fulfilling wealth-driven equilibria are approximately identical to the conventional dynamics studied by Brunnermeier and Sannikov (2014) and others. Thus, resolving the literature's puzzles requires us to go beyond wealth-driven equilibria and explore sentiment-driven equilibria.

[5] For instance, bubbles can relax credit constraints, allowing greater investment and thus justifying the existence of the bubble (Scheinkman and Weiss, 1986; Kocherlakota, 1992; Farhi and Tirole, 2012; Miao and Wang, 2018; Liu and Wang, 2014). Self-fulfilling credit dynamics can also arise with *unsecured* lending as opposed to collateralized (Gu et al., 2013; Azariadis et al., 2016).

feature any mechanical link between prices and constraints. (We say "more primitive" because equity constraints are present—either explicitly or implicitly—even in models with borrowing constraints. With unlimited outside equity, perfect risk-sharing could always be achieved and the effects of borrowing constraints circumvented.)

Bank runs, financial panics, and sudden stops are related to, but distinct from, our self-fulfilled fluctuations.[6] All of these phenomena similarly rely on financial frictions, are outcomes of coordination, and produce large fluctuations relative to fundamentals. However, whereas bank runs and its cousins are liability-side phenomena, self-fulfilled fire sales are pure asset-side phenomena. Furthermore, unlike runs, our mechanism does not require asset-market illiquidity or maturity mismatch. Finally, whereas runs are almost exclusively about large downside risk, our sentiment fluctuations also generate interesting boom-bust cycles.

Given our results hold even without ad-hoc borrowing constraints or runs, our paper illustrates that a much broader class of financial crisis models are subject to sunspots. We also do not rely on the more traditional multiplicity-inducing assumptions, such as overlapping generations,[7] non-convexities or externalities in technology,[8] asymmetric/imperfect information,[9] or multiple assets.[10]

Our focus on fear and volatility as drivers of self-fulfilling fluctuations closely relates to the "self-fulfilling risk panics" of Bacchetta et al. (2012). Benhabib et al. (2020) obtain a similar type of fluctuation by examining economies with either collateral or liquidity constraints, rather than the OLG setup of Bacchetta et al. (2012). Although we do not rely on common multiplicity-inducing features like OLG or collateral constraints, we expound on the deeper connection to these papers in Section 1.4.

[6] Mendo (2020) studies self-fulfilled panics that induce collapse of the financial sector. Gertler and Kiyotaki (2015) and Gertler et al. (2020) study bank runs in a similar class of models.

[7] The classic studies on OLG and multiplicity are Azariadis (1981) and Cass and Shell (1983). A more recent investigation, focusing on wealth redistribution across generations, is Farmer (2018).

[8] For example, see Azariadis and Drazen (1990) for multiplicity under threshold investment behavior. See Farmer and Benhabib (1994) for a multiplicity under increasing returns to scale.

[9] In a macro context, Piketty (1997) and Azariadis and Smith (1998) for self-fulfilling dynamics in the presence of screened/rationed credit. In a finance context, Benhabib and Wang (2015) and Benhabib et al. (2016, 2019) generate sunspot fluctuations in dispersed information models. Like us, Benhabib et al. (2015) pins down volatility by certain fundamentals of the economy. However, whereas their mechanism is static in nature, ours is intrinsically dynamic—this is why the "fundamentals" that determine our volatility include asset prices themselves, whereas their volatility is fully determined by deep structural parameters. For this reason, our self-fulfilling volatility is naturally time-varying.

[10] Hugonnier (2012), Gârleanu and Panageas (2021), and Khorrami and Zentefis (2023) all build "redistributive" sunspots that shift valuations among multiple positive-net-supply assets.

1 Model

Information structure. Time $t \geq 0$ is continuous. (We also study a discrete-time version of the model in Online Appendix F.) There are two types of uncertainty in the economy, modeled as two independent Brownian motions $Z := (Z^{(1)}, Z^{(2)})$. All random processes will be adapted to Z.[11] As will be clear below, the first shock $Z^{(1)}$ represents a *fundamental shock* in the sense that it directly impacts production possibilities, whereas the second shock $Z^{(2)}$ is a *sunspot shock* that is extrinsic to any economic primitives but nevertheless may impact endogenous objects. At the end of the paper, we will also consider extrinsic Poisson jumps as part of the information structure.

Technology and markets. There are two goods, a non-durable good (the numéraire, "consumption") and a durable good ("capital") that produces the consumption good. The aggregate supply of capital grows exogenously as

$$dK_t = K_t[gdt + \sigma dZ_t^{(1)}], \tag{1}$$

where g and $\sigma > 0$ are exogenous constants. The capital-quality shock $\sigma dZ^{(1)}$ is a standard way to introduce fundamental randomness in technology. Individual capital holdings evolve identically, except that capital may be traded frictionlessly between agents in the market.[12] The relative capital price is q_t and determined in equilibrium.

There are two types of agents, experts and households, who differ in their production technologies. Experts produce a_e units of the consumption good per unit of capital, whereas households' productivity is $a_h \in (0, a_e)$.

Financial markets consist solely of an instantaneously-maturing, risk-free bond that pays interest rate r_t is in zero net supply. The key financial friction: agents cannot issue equity when managing capital. It is inconsequential that the constraint be this extreme. Partial equity issuance, as long as there is some limit, will generate similar results on sunspot volatility (we discuss this further in Section 1.4).

Preferences and optimization. Given the stated assumptions, we can write the dynamic

[11] In the background, the Brownian motion Z exists on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, equipped with all the "usual conditions." All equalities and inequalities involving random variables are understood to hold almost-everywhere and/or almost-surely.

[12] Individual capital is thus a choice variable: if an agent holds capital k_t, its law of motion is

$$dk_t = gk_t dt + \sigma k_t dZ_t^{(1)} + d\Omega_t,$$

where the term $d\Omega_t$ corresponds to net purchases. To be clear, both g and $\sigma dZ_t^{(1)}$ affect agents' return-on-capital, whereas the net purchases term $d\Omega_t$ does not.

budget constraint of an agent of type ℓ (expert or household) as

$$dn_{\ell,t} = \Big[(n_{\ell,t} - q_t k_{\ell,t}) r_t - c_{\ell,t} + a_\ell k_{\ell,t}\Big] dt + q_t k_{\ell,t} dR_t, \tag{2}$$

where n_ℓ is the agent's net worth, c_ℓ is consumption, and k_ℓ is capital holdings. The last term $dR_t := \frac{d(q_t K_t)}{q_t K_t}$ is the capital and price appreciation while holding capital.

Experts and households have time-separable logarithmic utility, with discount rates ρ_e and $\rho_h \leq \rho_e$, respectively. All agents have rational expectations and solve

$$\sup_{c_\ell \geq 0, k_\ell \geq 0, n_\ell \geq 0} \mathbb{E}\Big[\int_0^\infty e^{-\rho_\ell t} \log(c_{\ell,t}) dt\Big] \tag{3}$$

subject to (2). Everything in optimization problem (3) is homogeneous in (c, k, n), so we can think of the expert and household as representative agents within their class.

Let us briefly discuss the solvency constraint $n_{\ell,t} \geq 0$ in (3). This constraint says that agents cannot borrow more than the market value of their capital, and since there are no other assets besides capital, one can think of $n_{\ell,t} \geq 0$ as the "natural borrowing limit." Intuitively, a sequence of negative shocks can completely destroy an agent's capital stock leaving them without any assets to repay their debts; hence, a net worth buffer must be maintained to assure debt repayments in the worst-case scenario. While assuming such a solvency constraint is relatively standard in infinite-horizon dynamic trading models, we analyze some microfoundations for this assumption in Appendix A, to provide more comfort that the solvency constraint is natural and minimal. In these microfoundations, we assume a No-Ponzi condition (eventual debt repayment) and a net worth lower bound which can be arbitrarily negative but finite.

Finally, to guarantee a stationary wealth distribution, we also allow a type-switching structure: experts retire and become households at rate δ_e, while households retire and become experts at rate δ_h. Technically, the presence of type-switching alters the objective function from (3), but this is irrelevant under the assumption of log utility, as optimal behavior will be as if solving (3)—we show this in Appendix B.1. To acknowledge the fact that type-switching shifts wealth across agent groups, which does not affect agents' individual net worth evolution, let N_e and N_h denote aggregate expert and household net worth. The dynamics of N_e and N_h include the effects of type-switching: $dN_e = N_e \frac{dn_e}{n_e} - \delta_e N_e dt + \delta_h N_h dt$ and $dN_h = N_h \frac{dn_h}{n_h} - \delta_h N_h dt + \delta_e N_e dt$. We reiterate that type-switching is unnecessary for our sunspot results and only serves to obtain stationarity in case we set $\rho_e = \rho_h$ (if $\rho_e > \rho_h + \sigma^2$, the wealth distribution will automatically be stationary even without type-switching). For example, the reader may wish to shut down type-switching ($\delta_e = \delta_h = 0$) and instead consider asymmetric discount rates ($\rho_e > \rho_h + \sigma^2$), and this is completely fine.

1.1 Equilibrium definition

The definition of competitive equilibrium is standard, following Brunnermeier and Sannikov (2014). To write a formal definition, denote the set of experts by the interval $\mathbb{I} = [0, \nu]$, for some $\nu \in (0,1)$ and index individual experts by $i \in \mathbb{I}$. Similarly, denote the set of households by $\mathbb{J} = (\nu, 1]$ with index j. If a type-switching structure exists, we necessarily have $\nu = \frac{\delta_h}{\delta_e + \delta_h}$ (i.e., the population size of experts is pinned down by switching rates) and the indexes of retiring experts/households are implicitly swapped with newly entering experts/households.

Definition 1. For any initial capital endowments $\{k^i_{e,0}, k^j_{h,0} : i \in \mathbb{I}, j \in \mathbb{J}\}$ such that $\int_{\mathbb{I}} k^i_{e,0} di + \int_{\mathbb{J}} k^j_{h,0} dj = K_0$, an *equilibrium* consists of stochastic processes—adapted to the filtered probability space generated by $\{Z_t : t \geq 0\}$—for capital price q_t, interest rate r_t, capital holdings $(k^i_{e,t}, k^j_{h,t})$, consumptions $(c^i_{e,t}, c^j_{h,t})$, and net worths $(n^i_{e,t}, n^j_{h,t})$, such that:

(i) initial net worths satisfy $n^i_{e,0} = q_0 k^i_{e,0}$ and $n^j_{h,0} = q_0 k^j_{h,0}$ for $i \in \mathbb{I}$ and $j \in \mathbb{J}$;

(ii) taking processes for q and r as given, each expert $i \in \mathbb{I}$ and household $j \in \mathbb{J}$ solves (3) subject to (2) and their solvency constraint;

(iii) consumption and capital markets clear at all dates, i.e.,

$$\int_{\mathbb{I}} c^i_{e,t} di + \int_{\mathbb{J}} c^j_{h,t} dj = a_e \int_{\mathbb{I}} k^i_{e,t} di + a_h \int_{\mathbb{J}} k^j_{h,t} dj \tag{4}$$

$$\int_{\mathbb{I}} k^i_{e,t} di + \int_{\mathbb{J}} k^j_{h,t} dj = K_t, \tag{5}$$

where K_t follows (1).

Note that the riskless bond market clears automatically by Walras' Law, which is why this condition is not included above.

1.2 Equilibrium characterization

We present a useful equilibrium characterization that aids all future analysis. First, conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t} dt + \sigma_{q,t} \cdot dZ_t]. \tag{6}$$

There are two potential avenues for random fluctuations. The standard term $\sigma_q \cdot \binom{1}{0}$ represents amplification (or dampening) of fundamental shocks, as in Brunnermeier and Sannikov (2014) and others. By contrast, the second element $\sigma_q \cdot \binom{0}{1}$ measures sunspot volatility that only exists because agents believe in it.

Given log utility and the scale-invariance of agents' budget sets, individual optimization problems are readily solvable. Optimal consumption satisfies the standard formula $c_\ell = \rho_\ell n_\ell$. Optimality conditions for capital holding by experts and households are

$$\frac{a_e}{q} + g + \mu_q + \sigma\sigma_q \cdot \binom{1}{0} - r = \frac{qk_e}{n_e}|\sigma_R|^2 \tag{7}$$

$$\frac{a_h}{q} + g + \mu_q + \sigma\sigma_q \cdot \binom{1}{0} - r \leq \frac{qk_h}{n_h}|\sigma_R|^2 \quad \text{(with equality if } k_h > 0), \tag{8}$$

where

$$\sigma_{R,t} := \sigma\binom{1}{0} + \sigma_{q,t} \tag{9}$$

denotes the shock exposure of capital returns. (Note that experts' optimality condition (7) assumes the solution is interior, i.e., $k_e > 0$. But this is clearly required in any equilibrium given experts earn a strictly higher expected return than households.) From these optimality conditions, notice that agents' capital holdings decisions are uniquely determined given the price process for q. The only additional optimality conditions are the transversality conditions

$$\lim_{T\to\infty} \mathbb{E}[e^{-\rho_\ell T}\frac{1}{c_{\ell,T}} n_{\ell,T}] = 0. \tag{10}$$

However, using $c_\ell = \rho_\ell n_\ell$, we see that (10) automatically holds. As a consequence of (10), our equilibria will always be bubble-free.[13]

Next, we aggregate. Due to financial frictions and productivity heterogeneity, both the distribution of wealth and capital holdings will matter in equilibrium. However, because all experts (and households) make the same scaled consumption c_ℓ/n_ℓ and portfolio choices k_ℓ/n_ℓ, the wealth and capital distributions may be summarized by experts' wealth share

$$\eta := \frac{N_e}{N_e + N_h} = \frac{N_e}{qK}$$

[13] Using transversality (10) and the consumption FOC $M_{\ell,t} = e^{-\rho_\ell t}(c_{\ell,t})^{-1}$, one can show that $q_t K_t = \mathbb{E}_t[\int_t^\infty \frac{M_s}{M_t} Y_s ds]$, where M is a consumption-weighted-average of expert and household SDFs M_e and M_h. Thus, capital is valued according to a present-value equation, and no bubbles exist.

and experts' capital share

$$\kappa := \frac{\int_{\mathbb{I}} k_e^i di}{K}.$$

Given agents' solvency and capital short-sales constraints, we must have $\eta \in [0,1]$ and $\kappa \in [0,1]$ in equilibrium. Substitute optimal consumption into goods market clearing (4), divide by aggregate capital K, and use the definitions of η and κ, to obtain

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h, \tag{PO}$$

where $\bar{\rho}(\eta) := \eta\rho_e + (1-\eta)\rho_h$ is the wealth-weighted average discount rate. Equation (PO) connects asset price q to output efficiency κ, which we call a *price-output* relation for short.

Using the definitions of η and κ, experts' and households' portfolio shares can be written $\frac{qk_e}{n_e} = \frac{\kappa}{\eta}$ and $\frac{qk_h}{n_h} = \frac{1-\kappa}{1-\eta}$. Then, differencing the optimal portfolio conditions (7)-(8), we obtain the *risk-balance* condition

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2\right]. \tag{RB}$$

Either experts manage the entire capital stock ($\kappa = 1$) or the excess return experts obtain over households, $(a_e - a_h)/q$, represents fair compensation for differential risk exposure, $\frac{\kappa-\eta}{\eta(1-\eta)}|\sigma_R|^2$. On the other hand, summing portfolio conditions (7)-(8), weighted by κ and $1-\kappa$, yields an equation for the riskless rate:

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - \left(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\right)|\sigma_R|^2. \tag{11}$$

Finally, by applying Itô's formula to experts' wealth share $\eta = N_e/(N_e + N_h)$, and using agents' net worth dynamics (2) along with contributions from type-switching, wealth share dynamics are given by

$$d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t} \cdot dZ_t, \quad \text{given} \quad \eta_0, \tag{12}$$

where

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2 + \delta_h - (\delta_e + \delta_h)\eta \tag{13}$$

$$\sigma_\eta = (\kappa - \eta)\sigma_R. \tag{14}$$

The initial wealth distribution $\eta_0 = \frac{\int_{\mathbb{I}} n^i_{e,0} di}{q_0 K_0} = \frac{\int_{\mathbb{I}} k^i_{e,0} di}{K_0}$ is given due being solely a function of the initial endowments of capital.

Lemma 1. *Given $\eta_0 \in (0,1)$, consider a process $(\eta_t, q_t, \kappa_t, r_t)_{t\geq 0}$ with dynamics for q_t and η_t described by (6) and (12), respectively. If $\eta_t \in [0,1]$, $\kappa_t \in [0,1]$, and equations (PO), (RB), (11), (13) and (14) hold for all $t \geq 0$, then $(\eta_t, q_t, \kappa_t, r_t)_{t\geq 0}$ corresponds to an equilibrium of Definition 1. Moreover, any distinct pair of such processes corresponds to distinct equilibria.*

Lemma 1 summarizes the full set of conditions characterizing equilibrium and is proved in Appendix B.2. In the rest of the paper, we use this lemma as a tool to simplify our search for equilibria.

Lastly, we make some mild parameter restrictions that will be applicable in the remainder of the paper.

Assumption 1. *Parameters satisfy (i) $0 < \frac{a_h}{\rho_h} < \frac{a_e}{\rho_e} < +\infty$; (ii) $\sigma^2 < \rho_e(1 - a_h/a_e)$; and (iii) either $\sigma^2 < \rho_e - \rho_h$, or $\delta_e, \delta_h > 0$.*

Assumption 1 part (i) makes the modest assumption that the capital price is higher if experts control 100% of wealth than if households control 100% of wealth. Part (ii), meant to make the problem interesting, ensures experts sometimes hold all capital, i.e., $\kappa = 1$. If fundamental risk is $\sigma^2 \geq \rho_e(1 - a_h/a_e)$, experts can never hold the entire capital stock, and the economy will always be in the region of inefficiency. Part (iii) ensures household survival: if experts consume at a rate sufficiently higher than households, or some type-switching exists, then experts do not asymptotically hold all wealth.

1.3 Types of equilibria

We categorize our equilibria into two types: fundamental and sunspot. Fundamental equilibria have two properties: (i) the sunspot shock $Z^{(2)}$ plays no role; and (ii) only the minimal set of state variables affects observables. Because of financial frictions and productivity heterogeneity, the expert wealth share η is a necessary state variable to summarize economic conditions. Other objects (e.g., q, r, κ) are either prices or control variables, so there is a sense in which η is the minimal state variable needed in this class of models. In other words, a fundamental equilibrium should only depend on η. Sunspot equilibria constitute all other equilibria, which we further categorize into two types depending on whether or not they are Markov in η.

Definition 2. A *Fundamental Equilibrium* (FE) is an equilibrium that is Markov in η and in which $\sigma_q \cdot \binom{0}{1} \equiv 0$. Any other equilibrium is a *Brownian Sunspot Equilibrium* (BSE). A

BSE that is Markov in η is called a *Wealth-driven BSE* (W-BSE). A BSE that is non-Markov in η is called a *Sentiment-driven BSE* (S-BSE).[14]

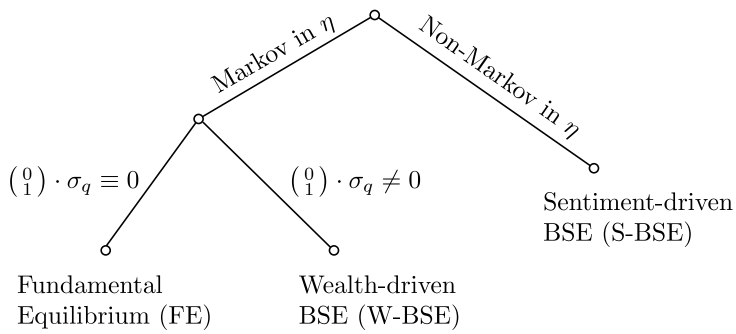

Figure 1: Types of equilibria.

Figure 1 displays the equilibrium taxonomy. The literature universally focuses on the FE of this model, e.g., Brunnermeier and Sannikov (2014). We discuss and analyze these fundamental equilibria in Online Appendix E.[15] The present paper is devoted to the S-BSEs, while a companion paper studies the W-BSEs (Khorrami and Mendo, 2024). We move directly to the S-BSEs for two reasons, based on the results in Khorrami and Mendo (2024). First, W-BSEs can only arise if $\sigma = 0$, or if fundamental risks are separately hedgeable in financial markets. Second, W-BSEs are well-approximated by a FE in the sense that the FE converges to the W-BSE as $\sigma \to 0$. These two results imply Markov equilibria in experts' wealth share η are either (a) pure FE or (b) look very much like pure FE. Consequently, the remainder of the paper studies S-BSEs in the hopes of uncovering new insights relative to the literature.

1.4 Benchmarks and discussion

Before proceeding to the main analysis, we analyze three benchmarks—frictionless equity issuance, homogeneous productivities, and zero fundamental uncertainty—that clarify the underpinnings of sentiment-driven equilibria.

[14] It will turn out that in some S-BSEs, the sunspot shock plays no role, i.e., $\sigma_q \cdot \binom{0}{1} \equiv 0$. However, we choose not to further sub-divide the S-BSEs into cases where the sunspot shock matters and where it doesn't, because that distinction turns out to be less relevant to the analysis. We therefore hope our use of the term "sunspot" in defining the types of equilibria is not confusing here.

[15] Online Appendix E provides some new results to this literature, including a multiplicity of fundamental equilibria when $\sigma > 0$. In particular, following the spirit of footnote 16 in Kiyotaki and Moore (1997), we show that there are two types of equilibria: a normal equilibrium in which negative shocks reduce asset prices (this is the one studied by the literature) and a "hedging equilibrium" in which, due to coordinated capital purchases/sales, asset prices and output respond oppositely to shocks.

Frictionless equity issuance. Suppose any agent, when managing capital, could issue unlimited equity to the market. In exchange for taking some exposure to the risk σ_R in capital returns, these outside equity contracts promise an expected excess return $\sigma_R \cdot \pi$ (here, π is the equilibrium risk price vector associated to the two shocks in Z). All agents can participate as buyers in this market. Since equity-issuance is unconstrained, it is straightforward to see that any capital owner must equate her expected excess returns on capital to $\sigma_R \cdot \pi$. (If $\sigma_R \cdot \pi$ were below an agent's expected excess capital returns, unlimited capital purchases financed by unlimited equity issuances would be an arbitrage; if $\sigma_R \cdot \pi$ were above, the agent would prefer to sell all their capital and invest solely in equity securities.) Experts always manage some capital, so

$$\frac{a_e}{q} + g + \mu_q + \sigma\sigma_q \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - r = \sigma_R \cdot \pi.$$

However, the analogous equation cannot hold for households, since their productivity is lower, $a_h < a_e$. Households will never manage capital in this economy, so $\kappa_t = 1$ at all times, hence $q_t = a_e/\bar{\rho}(\eta_t)$ by equation (PO). That q is solely a function of η rules out S-BSEs.[16] Thus, it is critical that capital is traded, i.e., κ varies.

For our main results, the friction in equity markets need not be as stark as the baseline model. Indeed, Online Appendix D.1 extends the baseline model to allow "partial equity issuance," subject to a constraint parameterized by $\chi \in [0, 1]$. In particular, suppose any agent can issue some equity up to a limit: he/she can offload up to $1 - \chi$ fraction of the risk associated to their capital stock as equity to a competitive financial market. The baseline model corresponds to $\chi = 1$ (i.e., zero issuance), while the frictionless model outlined above corresponds to $\chi = 0$ (i.e., unlimited issuance). We show that self-fulfilling volatility is possible for any $\chi > 0$, but the range of possible equilibrium asset prices shrinks as χ shrinks, and this range collapses to a singleton as $\chi \to 0$.

Homogeneous productivities. Consider our economy with $a_e = a_h = a$. Based on equation (PO), equal productivities immediately implies $q_t = a/\bar{\rho}(\eta_t)$. Again, q is solely a function of η, which rules out S-BSEs. Critically, sentiment-driven equilibria require real outcomes to depend on κ.

In fact, with equal productivities, equilibrium cannot support any endogenous dependence on shocks, i.e., one can show $\sigma_q \equiv 0$ when $a_e = a_h$.[17] This unveils a more

[16] In fact, q cannot be stochastic at all. Indeed, experts and households share identical risk preferences, so households will purchase the outside equity of experts in an amount that is consistent with perfect risk-sharing, meaning $\sigma_\eta \equiv 0$. Since $q_t = a_e/\bar{\rho}(\eta_t)$ is solely a function of η, which is deterministic, we have $\sigma_q \equiv 0$ as well. Shocks can play no amplifying role with frictionless equity markets.

[17] Plugging $a_e = a_h$ into equation (RB) implies either $\kappa = \eta$ or $|\sigma_R| = 0$. Either way, $\sigma_\eta = (\kappa - \eta)\sigma_R = 0$.

general point about the endogeneity of market incompleteness: one cannot necessarily add unspanned extrinsic shocks to an economy and declare markets incomplete. Even though this equal-productivity economy lacks insurance markets against $Z^{(2)}$ shocks, financial markets are *effectively complete*, in the sense that the economic structure imposes that $Z^{(2)}$ can have no impact on outcomes. What is required is a set of assumptions such that $Z^{(2)}$ has "real effects" in which case financial market incompleteness will have some bite. In our economy, all we require is $a_e > a_h$.

Discussion: imperfect risk-sharing and productivity heterogeneity. Based on the benchmarks above, let us explain the deep reasons why our model admits sentiment-driven equilibria. The fact that we require financial frictions and productivity heterogeneity is not surprising—these features are required even in the "financial accelerator" equilibria of Kiyotaki and Moore (1997) and Brunnermeier and Sannikov (2014). More interestingly, sentiment-driven equilibria require nothing more.

First, with limited equity issuance and lack of markets for insurance against sunspot shocks, capital is traded partly for risk-sharing purposes. In other words, risk can affect the capital ownership distribution (i.e., σ_R can affect κ). Second, productive heterogeneity permits "misallocation": the capital distribution can affect aggregate output, which translates into capital prices (i.e., κ can affect q).

Of course, all these endogenous variables are determined simultaneously, but it may be helpful to visualize, with the symbols of our model, the logic of multiplicity through the following chain of causality:

$$\sigma_R \Longrightarrow \kappa \Longrightarrow q. \tag{15}$$

Financial frictions modulate the first link ($\sigma_R \Rightarrow \kappa$), while productive heterogeneity modulates the second ($\kappa \Rightarrow q$). The current asset price q then depends on the distribution of future asset prices through σ_R. But what determines σ_R? Nothing, as long as we have both financial frictions and productive heterogeneity. S-BSEs, by removing the ad-hoc restriction that equilibria be Markov in η, remove an artificial anchor for σ_R and allow volatility to be coordination-driven.

Chain (15) also suggests a connection to the "self-fulfilling risk panics" of Bacchetta et al. (2012), further analyzed by Benhabib et al. (2020). Bacchetta et al. (2012) emphasize a negative relationship between asset prices and volatility, effectively collapsing the causal chain in equation (15) to $\sigma_R \Rightarrow q$. But digging deeper, Benhabib et al. (2020)

Then, applying Itô's formula to $q_t = a/\bar{\rho}(\eta_t)$, we obtain $q\sigma_q = -\frac{\rho_e - \rho_h}{\bar{\rho}(\eta)} q\sigma_\eta$, which equals zero.

explain that the key to risk panic equilibria is a causal dependence of the stochastic discount factor (SDF) on asset prices. Bacchetta et al. (2012) obtain a price-SDF link via OLG (see also Farmer, 2018, and Gârleanu and Panageas, 2021); Benhabib et al. (2020) show how a price-SDF link can also arise due to collateral or liquidity constraints. Our results are deeply connected—our price-output link (PO) necessarily implies a price-SDF link—but distinguished by the fact we do not rely on the common multiplicity-inducing features of OLG or ad-hoc borrowing constraints.

Discussion: zero fundamental uncertainty. One of the most striking results we will present is that non-fundamental equilibria can emerge even if $\sigma = 0$. While one could regard this as a simple limiting case as $\sigma \to 0$, some readers may expect a discontinuity in the results when σ literally equals 0. According to this logic, the riskless bond market—with no borrowing frictions—is enough to make financial markets complete when $\sigma = 0$, and so the First Welfare Theorem holds. Under the First Welfare Theorem, we would have generic equilibrium uniqueness.

For our economy without fundamental uncertainty, whether or not the financial market is complete or incomplete is actually *endogenous* and depends on whether asset prices q_t are volatile. Imagine an individual expert operating in a world where $\sigma_q \neq 0$. For him, equity-issuance constraints matter because outside equity is the only way to hedge capital price shocks. As stated by Chiappori and Guesnerie (1991), "the existence of a complete set of initial markets is not enough for having sun-complete markets. Insurance markets against sunspot should also be introduced to allow full insurance."

But is this statement vacuous? Why can't a researcher take any economic model and make its financial markets incomplete by simply conjecturing its asset price dynamics depend on some extrinsic shocks? The answer, suggested above by our benchmarks, is that the structure of most economies rules out any dependence of asset prices on extrinsic shocks. For example, we showed above that q cannot be stochastic with $a_e = a_h$. In such cases, even if extrinsic shocks are strictly speaking uninsurable, markets are *effectively complete* because equilibrium cannot support extrinsic shocks to asset prices.

An alternative line of thinking suggests agents should ignore shocks to q when $\sigma = 0$. Whereas fundamental shocks directly impact capital, extrinsic shocks to prices only affect net worth on paper. For example, consider the following buy-and-hold strategy: borrow using the riskless bond market; use the proceeds to purchase capital; use the cash flows from capital to repay debts over time; ignore any capital price fluctuations and never sell the capital; and consume after all debts are repaid. Assuming no exogenous growth ($g = 0$) for simplicity, this trading strategy has cash flows $\{a_e - r_t b_t\}_{t \geq 0}$, where the debt balance b_t satisfies $db_t = -(a_e - r_t b_t - c_t)dt$ with $b_0 = q_0$. The consumption associated with this strategy is $c_t = \mathbf{1}_{t>\tau} a_e$, where $\tau := \inf\{t : b_t \geq 0\}$ is the time when all debts are repaid. Since this consumption is non-negative, and zero initial investment was made, such a strategy constitutes an arbitrage if it is feasible. Furthermore, if all experts behaved in this way, capital prices would not be volatile or ever fall below their efficient value.

The general problem with such strategies that "ignore market prices" is that debts can become arbitrarily large. When the interest rate rises, the example strategy above produces negative cash flows. Agents must increase their borrowing to continue holding capital. With positive probability, this happens so often and for so long that either debts approach infinity, or default occurs eventually. If markets impose the requirements that net worth remains lower bounded and all debts are eventually repaid, such a strategy is ruled out. This is the content of Appendix A, where we show more generally that a net worth lower bound and a No-Ponzi constraint are equivalent to a solvency constraint $n_t \geq 0$ that rules out all arbitrage trades. In other words, the "ignore market prices" trade is not feasible, which is why sentiment-driven equilibria are not ruled out even when $\sigma = 0$.

2 Sentiment-driven equilibria

We endeavor here to analyze a rich class of equilibria that are not Markov in η, the S-BSEs. Below, we construct and provide detailed characterization of such equilibria.

Because the capital price q is the critical endogenous object (one may think of q as the "co-state" variable), equilibria which are not Markov in η share the defining characteristic that a variety of different asset prices can prevail for a given wealth distribution. Since η captures all fundamental information in this economy, one can think of "sentiment" as responsible for generating the multiplicity of asset prices corresponding to the same η. This is why Definition 2 refers to this class of equilibria as Sentiment-driven BSEs.

The usual approach to constructing sunspot equilibria is to first analyze the non-stochastic equilibria of a model, identify a fundamental indeterminacy, and then add sunspot shocks that essentially randomize over the multiplicity of fundamental equilibria. Before diving into the details, we remark on how and why our construction must differ from this usual approach.

Remark 1 (Stability and multiplicity: connection to literature)**.** *Stability is the critical property enabling sunspots in deterministic dynamical systems. For example, recall the neoclassical* *growth model, in which capital and consumption are the state and co-state variables, respectively, and only one value of initial consumption is consistent with a non-explosive equilibrium. By contrast, OLG versions of the growth model can feature a stable steady state, to which many alternative values of initial consumption would converge (Azariadis, 1981; Cass and Shell, 1983). This literature generates stochastic sunspot equilibria by basically randomizing over the multiplicity of transition paths.*

S-BSEs will also feature a type of stability, whereby for a fixed initial wealth distribution η_0, many initial values of the co-state q_0 can be consistent with non-explosive behavior. But the analogy to deterministic models breaks down in an important sense: Online Appendix D.2 shows that the deterministic steady state of our class of models is only saddle-path stable. In other words, in the deterministic equilibrium of our model, q is pinned down to be a function of η. Given η_0, there is a single transition path to steady state, so we cannot *obtain volatility by randomizing over a multiplicity of deterministic transition paths. For the same reason, we cannot hard-wire arbitrary amounts of volatility for any combination (η, q). Rather, as will soon be clear, our model uniquely determines return volatility $|\sigma_R|$ for each (η, q), reminiscent of the endogenously-determined sentiment distribution in Benhabib et al. (2015).*

2.1 Construction of S-BSEs

Now, we provide a sketch of an explicit construction of an S-BSE. Remember the goal from Lemma 1: given η_t, we want to find $(\mu_{\eta,t}, \sigma_{\eta,t}, \mu_{q,t}, \sigma_{q,t}, q_t, \kappa_t, r_t)$ satisfying equations (PO), (RB), (11), and (13)-(14) for all $t \geq 0$ and such that $\eta_t, \kappa_t \in [0, 1]$.

First, let us count the number of equations and unknowns. The equations are (PO), (RB), (11), (13), and (14)—these are 6 equations (recall that (14) involves two equations) that hold at each time t. Given η_t at a particular point in time, the unknowns are the wealth share dynamics (μ_η, σ_η), the level and dynamics of capital prices (q, μ_q, σ_q), the capital share κ, and the interest rate r—these are 9 unknowns (recall σ_η and σ_q are 2-by-1 vectors). Thus, we seem to have 3 degrees of freedom in constructing equilibrium. A Fundamental Equilibrium, universally studied by the literature, additionally imposes that equilibria be Markov in η. Such a Markovian restriction eliminates the 3 degrees of freedom: applying Itô's formula to $q(\eta)$ delivers 3 additional conditions for σ_q and μ_q. But in an S-BSE, q_t is not simply a function of η_t, so the 3 Itô conditions are dropped. Instead, (σ_q, μ_q) are determined by coordination.

The specific construction we outline below has the property that all equilibrium objects are functions of (η_t, q_t). We are using one degree of freedom in making q a "state variable” in the equilibrium. It will turn out that the relevant domain for (η, q) is

$$\mathcal{D} := \{(\eta, q) : 0 < \eta < 1,\, q^L(\eta) < q \leq q^H(\eta)\}, \tag{16}$$

$$\text{where} \quad q^H(\eta) := a_e/\bar{\rho}(\eta)$$

$$q^L(\eta) := [\eta a_e + (1-\eta)a_h]/\bar{\rho}(\eta).$$

From the price-output relation (PO), notice that q^H corresponds to the capital price when $\kappa = 1$, whereas q^L corresponds to the capital price when $\kappa = \eta$. Equilibrium must have $\kappa \leq 1$ (Lemma 1) and $\kappa > \eta$, the latter because a solution to equation (RB) will not exist otherwise. These restrictions are captured by ensuring (η, q) remains in $\mathcal{D}$. Figure 2 illustrates this set.

The first step in the construction is to reduce the system. Imagine we know the values of $(\eta, q, \sigma_q, \mu_q)$. Price-output relation (PO) determines κ as a function of (η, q) and nothing else, given by

$$\kappa(\eta, q) := \frac{q\bar{\rho}(\eta) - a_h}{a_e - a_h}. \tag{17}$$

Substituting this result for κ, equation (11) then fully determines r. Equations (13)-(14), after plugging in the result for κ, fully determine (σ_η, μ_η). At this point, given (η, q), the remaining unknowns are (σ_q, μ_q) and the remaining equation is (RB).

When capital is efficiently allocated (i.e., $\kappa = 1$), we have $q = q^H(\eta)$ as an explicit function of η. Hence, both σ_q and μ_q are determined by Itô’s formula. But when $q < q^H(\eta)$ (i.e., $\kappa < 1$), we have much more flexibility. Equation (RB) requires

$$|\sigma_R| = \sqrt{\frac{\eta(1-\eta)}{\kappa(\eta, q) - \eta} \frac{a_e - a_h}{q}}, \quad \text{if} \quad q < q^H(\eta). \tag{18}$$

In other words, given (η, q), the level of return volatility is pinned down. But notice that this only restricts the norm of $\sigma_q = \sigma_R - \sigma\binom{1}{0}$, not each of its components separately. We will revisit this indeterminacy in the components of σ_q below.

Similarly, there is as yet no restriction on μ_q despite using all 6 equilibrium equations. All that remains is to show that $(\eta_t, q_t)_{t \geq 0}$ remains in $\mathcal{D}$ almost-surely, and this will provide some mild restrictions on μ_q. The importance of proving that $(\eta_t, q_t)_{t \geq 0}$ remains in $\mathcal{D}$ is to ensure that no optimality or market clearing conditions are violated along the proposed equilibrium path. For example, equation (18) is only well-defined for $\kappa_t > \eta_t$, or equivalently $q_t > q^L(\eta_t)$. Also, Lemma 1 requires $\kappa_t \leq 1$ and $\eta_t \in [0, 1]$, which only hold on $\mathcal{D}$.

To ensure that (η_t, q_t) remains in $\mathcal{D}$, all we need to impose are *boundary conditions* on μ_q. The idea is that (η_t, q_t) can only escape $\mathcal{D}$ through its boundaries, and so μ_q is only restricted near these boundaries. In particular, we only need some force strong enough to push (η_t, q_t) back toward the interior of $\mathcal{D}$. For example, when $q < q^H(\eta)$, we can set μ_q to *any* C^1 function with a boundary condition like the following:

$$\inf_{\eta \in (0,1)} \lim_{q \searrow q^L(\eta)} \left[q - q^L(\eta)\right] \mu_q(\eta, q) = +\infty. \tag{19}$$

Condition (19) says that the drift of q diverges fast enough in order to prevent q from hitting $q^L(\eta)$. This lower boundary, in particular, has the technical issue that σ_q in (18) explodes near it, so the formal proof in Appendix B.3 actually imposes a slightly stronger condition whereby μ_q diverges slightly above $q^L(\eta)$. The conditions at the upper boundary $q^H(\eta)$ are slightly more complicated because the economy is actually allowed to visit this upper boundary—these technical details are all addressed in Appendix B.3. The important takeaway is that equilibrium only imposes boundary conditions on μ_q and leaves it indeterminate in the interior of $\mathcal{D}$.

Methodologically, our formal proof employs stochastic stability theory to show that this construction yields a non-degenerate stationary distribution for $(\eta_t, q_t)_{t \geq 0}$. Appendix B.4 states and proves the appropriate version of a stochastic stability lemma that we use. In particular, the key object is the infinitesimal generator $\mathscr{L}$ of the joint process $(\eta_t, q_t)_{t \geq 0}$ induced by equilibrium. And the key task is to find a positive (Lyapunov) function v, which diverges at the boundaries of $\mathcal{D}$, such that $\mathscr{L}v \to -\infty$ at the boundaries of $\mathcal{D}$. This mathematical condition exactly captures the intuition that boundary conditions on the dynamics are sufficient for stationarity. (The ability to leverage stochastic stability theory to analyze boundary conditions is precisely the simplification offered by our continuous-time setup. That said, Online Appendix F also constructs an example sentiment-driven equilibrium in a discrete-time version of our model.)

Theorem 1 (Existence)**.** *Let Assumption 1 hold. Then, there exists an S-BSE in which $(\eta_t, q_t)_{t \geq 0}$ remains in $\mathcal{D}$ almost-surely and possesses a non-degenerate stationary distribution.*

Theorem 1 is formally proved in Appendix B.3 with an explicit S-BSE construction that addresses several of the minor technical issues raised in the preceding discussion.

Figure 2 plots the admissible set of η and q, along with return volatility $|\sigma_R|$ (indicated by shading) at each point in the space $\mathcal{D}$. For reference, we also place two Fundamental Equilibria (FE): the W-BSE (which is the limiting FE as $\sigma \to 0$) and an FE with $\sigma = 0.1$.

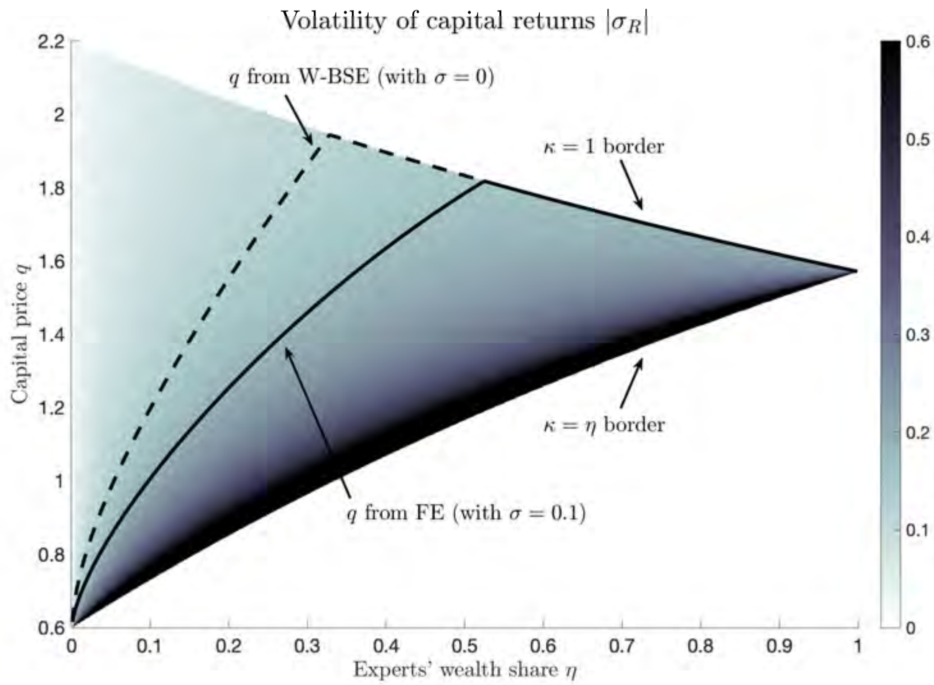

Figure 2: Colormap of volatility $|\sigma_R|$ as a function of (η, q), in the region $\mathcal{D} := \{(\eta, q) : \eta \in (0,1)$ and $\eta a_e + (1-\eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$. Volatility is truncated for aesthetic purposes (because $|\sigma_R| \to \infty$ as $\kappa \to \eta$). For reference, also included are the W-BSE with $\sigma = 0$ and the Fundamental Equilibrium (FE) with $\sigma = 0.1$. Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$.

These equilibria attain only 10-20% volatility, a tiny amount relative to what S-BSEs can do. In fact, we have the following formal result.

Corollary 1 (Volatility indeterminacy)**.** *Given wealth share $\eta \in (0,1)$, let $\mathcal{Q}(\eta)$ denote the set of possible S-BSE values of q, and let $\mathcal{V}(\eta)$ denote the associated set of possible S-BSE values of return variance $|\sigma_R(\eta, q)|^2$. Then, $\mathcal{Q}(\eta)$ is an interval with*

$$\inf \mathcal{Q}(\eta) = q^L(\eta)$$
$$\sup \mathcal{Q}(\eta) = q^H(\eta)$$

and $\mathcal{V}(\eta)$ consists of at most two intervals, with

$$\inf \mathcal{V}(\eta) = \min \left[\eta\bar{\rho}(\eta)\frac{a_e - a_h}{a_e}, \sigma^2(\bar{\rho}(\eta)/\rho_e)^2\right]$$
$$\sup \mathcal{V}(\eta) = +\infty.$$

In an S-BSE, return variance $|\sigma_R|^2$ is pinned down once we know both η and q together; see equation (18). But q can take any value in the interval $\mathcal{Q}(\eta)$ for each η, which implicitly defines a set $\mathcal{V}(\eta)$ of values for $|\sigma_R|^2$. Corollary 1 shows that the range of possible return volatilities is large, in fact unbounded above.

2.2 Economic intuition behind S-BSEs

Next, we explain our S-BSEs more intuitively. We first offer an interpretation of our equilibrium as driven by *uncertainty shocks*. Then, we take a dynamical-system perspective to understand why self-fulfilling volatility is possible.

Uncertainty shocks. Given a wealth distribution η and a level of return volatility $|\sigma_R|$, the capital market is equilibrated at each time via the risk-balance condition (RB) and the price-output relation (PO), restated here for convenience:

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2\right] \tag{RB}$$

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h. \tag{PO}$$

The left panel of Figure 3 shows how the intersection of these two curves determines the capital allocation κ and the capital price q. The downward-sloping risk-balance (RB) can be thought of as experts' relative capital demand: for a fixed level of wealth η and return volatility $|\sigma_R|$, experts will only hold more capital if it is cheaper, thereby offering a higher expected return. (Of course, households also want to buy capital when it is cheaper, but this force is relatively stronger for experts because of their productivity advantage.) The upward-sloping price-output (PO) is a capital supply curve: experts' capital provision raises allocative efficiency and capital valuations.

But whereas η is a state variable that can be rightly treated as fixed in this static sense, return volatility $|\sigma_R|$ is not. The right panel of Figure 3 shows what changes if there is a sudden rise in *fear*, manifested as higher perceived volatility $|\sigma_R|$. Experts, being risk-averse, are less willing to hold capital when volatility is high. This is illustrated as a leftward shift in the risk-balance curve from the solid to the dashed line. After this "fire sale," capital is allocated less efficiently, and asset prices are lower.

So far, nothing rules out this arbitrary rise in fear, and $|\sigma_R|$ appears indeterminate. Mathematically, fixing η, equations (RB) and (PO) constitute two equations in the three unknowns $(\kappa, q, |\sigma_R|)$. The indeterminacy in $|\sigma_R|$ translates into an indeterminacy in q, which can be seen by combining (RB) and (PO) to eliminate κ and obtain the negative

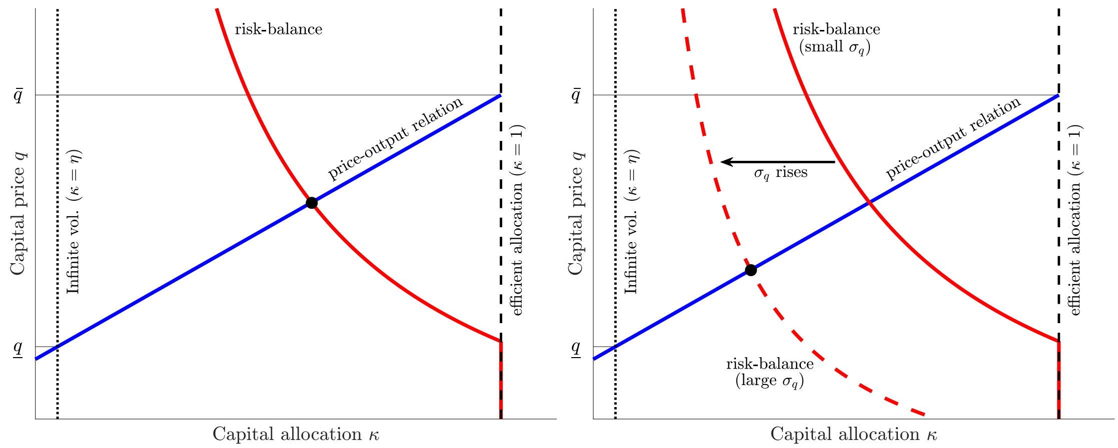

Figure 3: An uncertainty shock. Both panels plot the risk-balance condition (RB) and price-output relation (PO) for a fixed level of $\eta = 0.2$. The horizontal lines labeled $\bar{q}$ and $\underline{q}$ refer to maximal and minimal possible values of the capital price, respectively, corresponding to an efficient capital allocation ($\kappa = 1$) and an infinite-volatility allocation ($\kappa = \eta$). *Left panel:* equilibrium with $|\sigma_R| = 0.13$. *Right panel:* equilibrium after a shift to $|\sigma_R| = 0.20$. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.10$.

price-variance association:

$$|\sigma_R|^2 = \frac{\eta(1-\eta)(a_e - a_h)^2}{q\bar{\rho}(\eta) - \eta a_e - (1-\eta)a_h} \frac{1}{q} \quad \text{when } \kappa < 1. \tag{20}$$

In our construction leading up to Theorem 1, we treated (η, q) as state variables and determined all other equilibrium objects as functions of (η, q). The preceding story about fear suggests that one can also think of S-BSEs as being driven by uncertainty shocks—time-varying beliefs about volatility $|\sigma_R|$—an interpretation which is supported by the one-to-one mapping between q and $|\sigma_R|$ in equation (20).

Bounce-back beliefs and dynamic stability. Based on the short-run conditions (RB) and (PO), equilibrium seems to support a multiplicity of prices q for a fixed η. To understand the long-run beliefs that sustain this multiplicity, it will be helpful to take a dynamical-systems perspective, as suggested in Remark 1.

Let us think of $(\eta_t, q_t)_{t \geq 0}$ as a stochastic dynamical system. As in deterministic dynamical systems, a pair (η_t, q_t) will only be an equilibrium if it does not lead to explosive paths. Thus, beliefs must be such that (η_t, q_t) will mean-revert, or bounce back, from extreme states. What does this entail?

To fix ideas, consider the following explosive path. Suppose a fear shock raises volatility $|\sigma_q|$ and lowers asset prices q. Under higher volatility, any subsequent fear shocks would have a larger direct impact on q, further raise volatility $|\sigma_q|$, and so on, ad infinitum. Thus, with enough such fear shocks, we will have $q \searrow q^L(\eta)$ and $|\sigma_q| \nearrow +\infty$

(see equation (RB) and take $\kappa \to \eta$).

For this fear-driven path to be an equilibrium, agents must believe that, at least eventually, q will recover and $|\sigma_q|$ will fall. In other words, agents must believe μ_q will increase enough to buoy q if prices ever fall too low. This is an example of what we label *bounce-back beliefs*.

Bounce-back beliefs can be justified, because μ_q is not pinned down by any other equilibrium considerations. Importantly, optimal capital holdings are a function of the *risk premium*. This is clearly visible in the optimal portfolio FOCs (7)-(8), where only the spread $\mu_q - r$ appears. Consequently, even given a price q and diffusion σ_q, only the spread $\mu_q - r$ is pinned down in equilibrium, as equation (11) shows; μ_q and r are not separately determined.

Translating agents' bounce-back beliefs into specific mathematical conditions on μ_q is straightforward. Because $(\eta_t, q_t)_{t\geq 0}$ evolves in a diffusive fashion, stability criteria conveniently boil down to boundary behavior of the dynamical system. Imposing conditions on μ_q at the boundaries of the domain $\mathcal{D}$ (i.e., the triangle in Figure 2) is sufficient to ensure a stochastically stable system. For example, we can impose that $\mu_q \to +\infty$ if q falls too low, and $\mu_q \to -\infty$ if q rises too high.

In a sense, the mean-reversion embedded in bounce-back beliefs is precisely the mechanism of self-fulfillment in our model. Fear can push asset prices very low precisely because a recovery is expected. Prices can rise in a sentiment-driven boom precisely because agents know the boom will eventually subside.

2.3 The three indeterminacies

Recall that there are three indeterminacies in S-BSEs:

(i) The level of volatility $|\sigma_R|$ is only pinned down by (η, q) but not by η alone;

(ii) The two components of σ_q are indeterminate given (η, q);

(iii) The drift μ_q is indeterminate given (η, q), except at the boundaries of $\mathcal{D}$.

The first indeterminacy was covered by Corollary 1. These second and third indeterminacies are formalized and explained in the next two corollaries.

Corollary 2 (Decoupling)**.** *The economy can be arbitrarily coupled or decoupled from fundamentals in the following sense. Let $\phi(\eta, q) \in [0, 1]$ be any C^1 function. An equilibrium exists such that when $\kappa < 1$, a fraction $\phi(\eta, q)$ of return variance $|\sigma_R|^2$ is due to the fundamental shock.*

S-BSEs do not pin down the fraction of volatility stemming from the fundamental and sunspot shocks, $Z^{(1)}$ and $Z^{(2)}$, respectively. The reason: when trading, agents only care about total return variance, not its source. Mathematically, the return volatility $|\sigma_R|$ is pinned down in (18), but σ_R itself has two components that can make indeterminate contributions to equilibrium. Asset prices and economic activity can be either closely linked to fundamentals, or completely decoupled from them, and this decoupling can be time-varying in arbitrary ways. Nevertheless, Section 3 presents perhaps the most natural example of an S-BSE, in which volatility and fundamentals must decouple as total volatility rises.

The theoretical possibility that $\phi = 1$ in Corollary 2 helps illustrate that our multiplicity does not require any extrinsic force. Even with $Z^{(2)}$ playing no role, it is possible for agents to coordinate purely on endogenous objects in a self-fulfilling way.

Corollary 3 (Drift indeterminacy)**.** *The economy can feature any degree of persistence or transience in the following sense. Let $m(\eta, q)$ be any C^1 function. An equilibrium exists with $\mathbb{P}[\mu_{q,t} = m(\eta_t, q_t) \mid \kappa_t < 1]$ arbitrarily close to one. Furthermore, the inefficiency probability $\mathbb{P}[\kappa_t < 1]$ can take any value between zero and one.*

As suggested earlier, the proof of Theorem 1 only imposes boundary conditions on μ_q, allowing almost any behavior in the interior of the state space. For example, asset prices could almost always behave like a random walk (corresponding to $\mu_q \approx 0$ in the interior), with just enough mean-reversion in extreme states to keep things stationary; in such a design, extreme states become arbitrarily close to reflecting boundaries. Alternatively, fluctuations could be much more transitory in nature. In Section 3, we harness the indeterminacy in μ_q to address predictability of busts and speed of recovery.

Remark 2 (Dynamics and indeterminacies)**.** *Indeterminacies arise because beliefs about capital price dynamics influence real outcomes such as capital allocation. In this model we have two prices—capital price q and interest rate r—and two (non-redundant) market clearing conditions. However, we need to solve not only for current prices but also for future capital price behavior, which is summarized by the diffusion $\sigma_q \in \mathbb{R}^2$ and drift $\mu_q \in \mathbb{R}$ terms.*[18] *Optimality imposes a tight (negative) link between q and $|\sigma_q|$, while long-run stability imposes some mild conditions on μ_q in extreme states. Besides those restrictions, (σ_q, μ_q) are indeterminate.*

[18]The logic in a discrete time model is analogous: the indeterminacies will be associated to the distribution of capital price tomorrow. This distribution is an infinite dimensional object, which makes it challenging to prove the existence of our sentiment-driven equilibria in discrete time models. Online Appendix F provides a discrete-time example of a sentiment-driven equilibrium by specializing to a binomial tree for capital prices. We purposely design this binomial example with a trading interval Δ such that our Brownian model is recovered as $\Delta \to 0$.

2.4 Manipulating beliefs with policy

At this point, it should be clear that beliefs about the future behavior of asset prices are not determined by equilibrium conditions alone but also by coordination. Here, we ask: assuming policymakers can manipulate these beliefs, in what way could outcomes improve? Future research might investigate how policies affect beliefs, which policies are most effective in doing so, and which types of commitment devices are required.

For simplicity, we consider a policymaker that pledges to support asset prices at some lower level $\underline{q}(\eta)$. One could think about policy pledges to make future asset purchases, because of the intuitive idea that asset purchases introduce demand pressure that pushes up prices. But we do not explicitly model any intervention, and instead assume that the policymaker can convince agents that asset prices will be supported. One can interpret this as sufficient credibility attached to the policymaker's ability to affect asset prices.

To be concrete, suppose agents perceive $\underline{q}(\eta)$ as a *reflecting boundary* for asset prices, i.e., beliefs are such that $q_t \geq \underline{q}(\eta_t)$. By rational expectations, prices will in fact always obey this lower bound, but no intervention need occur. Instead, the policy promise induces self-fulfilling dynamics: agents believe prices will be reflected at $\underline{q}(\eta)$ and trade capital to make it so. In this sense, the reflecting boundary is an extreme case of the bounce-back beliefs described earlier.

Reflection introduces a new term to price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t} \cdot dZ_t + dP_t],$$

where P is the barrier process that increases only to keep $q_t \geq \underline{q}(\eta_t)$. Absence of arbitrage requires the riskless bond return to be $r_t dt + dP_t$, such that the excess return on capital is unaffected by dP_t (c.f., Appendix B of Karatzas and Shreve, 1998). Consequently, agents' FOCs on capital holding remain unaffected, and both the risk-balance condition (RB) and equation (11) for r_t still hold.

Finally, the policy has no impact on the dynamics of η_t, which still take the diffusive form (12). Indeed, excess capital returns feature no dP_t component, so expert and household return-on-wealth contain identical contributions from dP_t, implying $d\eta_t$ contains no dP_t term. This is a clear indication that our policy has no "fundamental impact." He and Krishnamurthy (2013), by contrast, analyze policies with only fundamental effects and no belief effects (they study borrowing subsidies, asset purchases, and equity injections). For them, policy effects on wealth distribution dynamics are critical.

We have thus constructed an equilibrium with $q_t \geq \underline{q}(\eta_t)$ at all times, for an arbitrarily designed lower boundary $\underline{q}$. For example, $\underline{q}(\eta)$ could be designed to keep volatility below some threshold, e.g., $|\sigma_R| \leq \Sigma^*$, and volatility would sometimes approach but never exceed that threshold. Away from the boundary $\underline{q}(\eta)$, equilibrium is identical to the one constructed in Theorem 1; this property is an artifact of log utility, for which only the local dynamics of asset prices matter. Policies that truncate the lower tail of asset prices are clearly helpful (and in fact, policy ideally wants to keep κ as close to 1 as possible), but with log utility the truncation is the entirety of their impact. With more general utility functions, how much can promises to remove tomorrow's tail risk affect today's asset price dynamics by "calming the market?" We leave this question for future research.

3 Resolving puzzles with sentiment

We have just demonstrated that sunspot equilibria, which are endemic to this class of models, in principle can support rich dynamics. Now, we solve some concrete examples to illustrate several substantive results along these lines.

3.1 Explicit construction with a sentiment state variable

In contrast to the previous section, where (η, q) was the state variable, here we implement our sentiment-driven equilibria with an auxiliary state variable s and with q as a function of η and s. Being explicit about a sentiment state variable is useful for several reasons. First, this equilibrium construction will be pedagogically more familiar to the literature on sunspots. Second, the sentiment state dynamics can be modeled as locally uncorrelated with fundamental shocks, which brings some clarity. Third, this setting happens to facilitate building sunspot equilibria in which experts fully de-lever as their wealth shrinks, i.e., $\kappa \to 0$ as $\eta \to 0$, for which there are natural justifications.

Let s be a pure sunspot that is irrelevant to economic fundamentals and loads on only the second shock (recall $Z^{(1)}$ affects capital and $Z^{(2)}$ does not):

$$ds_t = \mu_{s,t} dt + \sigma_{s,t} \binom{0}{1} \cdot dZ_t, \quad s_t \in \mathcal{S}. \tag{21}$$

(Online Appendix D.4 solves additional examples with sentiment correlated to fundamentals, i.e., with $ds = \mu_s dt + \sigma_s^{(1)} dZ^{(1)} + \sigma_s^{(2)} dZ^{(2)}$.) We will also find some use in introducing auxiliary state variables that can affect the drift $\mu_{s,t}$. This is possible to do in a very flexible way, due to the drift indeterminacy result of Corollary 3. Let $x_t \in \mathcal{X}$ be an arbitrary bounded diffusion,

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t) \cdot dZ_t,$$

which (only) affects the sentiment drift, through $\mu_{s,t} = \mu_s(\eta_t, s_t, x_t)$.

Definition 3. A *Markov S-BSE* in states $(\eta, s, x) \in (0,1) \times \mathcal{S} \times \mathcal{X}$ consists of functions $(q, \kappa, r, \sigma_\eta, \mu_\eta, \sigma_s) : (0,1) \times \mathcal{S} \mapsto \mathbb{R}$, and $\mu_s : (0,1) \times \mathcal{S} \times \mathcal{X} \mapsto \mathbb{R}$, all C^2 almost-everywhere, such that the process $(\eta_t, q(\eta_t, s_t), \kappa(\eta_t, s_t), r(\eta_t, s_t))_{t \geq 0}$ is an S-BSE.

Remark 3 (Endogenous sentiment dynamics)**.** *Note that the statement of Definition 3 allows (σ_s, μ_s) to be endogenous, in the sense that they could depend on the wealth distribution η. Our examples in this section purposefully entertain this endogeneity, partly because we think of this as the more interesting and realistic situation. Why? As shown in Section 2, dynamics depend explicitly on q in an S-BSE. Thus, it is completely sensible for agents in our S-BSEs to use asset prices directly in forecasting; in particular, sentiment dynamics (σ_s, μ_s)—which are nothing but belief dynamics—themselves should condition on q. But q will depend on both s and η, implying sentiment dynamics (σ_s, μ_s) depend on η too, through q. That said, Online Appendix D.5 verifies that similar types of sunspot equilibria can be constructed with exogenous sentiment dynamics, i.e., (σ_s, μ_s) are only functions of s, not η.*

The Markov assumption in Definition 3 allows us to specialize equilibrium conditions. By applying Itô's formula to $q(\eta, s)$, we obtain the capital price volatility σ_q in terms of σ_η, namely

$$q\sigma_q = \sigma_\eta \partial_\eta q + \sigma_s \partial_s q.$$

From equation (14), we also have σ_η in terms of σ_q. Solving this two-way feedback, we obtain

$$\sigma_q = \frac{\binom{1}{0}(\kappa - \eta)\sigma \partial_\eta \log q + \binom{0}{1}\sigma_s \partial_s \log q}{1 - (\kappa - \eta)\partial_\eta \log q}. \tag{22}$$

Using (22) in (RB), we obtain the following equation linking capital prices, the capital distribution, and sentiment volatility:

$$0 = \min\Big[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\Big(\frac{\sigma^2 + (\sigma_s \partial_s \log q)^2}{(1 - (\kappa - \eta)\partial_\eta \log q)^2}\Big)\Big]. \tag{23}$$

Our strategy to find a Markov S-BSE is to guess a capital price function $q(\eta, s)$ and then use equation (23) to "back out" the sunspot volatility σ_s that justifies it. We will perform a construction such that sunspots only increase volatility relative to the fundamental equilibrium, to highlight their potential for resolving puzzles. For this reason, we sometimes refer to s as *rational pessimism*.

More specifically, suppose a fundamental equilibrium, where sunspots do not matter, exists with equilibrium capital price q^{FE} (see Online Appendix E for details on the fundamental equilibria). We will think of q^{FE} as the "best-case" capital price, because despite featuring amplification, q^{FE} inherits no sunspot volatility. Conversely, think of the capital price q^{∞} associated to an infinite-volatility equilibrium as the "worst-case" capital price (substitute $|\sigma_R| \to \infty$ into (20) to find $q^{\infty} := \frac{\eta a_e + (1-\eta) a_h}{\bar{\rho}}$).

Our strategy is essentially to treat the sentiment variable s as a device to shift continuously between the best-case q^{FE} and the worst-case q^{∞}. Mathematically, we conjecture a capital price that is approximately a weighted average of q^{FE} and q^{∞}, with weights $1-s$ and s. The novelty of our approach here is to then use equation (23) to solve for sunspot volatility σ_s, which will generically depend on experts' wealth share η. In terms of Figure 2, the economy will live in the sub-region bounded by the solid FE line and the $\kappa = \eta$ border (and notice this implies the full-deleveraging condition $\kappa \to 0$ as $\eta \to 0$). In the proposition below, we verify that such a construction is indeed an equilibrium.

Proposition 1. *Let Assumption 1 hold, and assume a fundamental equilibrium exists for each $\sigma \geq 0$ small enough. Then, for all $\sigma \geq 0$ small enough, there exists a Markov S-BSE with capital prices arbitrarily close to $(1-s)q^{FE}(\eta) + sq^{\infty}(\eta)$. In this equilibrium, μ_s is indeterminate except near the boundaries of $(0,1) \times \mathcal{X} \times \mathcal{S}$.*

We construct a numerical example closely following Proposition 1, which we will use in subsequent sections. The left panel of Figure 4 shows the capital price function. A rise in rational pessimism s reduces the capital price q, independently of wealth share η (although η will also endogenously respond to s-shocks).

The middle panel of Figure 4 displays capital return volatility, which can be substantially greater than in the fundamental equilibrium. Implied by capital return volatility is an underlying sunspot shock size σ_s, which is displayed in the right panel of Figure 4. Sunspot dynamics become more volatile both as experts become poor (η shrinks) and as the economy approaches the worst-case equilibrium (s rises). The dependence of σ_s on η is the notion of endogenous beliefs that can occur in S-BSEs.

3.2 Non-fundamental crises and large amplification

We now show how our model with sentiment shocks can help resolve some empirical issues related to financial crises and recoveries.

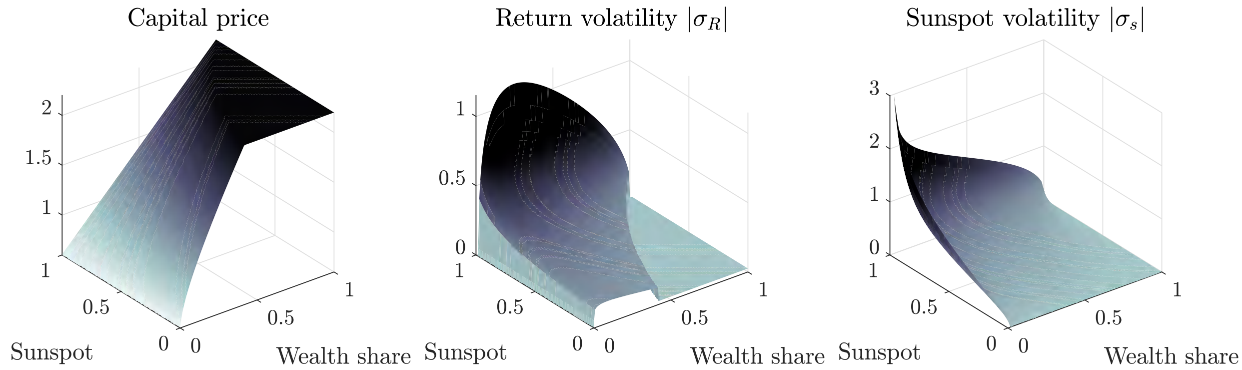

Figure 4: Capital price q, volatility of capital returns $|\sigma_R|$, and sunspot shock volatility σ_s. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$.

First, Figure 5 compares impulse responses to a large negative balance-sheet shock (i.e., decline in η) versus a wave of pessimism (i.e., increase in s). The shock sizes are chosen so that the initial drop in capital price $q_0 - q_{0-}$ is roughly the same. "Balance-sheet recessions" (decline in η) feature a modest increase in volatility followed by relatively slow recoveries, as experts can only rebuild their balance sheets by earning profits over time. By contrast, "pessimism crises" (increase in s) feature large temporary volatility spikes and can have accelerated recoveries (depending on the choice of μ_s). The dynamics after a pessimism shock—both the rise in volatility and speed of recovery—are closer to empirical evidence.[19] Our results on recovery speeds are related to Maxted (2023), who shows how extrapolative beliefs can help this class of models match such evidence, but with our rational sentiment in place of his behavioral sentiment.

To establish some more confidence in these results, we present the following two propositions which together show that amplification can be arbitrarily high (Proposition 2) as long as sentiment shocks are the source (Proposition 3). Given the literature's struggle to identify a "smoking gun" (e.g., TFP shocks, capital efficiency shocks) for financial crises, we view this set of results as a helpful insight. The importance of sentiment s, relative to experts' wealth share η, also echoes the empirical results suggesting financial crises are not associated with pre-crisis levels of bank capital (Jordà et al., 2021).

Proposition 2 (Arbitrary volatility)**.** *Given a target variance $\Sigma^* > 0$ and any parameters*

[19] During the 2008 financial crisis and 2020 COVID-19 episode in the US, implied volatility from option markets spiked by magnitudes on the order of 60%. For a rough idea of what the data says about crisis recoveries, see Jordà et al. (2013) and Reinhart and Rogoff (2014) for output, and see Muir (2017) and Krishnamurthy and Muir (2017) for credit spreads and stock prices. Across these many measures, and using broad-based international panels, crisis recovery times tend to range from 4-6 years on average.

Of course, note that η responds to s-shocks, i.e., σ_η has a non-zero second component. Thus, a true sentiment-driven crisis features dynamics that are a blend of the two IRFs in Figure 5. Figure 5 shows a pure shock to s, without the endogenous co-movement in η, for theoretical clarity.

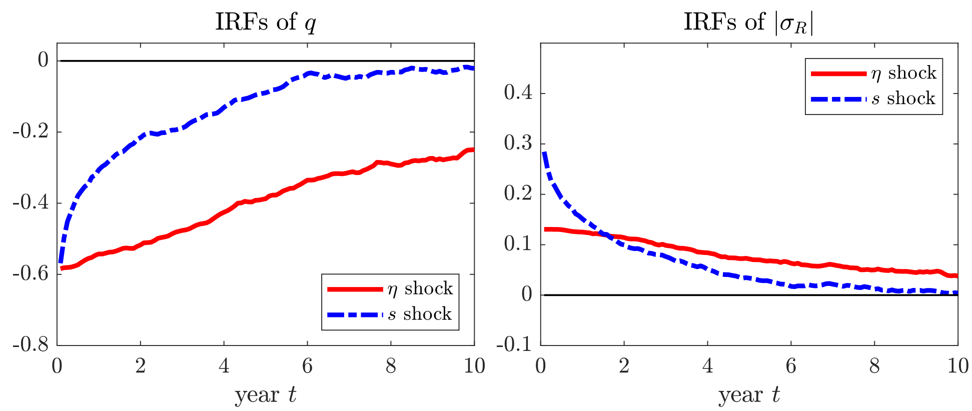

Figure 5: Bust IRFs of capital price q and return volatility $|\sigma_R|$. The IRFs labeled "η shock" are responses to a decrease in η from $\eta_{0-} = 0.5$ to $\eta_0 = 0.25$, holding s_0 fixed at 0.1. The IRFs labeled "s shock" are responses to an increase in s from $s_{0-} = 0.1$ to $s_0 = 0.8$, holding η_0 fixed at 0.5. These shock sizes are chosen such that the initial response of q are approximately equal. Note that η_0 would respond to an "s shock," since σ_η has a non-zero second element, but we keep it fixed here. IRFs are computed as averages across 500 simulations at daily frequency, with the outcomes above then averaged to the monthly level. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$. In this example, we set the sunspot drift $\mu_s = 0.0002s^{-1.5} - 0.0002(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$. This choice ensures $s_t \in (0, s_{\max})$ with probability 1.

satisfying the assumptions of Proposition 1, there exists a Markov S-BSE with stationary average return variance exceeding the target, i.e., $\mathbb{E}[|\sigma_R|^2] > \Sigma^*$.

Proposition 3 (Decoupling)**.** *In the Markov S-BSEs of Proposition 1, both the fraction of return volatility due to sentiments* $|\binom{0}{1} \cdot \sigma_R| / |\sigma_R|$ *and total return volatility* $|\sigma_R|$ *increase with* s.

3.3 Booms predict crises

We now use the same framework to cast light on empirical findings suggesting that financial crises are predictable, in particular by large credit and asset price booms (Reinhart and Rogoff, 2009; Jordà et al., 2011, 2013, 2015a,b; Mian et al., 2017) that feature below-average credit spreads (Krishnamurthy and Muir, 2017; López-Salido et al., 2017; Baron and Xiong, 2017).

To do this, we make use of the auxiliary variable x that can affect the sentiment drift. Following some models of extrapolative beliefs (Barberis et al., 2015; Maxted, 2023), define an exponentially-declining weighted average of sentiment shocks:

$$x_t := x_0 + \sigma_x \int_0^t e^{-\beta_x (t-u)} dZ_u^{(2)}. \tag{24}$$

The variable x measures the stock of past pessimism. Assume the drift of s depends on

x via

$$\mu_{s,t} = b_x x_t + \hat{\mu}_s(s_t) \quad \text{with} \quad b_x \leq 0.$$

Similar to Section 3.2, the term $\hat{\mu}_s$ will be designed to induce stationarity in s_t. The new term $b_x x$ induces the following dynamics: after a wave of optimism ($dZ_t^{(2)} < 0$), s_t and x_t will be low, but this raises $\mu_{s,t}$ and shifts up the conditional distributions of future pessimism s_{t+h}. If the constant b_x is large enough, the shift can generate dynamics reminiscent of "overshooting," in which an optimism-driven boom raises bust probabilities. Differently from the extrapolative belief literature, the beliefs implied by these sentiment dynamics are completely rational.

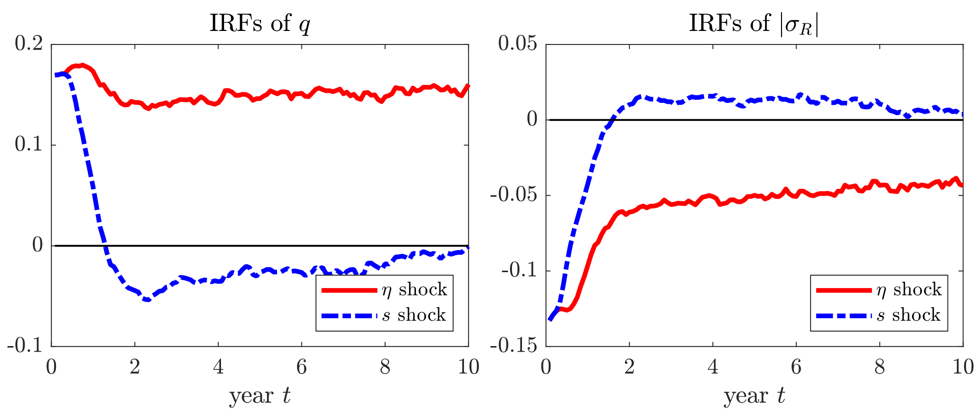

Figure 6: Boom IRFs of capital price q and return volatility $|\sigma_R|$. The IRFs labeled "η shock" are responses to an increase in η from $\eta_{0-} = 0.5$ to $\eta_0 = 0.7$, holding s_0 fixed at 0.4. The IRFs labeled "s shock" are responses to a decrease in s from $s_{0-} = 0.4$ to $s_0 = 0.1$, holding η_0 fixed at 0.5. These shock sizes are chosen such that the initial response of q are approximately equal. Note that η_0 would respond to an "s shock," since σ_η has a non-zero second element, but we keep it fixed here. IRFs are computed as averages across 2000 simulations at daily frequency, with the outcomes above then averaged to the monthly level. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$. In this example, we set the sunspot drift $\mu_s = b_x x + 0.0001 s^{-1.5} - 0.0001(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$, $b_x = -25$, $\beta_x = 0.1$, and $\sigma_x = 0.025$. The parameters (β_x, σ_x) are approximately the values used for the mean-reversion and volatility of the diagnostic belief process in Maxted (2023).

Figure 6 displays IRFs consistent with this overshooting logic. Sentiment-driven booms predict future busts: an optimism shock raises asset prices and lowers volatility for 1-2 years, but predicts lower prices and higher volatility afterward. (This number of years depends on b_x.) By contrast, a boom driven by expert wealth counterfactually predicts high prices, lower volatility, and lower fragility at all horizons.

To connect to the empirical literature, we conduct an event study in Figure 7. We simulate our model (which thus features contributions from both fundamental and sunspot shocks), identify crises in the simulated data, and plot average outcomes in the years before and after crisis. Crises are identified as the worst 3rd percentile of yearly output drops; other tail outcomes will produce similar graphs. We see that conditions are improving up to 1 year before the crisis, with risk premia below average and *declining*. The crisis emerges suddenly and features spikes in all variables. Although we do not report it here, such dynamics cannot be produced in the non-sunspot equilibria of the model.

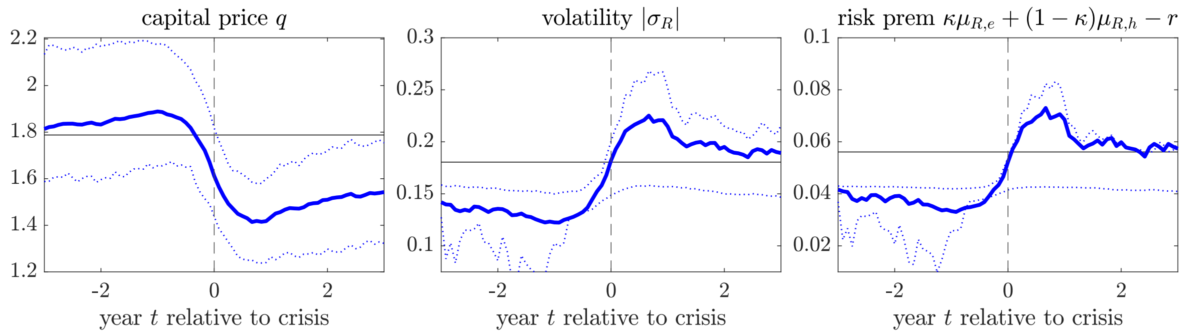

Figure 7: Event studies around financial crises. Crises are defined as the bottom 3rd percentile of year-to-year log output declines. Data is generated via a 10,000 year simulation at the daily frequency, with the outcomes above then averaged to the monthly level. The solid blue line is the mean path, and the dotted blue lines represent the 25th and 75th percentiles. The thin horizontal line represents the unconditional average. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$. In this example, we set the sunspot drift $\mu_s = b_x x + 0.0002 s^{-1.5} - 0.0002(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$, $b_x = -25$, $\beta_x = 0.1$, and $\sigma_x = 0.025$. The parameters (β_x, σ_x) are approximately the values used for the mean-reversion and volatility of the diagnostic belief process in Maxted (2023).

3.4 Sentiment-based jumps

In our final exercise, we show how similar substantive results—large and sudden crises that are preceded by booms featuring low volatility and risk premia—also hold in alternative equilibria with sentiment-based jumps. There are three reasons why jump-type fluctuations are an interesting avenue to explore vis á vis the puzzles in this literature. First, jumps are large and sudden by definition, helping resolve the trouble with limited amplification. Second, the larger jumps that characterize a financial crisis can only happen from a moderate or good state that characterizes a boom. Third, introducing jumps reveals an additional indeterminacy that can be useful in exacerbating the previous point, namely the jump probability can be coordinated on in a way that makes jumps more likely in good times.

Consider a broader class of solutions for the baseline model where capital price can also respond to an extrinsic jump shock, i.e.,

$$\frac{dq_t}{q_{t-}} = \mu_{q,t-} dt + \sigma_{q,t-} \cdot dZ_t - \ell_{q,t-} dJ_t,$$

where J is a Poisson process with intensity λ. For simplicity, we restrict attention to equilibria where the jump size ℓ_q is pre-determined, in particular a function of (η, q), and we focus on adverse jumps with $\ell_q \geq 0$.

We sketch the solution of a jumpy equilibrium (with more details in Online Appendix C.3). The risk-balance condition (RB) is modified to read

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\left(|\sigma_R|^2 + \frac{\lambda \ell_q^2}{\left(1 - \frac{\kappa}{\eta}\ell_q\right)\left(1 - \frac{1-\kappa}{1-\eta}\ell_q\right)}\right)\right]. \tag{RBJ}$$

The additional terms involving ℓ_q arise because there is a jump risk premium. The price-output relation remains (PO).

By adding a new source of risk, we have an additional degree of freedom. The risk-balance condition disciplines overall risk—the term in parentheses of (RBJ) is pinned down given (η, q)—but the split between the Brownian and Poisson shocks is indeterminate. We have tremendous flexibility in our choice of ℓ_q.

It is easy to avoid stability concerns: just set $\ell_q = 0$ near the boundaries of the equilibrium region (i.e., the triangle in Figure 2). Doing this, the stability analysis remains unchanged from Theorem 1, since near the boundaries the economy behaves as if it is only hit by Brownian shocks.

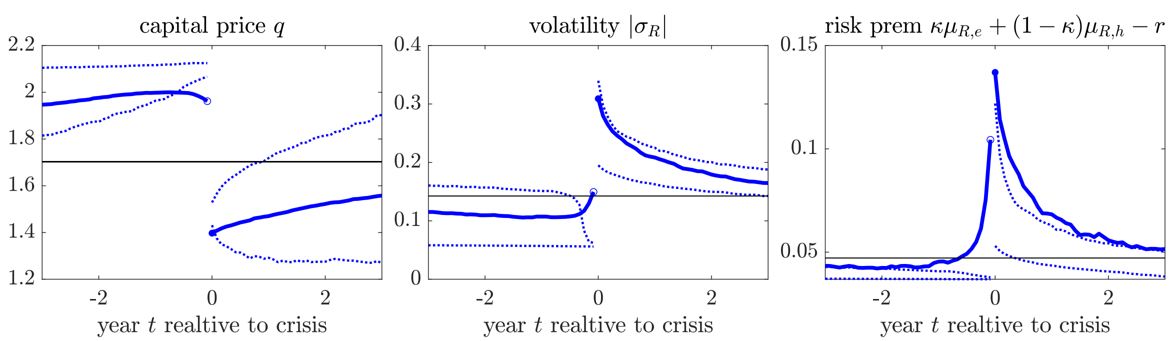

Figure 8: Event studies around financial crises in the jump-diffusion model. Crises are defined as the bottom 3rd percentile of year-to-year log output declines. Data is generated via a 100,000 year simulation at monthly frequency. The solid blue line is the mean, and dotted blue lines represent 25th and 75th percentiles. The horizontal black line is the unconditional mean. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$. We reflect (η, q) near boundaries of $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1-\eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$. Away from the boundaries, we set $\mu_q = 0.1(q^{\text{mid}}(\eta) - q)$, where q^{mid} corresponds to $\kappa(q^{\text{mid}}, \eta) = 0.8$.

Figure 8 shows a financial crisis event study from simulated data of the jump model.

We make the following choice for jump sizes

$$\ell_q = \begin{cases} 0.95\ell_q^{\max}, & \text{if } \kappa > 0.9 \text{ and } 0.95\ell_q^{\max} > 0.2 \\ 0, & \text{otherwise,} \end{cases}$$

where $\ell_q^{\max}$ is the maximum allowable jump consistent with equilibrium (derived in the appendix). Thus, we focus attention on an economy with large jumps (greater than 20%) that are additionally only realized from high-κ states.[20]

Because we focus on large jumps and only allow them in high-κ states, crises tend to arrive after a sequence of positive fundamental Brownian shocks. Accordingly, in the years before the crisis, asset prices are high, and both volatility and risk premia are below their usual level. Similar to Figure 7, volatility and risk premia tend to decline in the years prior to crisis. Crises arrive suddenly and generate large movements in observables, because simulated crises often coincide with realizations of a jump.

4 Conclusion

We have shown that macroeconomic models with financial frictions may inherently permit sunspot volatility. The types of models we study are extremely common in macroeconomics, so this phenomenon cannot be ignored.

On the bright side, our paper demonstrates how a fully-rational notion of "sentiments" can be a powerful input into macro-finance dynamics. Time-varying uncertainty drives all dynamics in our sentiment-driven fluctuations. Sharp volatility spikes and belief-driven boom-bust cycles are among the many interesting possibilities raised by our framework. While ours is not a full-blown quantitative analysis, we aim to show that rational sentiment can help on these dimensions.

On the hazier side, our results suggest a modicum of caution. Many researchers employ numerical techniques to solve and analyze DSGE models that are built upon the core assumptions in our paper—these procedures implicitly select an equilibrium, without any explicit justification. In Online Appendix D, we have considered some simple refinements, based on small amounts of idiosyncratic risk and limited commitment, but these refinements only stipulate the full-deleveraging boundary condition $\lim_{\eta \to 0} \kappa = 0$,

[20] In unreported results, we also solved an example without the $\kappa > 0.9$ restriction, i.e., where we set $\ell_q = 0.95\ell_q^{\max}\mathbf{1}_{\{0.95\ell_q^{\max} > 0.2\}}$. The results are similar to Figure 8—because large jumps still tend to happen from good states—but slightly muted.

which barely trims indeterminacies. A deeper analysis of refinements, perhaps leveraging global-games approaches or adaptive learning, still remains to be done.

What about policy?[21] Caveated by the need for further study on refinements, we can offer some initial thoughts. Some traditional policies become less effective in sunspot equilibria. For example, deposit insurance has less bite because run-like behavior can occur solely due to fire-sale coordination, i.e., on the asset side rather than the liability side. Sunspot equilibria also decouple financial crises from bank balance sheets and wealth, which defangs capital requirements, bailouts, and the like. On the other hand, policies that manipulate beliefs can be effective (Section 2.4 briefly investigates this possibility). Future research might better explain which policy designs have the power to manipulate beliefs in this way. Given the framework we study relies on fire sales, asset purchases (or future commitments to them) are one interesting candidate.

References

S Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684, 1994.

Costas Azariadis. Self-fulfilling prophecies. *Journal of Economic Theory*, 25(3):380–396, 1981.

Costas Azariadis and Allan Drazen. Threshold externalities in economic development. *The Quarterly Journal of Economics*, 105(2):501–526, 1990.

Costas Azariadis and Bruce Smith. Financial intermediation and regime switching in business cycles. *The American Economic Review*, pages 516–536, 1998.

Costas Azariadis, Leo Kaas, and Yi Wen. Self-fulfilling credit cycles. *The Review of Economic Studies*, 83(4):1364–1405, 2016.

Philippe Bacchetta, Cédric Tille, and Eric Van Wincoop. Self-fulfilling risk panics. *The American Economic Review*, 102(7):3674–3700, 2012.

Nicholas Barberis, Robin Greenwood, Lawrence Jin, and Andrei Shleifer. X-capm: An extrapolative capital asset pricing model. *Journal of Financial Economics*, 115(1):1–24, 2015.

Matthew Baron and Wei Xiong. Credit expansion and neglected crash risk. *The Quarterly Journal of Economics*, 132(2):713–764, 2017.

[21]Many studies in the recent literature have moved toward policy analysis (Phelan, 2016; Dávila and Korinek, 2018; Drechsler et al., 2018; Di Tella, 2019; Silva, 2017; Elenev et al., 2021; Begenau, 2020; Begenau and Landvoigt, 2021; Klimenko et al., 2016).

Juliane Begenau. Capital requirements, risk choice, and liquidity provision in a business-cycle model. *Journal of Financial Economics*, 136(2):355–378, 2020.

Juliane Begenau and Tim Landvoigt. Financial regulation in a quantitative model of the modern banking system. Unpublished working paper, 2021.

Jess Benhabib and Pengfei Wang. Private information and sunspots in sequential asset markets. *Journal of Economic Theory*, 158:558–584, 2015.

Jess Benhabib, Pengfei Wang, and Yi Wen. Sentiments and aggregate demand fluctuations. *Econometrica*, 83(2):549–585, 2015.

Jess Benhabib, Xuewen Liu, and Pengfei Wang. Sentiments, financial markets, and macroeconomic fluctuations. *Journal of Financial Economics*, 120(2):420–443, 2016.

Jess Benhabib, Xuewen Liu, and Pengfei Wang. Financial markets, the real economy, and self-fulfilling uncertainties. *The Journal of Finance*, 74(3):1503–1557, 2019.

Jess Benhabib, Xuewen Liu, and Pengfei Wang. Self-fulfilling risk panics: An expected utility framework. Unpublished working paper, 2020.

Markus K Brunnermeier and Yuliy Sannikov. A macroeconomic model with a financial sector. *The American Economic Review*, 104(2):379–421, 2014.

Markus K Brunnermeier and Yuliy Sannikov. Macro, money, and finance: A continuous-time approach. *Handbook of Macroeconomics*, 2:1497–1545, 2016.

Ricardo J Caballero and Alp Simsek. A risk-centric model of demand recessions and speculation. *The Quarterly Journal of Economics*, 135(3):1493–1566, 2020.

David Cass and Karl Shell. Do sunspots matter? *Journal of Political Economy*, 91(2): 193–227, 1983.

Pierre Andre Chiappori and Roger Guesnerie. Sunspot equilibria in sequential markets models. *Handbook of Mathematical Economics*, 4:1683–1762, 1991.

Eduardo Dávila and Anton Korinek. Pecuniary externalities in economies with financial frictions. *The Review of Economic Studies*, 85(1):352–395, 2018.

Sebastian Di Tella. Uncertainty shocks and balance sheet recessions. *Journal of Political Economy*, 125(6):2038–2081, 2017.

Sebastian Di Tella. Optimal regulation of financial intermediaries. *The American Economic Review*, 109(1):271–313, 2019.

Itamar Drechsler, Alexi Savov, and Philipp Schnabl. A model of monetary policy and risk premia. *The Journal of Finance*, 73(1):317–373, 2018.

Philip H Dybvig and Chi-Fu Huang. Nonnegative wealth, absence of arbitrage, and feasible consumption plans. *The Review of Financial Studies*, 1(4):377–401, 1988.

Vadim Elenev, Tim Landvoigt, and Stijn Van Nieuwerburgh. A macroeconomic model with financially constrained producers and intermediaries. *Econometrica*, 89(3):1361–1418, 2021.

Emmanuel Farhi and Jean Tirole. Bubbly liquidity. *The Review of Economic Studies*, 79(2): 678–706, 2012.

Roger EA Farmer. Pricing assets in a perpetual youth model. *Review of Economic Dynamics*, 30:106–124, 2018.

Roger EA Farmer and Jess Benhabib. Indeterminacy and increasing returns. *Journal of Economic Theory*, 63:19–41, 1994.

David M Frankel, Stephen Morris, and Ady Pauzner. Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory*, 108(1):1–44, 2003.

Nicolae Gârleanu and Stavros Panageas. What to expect when everyone is expecting: Self-fulfilling expectations and asset-pricing puzzles. *Journal of Financial Economics*, 140(1):54–73, 2021.

Mark Gertler and Nobuhiro Kiyotaki. Financial intermediation and credit policy in business cycle analysis. *Handbook of Monetary Economics*, 3(3):547–599, 2010.

Mark Gertler and Nobuhiro Kiyotaki. Banking, liquidity, and bank runs in an infinite horizon economy. *The American Economic Review*, 105(7):2011–43, 2015.

Mark Gertler, Nobuhiro Kiyotaki, and Andrea Prestipino. A macroeconomic model with financial panics. *The Review of Economic Studies*, 87(1):240–288, 2020.

Chao Gu, Fabrizio Mattesini, Cyril Monnet, and Randall Wright. Endogenous credit cycles. *Journal of Political Economy*, 121(5):940–965, 2013.

John C Harsanyi and Reinhard Selten. A general theory of equilibrium selection in games. *MIT Press Books*, 1, 1988.

Zhiguo He and Arvind Krishnamurthy. A model of capital and crises. *The Review of Economic Studies*, 79(2):735–777, 2012.

Zhiguo He and Arvind Krishnamurthy. Intermediary asset pricing. *The American Economic Review*, 103(2):732–70, 2013.

Zhiguo He and Arvind Krishnamurthy. A macroeconomic framework for quantifying systemic risk. *American Economic Journal: Macroeconomics*, 11(4):1–37, 2019.

Julien Hugonnier. Rational asset pricing bubbles and portfolio constraints. *Journal of Economic Theory*, 147(6):2260–2302, 2012.

Òscar Jordà, Moritz Schularick, and Alan M Taylor. Financial crises, credit booms, and external imbalances: 140 years of lessons. *IMF Economic Review*, 59(2):340–378, 2011.

Òscar Jordà, Moritz Schularick, and Alan M Taylor. When credit bites back. *Journal of Money, Credit and Banking*, 45(s2):3–28, 2013.

Òscar Jordà, Moritz Schularick, and Alan M Taylor. Betting the house. *Journal of International Economics*, 96:S2–S18, 2015a.

Òscar Jordà, Moritz Schularick, and Alan M Taylor. Leveraged bubbles. *Journal of Monetary Economics*, 76:S1–S20, 2015b.

Òscar Jordà, Björn Richter, Moritz Schularick, and Alan M Taylor. Bank capital redux: solvency, liquidity, and crisis. *The Review of Economic Studies*, 88(1):260–286, 2021.

Ioannis Karatzas and Steven E Shreve. *Methods of Mathematical Finance*, volume 39. Springer Science & Business Media, 1998.

Rafail Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer Science & Business Media, 2011.

Paymon Khorrami and Fernando Mendo. Dynamic self-fulfilling fire sales. Unpublished working paper, 2024.

Paymon Khorrami and Alexander Zentefis. Segmentation and beliefs: A theory of self-fulfilling idiosyncratic risk. Unpublished working paper, CESifo, 2023.

Nobuhiro Kiyotaki and John Moore. Credit cycles. *Journal of Political Economy*, 105(2): 211–248, 1997.

Nataliya Klimenko, Sebastian Pfeil, Jean-Charles Rochet, and Gianni De Nicolò. Aggregate bank capital and credit dynamics. Unpublished working paper. University of Zurich, 2016.

Nataliya Klimenko, Sebastian Pfeil, and Jean-Charles Rochet. A simple macroeconomic model with extreme financial frictions. *Journal of Mathematical Economics*, 68:92–102, 2017.

Narayana R Kocherlakota. Bubbles and constraints on debt accumulation. *Journal of Economic Theory*, 57(1):245–256, 1992.

Arvind Krishnamurthy and Wenhao Li. Dissecting mechanisms of financial crises: Intermediation and sentiment. Unpublished working paper, 2020.

Arvind Krishnamurthy and Tyler Muir. How credit cycles across a financial crisis. Unpublished NBER working paper, 2017.

Nicolai V Krylov. Diffusion on a plane with reflection. construction of the process. *Siberian Mathematical Journal*, 10(2):244–252, 1969.

Nicolai V Krylov. On weak uniqueness for some diffusions with discontinuous coefficients. *Stochastic Processes and their Applications*, 113(1):37–64, 2004.

Zheng Liu and Pengfei Wang. Credit constraints and self-fulfilling business cycles. *American Economic Journal: Macroeconomics*, 6(1):32–69, 2014.

David López-Salido, Jeremy C Stein, and Egon Zakrajšek. Credit-market sentiment and the business cycle. *The Quarterly Journal of Economics*, 132(3):1373–1426, 2017.

Rodolfo Manuelli and James Peck. Sunspot-like effects of random endowments. *Journal of Economic Dynamics and Control*, 16(2):193–206, 1992.

Peter Maxted. A macro-finance model with sentiment. *Review of Economic Studies (forthcoming)*, page rdad023, 2023.

Fernando Mendo. Risky low-volatility environments and the stability paradox. Unpublished working paper, 2020.

Atif Mian, Amir Sufi, and Emil Verner. Household debt and business cycles worldwide. *The Quarterly Journal of Economics*, 132(4):1755–1817, 2017.

Jianjun Miao and Pengfei Wang. Asset bubbles and credit constraints. *The American Economic Review*, 108(9):2590–2628, 2018.

Alan Moreira and Alexi Savov. The macroeconomics of shadow banking. *The Journal of Finance*, 72(6):2381–2432, 2017.

Tyler Muir. Financial crises and risk premia. *The Quarterly Journal of Economics*, 132(2): 765–809, 2017.

Gregory Phelan. Financial intermediation, leverage, and macroeconomic instability. *American Economic Journal: Macroeconomics*, 8(4):199–224, 2016.

Thomas Piketty. The dynamics of the wealth distribution and the interest rate with credit rationing. *The Review of Economic Studies*, 64(2):173–189, 1997.

Carmen M Reinhart and Kenneth S Rogoff. *This time is different: Eight centuries of financial folly*. Princeton University Press, 2009.

Carmen M Reinhart and Kenneth S Rogoff. Recovery from financial crises: Evidence from 100 episodes. *The American Economic Review*, 104(5):50–55, 2014.

Jose A Scheinkman and Laurence Weiss. Borrowing constraints and aggregate economic activity. *Econometrica*, 54(1):23–45, 1986.

Dejanir Silva. The risk channel of unconventional monetary policy. Unpublished working paper. University of Illinois., 2017.

Online Appendix:
Rational Sentiments and Financial Frictions
Paymon Khorrami and Fernando Mendo
June 19, 2024

A Solvency constraint as the natural borrowing limit

Here, we discuss the solvency constraint $n_t \geq 0$, which serves as the natural borrowing limit in our framework. The idea of a natural borrowing limit is that agents can borrow at most the present-value of their future income if they want to consume non-negative amounts and also not run a Ponzi scheme (see, e.g., Aiyagari, 1994). In our context, the only asset is capital, and the stream of its future dividends represents future income. Thus, if the income stream is valued at $q_t k_t$ for k_t units of capital holdings, it is sensible that an agent should be able to borrow at most this amount: $b_t \leq q_t k_t$. Since net worth is defined as assets minus liabilities, $n_t = q_t k_t - b_t$, this implies $n_t \geq 0$.

Below, we explore three microfoundations for the solvency constraint $n_t \geq 0$, all of which hopefully clarify that this constraint is "natural" in some sense. In these derivations, we allow the possibility of zero fundamental volatility, $\sigma = 0$, for generality.

A.1 Finite-horizon approximation

The first microfoundation is the easiest and most obvious, but also the most ad-hoc. We suppose there is a strictly increasing sequence of deterministic times $\{T_j\}_{j=1}^{\infty}$, with possibly arbitrarily large gaps $T_{j+1} - T_j$, such that net worth must be non-negative at those times:

$$n_{T_j} \geq 0 \quad \text{almost-surely for each } T_j. \tag{NPC-1}$$

This says that unsecured debts—debt in excess of the present value of capital holdings—must be fully repaid at some future date. Such a constraint rules out finite-horizon Ponzi schemes.

Furthermore, we assume that agents must satisfy

$$e^{-\int_0^t r_s ds} n_t \geq -\underline{n}, \tag{NLB-1}$$

where $\underline{n}$ can be arbitrarily large but finite. Constraint (NLB-1) is an example of the requirement that portfolios be "tame" (see Karatzas and Shreve, 1998, Chapter 1, Definition 2.4). In dynamic trading models, the point of tame portfolios is to rule out certain trivial arbitrage opportunities like "doubling strategies" (c.f., Karatzas and Shreve, 1998, Chapter 1, Example 2.3). Thus, no equilibrium could exist without a requirement like (NLB-1), which is why we view these constraints as a minimal requirement.[22] Furthermore, the lower bound $\underline{n}$ can be arbitrarily large, which permits any trading strategy that doesn't leave the agent infinitely indebted.

In this environment, we have the following result which is standard in the literature (e.g., Theorem 1 of Dybvig and Huang, 1988).

Lemma A.1. *Let (NPC-1) hold for some sequence $\{T_j\}_{j=1}^{\infty}$. Assume (NLB-1) holds for all t. Then, every agent must obey $n_t \geq 0$.*

Proof of Lemma A.1. See the proof of Lemma A.3 below. In that proof, we simply use the inequality $n_T \geq 0$ in equation (A.10), where $T \in \{T_j\}_{j=1}^{\infty}$. □

A.2 Infinite-horizon borrowing limits

The other three microfoundations, instead, assume only that unsecured debts must be repaid *eventually*. That is, there will be an asymptotic No-Ponzi condition.

To set up the environment and the constraints, consider an agent with net worth n_t who may choose any consumption and trading strategy $\{c_t, k_t\}_{t\geq 0}$ that satisfies appropriate mild integrability conditions. The dynamic budget constraint of this agent takes the form

$$dn_t = \Big[r_t n_t - c_t + q_t k_t(\mu_{R,t} - r_t)\Big]dt + q_t k_t \sigma_{R,t} \cdot dZ_t, \quad n_0 \text{ given}, \tag{A.1}$$

where $\mu_{R,t}$ is that agent's expected return on capital (which differs between experts and households). Given these trading opportunities, let M_t be the state-price density faced

[22] An alternative constraint that achieves the same result as (NLB-1) is to impose an integrability condition on the trading strategies agents can do:

$$\tilde{\mathbb{E}}\Big[\int_0^\infty e^{-2\int_0^t r_s ds}(q_t k_t)^2 |\sigma_{R,t}|^2 dt\Big] < \infty,$$

where $\tilde{\mathbb{E}}$ represents the risk-neutral expectation in the model. Dybvig and Huang (1988), Theorems 4 and 5, prove that the lower bound (NLB-1) and the integrability condition above are essentially equivalent in this environment: they both rule out arbitrage and permit essentially the same trading strategies. We work with the uniform net worth lower bound because it will translate better into our infinite-horizon proofs in Section A.2.

by this agent:

$$M_t = \exp\Big[-\int_0^t \Big(r_s + \frac{1}{2}|\pi_s|^2\Big)ds - \int_0^t \pi_s \cdot dZ_s\Big], \tag{A.2}$$

$$\text{where} \quad \sigma_{R,t} \cdot \pi_t = \mu_{R,t} - r_t. \tag{A.3}$$

Note that equation (A.3) defines π_t as the agent's market price of risk process, which again is agent-specific in our model. Because we will refer to it very often, define the exponential local martingale

$$\tilde{M}_t := \exp\Big[-\frac{1}{2}\int_0^t |\pi_s|^2 ds - \int_0^t \pi_s \cdot dZ_s\Big]. \tag{A.4}$$

The process $\tilde{M}_t$, provided it is a true martingale, will be used to define the risk-neutral probability measure $\tilde{\mathbb{P}}$. (In an infinite-horizon model, there is some additional subtlety to the construction of the risk-neutral measure, which we will explain in the proof of Lemma A.3 below.)

Given this environment, we consider two different formulations of the asymptotic No-Ponzi condition. In the first formulation, we assume that agents must obey

$$\liminf_{T\to\infty} M_T n_T \geq 0 \quad \mathbb{P}\text{-almost-surely.} \tag{NPC-2}$$

(this is weaker than the condition $\liminf_{T\to\infty} n_T \geq 0$ because of the fact that $M_T > 0$). In the second formulation, we assume that agents obey

$$\liminf_{T\to\infty} e^{-\int_0^T r_t dt} n_T \geq 0 \quad \tilde{\mathbb{P}}\text{-almost-surely,} \tag{NPC-3}$$

where $\tilde{\mathbb{P}}$ denotes the risk-neutral probability measure. The intuitive idea behind constraints (NPC-2) and (NPC-3) is as follows. By taking expectations of (NPC-2) and (NPC-3), we have that $\mathbb{E}_t[M_\infty n_\infty] \geq 0$ and $\tilde{\mathbb{E}}_t[e^{-\int_0^\infty r_t dt} n_\infty] \geq 0$, respectively. Therefore, these constraints imply that the present-value of unsecured debts must vanish eventually, ruling out arbitrarily large debts asymptotically. However, by themselves, neither (NPC-2) nor (NPC-3) is sufficient to induce the solvency constraint $n_t \geq 0$.

We impose, in addition, a uniform lower bound on net worth, but with two different functional forms. In the first formulation, we impose a lower bound on the present-value of net worth,

$$M_t n_t \geq -\underline{n}, \tag{NLB-2}$$

where $\underline{n}$ can be arbitrarily large but finite. In the second microfoundation, we impose a lower bound on net worth directly,

$$e^{-\int_0^t r_s ds} n_t \geq -\underline{n}, \tag{NLB-3}$$

where again $\underline{n}$ can be arbitrarily large but finite. Constraints (NLB-2) and (NLB-3) are again "tame" portfolio requirements that rule out certain trivial arbitrages like doubling strategies.

Now, we provide two proofs that the solvency constraint holds.

Lemma A.2. *Let (NPC-2) and (NLB-2) hold. Then, every agent must obey $n_t \geq 0$.*

Lemma A.3. *Let (NPC-3) and (NLB-3) hold. Suppose $\tilde{M}_t$ is a martingale. Then, every agent must obey $n_t \geq 0$.*

Remark 4. *We make a brief remark about the assumption that $\tilde{M}_t$ be a martingale in the latter lemma. This assumption should be regarded as relatively minor. Indeed, a sufficient condition for $\tilde{M}_t$ to be a martingale is that $\sup_t |\pi_t| < \infty$, i.e., risk prices be uniformly bounded. It is straightforward to verify that equilibrium risk prices only diverge at the boundary where $\eta \to 0$ and $\kappa/\eta \to +\infty$, so what we need is for state dynamics prevent the economy from approaching this boundary.*[23] *This can be done: an example of such an equilibrium construction is presented in Proposition 1, in which risk prices are indeed uniformly bounded.*

Proof of Lemma A.2. The general strategy of the proof is to derive a static budget constraint, and then use this budget constraint to prove that $n_t \geq 0$.

Apply Itô's formula to the process

$$H_t := M_t n_t + \int_0^t M_s c_s ds,$$

then use the dynamic budget constraint (A.1) and equation (A.3) for π_t, to obtain

$$H_T - H_t = M_T n_T - M_t n_t + \int_t^T M_s c_s ds = \int_t^T M_s \Big(q_s k_s \sigma_{R,s} - n_s \pi_s \Big) \cdot dZ_s. \tag{A.5}$$

[23] Indeed, (squared) expert risk prices are given by $|\pi|^2 = (\frac{\kappa}{\eta})^2 |\sigma_R|^2$, which after using the equilibrium value of $|\sigma_R|^2$ when $\kappa < 1$ gives us $|\pi|^2 = (\frac{\kappa}{\eta})^2 \frac{\eta(1-\eta)}{\kappa-\eta} \frac{a_e - a_h}{q}$. This is bounded except at the boundary $\eta \to 0$ and $\kappa \to \bar{\kappa} > 0$. At this boundary, the risk price behaves like $|\pi|^2 \sim \eta^{-1}\bar{C}$, where $\bar{C} := \frac{\bar{\kappa}(a_e - a_h)}{a_h + \bar{\kappa}(a_e - a_h)} \rho_h$.

This shows that H_t is a local martingale. Furthermore, the lower bound (NLB-2) and the non-negativity of consumption imply $H_t \geq -\underline{n}$ and so H_t is a super-martingale. Taking time-t expectations of (A.5), we thus have

$$\mathbb{E}_t\Big[M_T n_T\Big] + \mathbb{E}_t\Big[\int_t^T M_s c_s ds\Big] \leq M_t n_t. \tag{A.6}$$

Because consumption is non-negative, the monotone convergence theorem implies

$$\lim_{T\to\infty} \mathbb{E}_t\Big[\int_t^T M_s c_s ds\Big] = \mathbb{E}_t\Big[\int_t^\infty M_s c_s ds\Big].$$

For the terminal wealth term, the lower bound (NLB-2) implies $(M_T n_T)_{T\geq\infty}$ is a uniformly lower-bounded family of random variables, so by Fatou's lemma we have

$$\liminf_{T\to\infty} \mathbb{E}_t\Big[M_T n_T\Big] \geq \mathbb{E}_t\Big[\liminf_{T\to\infty} M_T n_T\Big].$$

Using asymptotic No-Ponzi condition (NPC-2), the right-hand-side term is non-negative. Using these limiting results in (A.6), we have

$$\mathbb{E}_t\Big[\int_t^\infty M_s c_s ds\Big] \leq M_t n_t. \tag{A.7}$$

Equation (A.7) is the usual "static" budget constraint. From (A.7), the fact that consumption is non-negative, and the fact that the state-price density is strictly positive, we immediately obtain $n_t \geq 0$. Since time t was arbitrary, this must hold for all times. □

Proof of Lemma A.3. This proof proceeds slightly differently than Lemma A.2. Indeed, since there is no obvious lower bound that can be applied to $M_T n_T$ in equation (A.6), the proof becomes more technical and complex. The general strategy is to examine the dynamics of $e^{-\int_0^t r_s ds} n_t$, which is lower-bounded, rather than $M_t n_t$.

There are two complications. First, to continue to use martingale methods, we must examine the dynamics of $e^{-\int_0^t r_s ds} n_t$ under the risk-neutral measure $\tilde{\mathbb{P}}$ rather than the true probability $\mathbb{P}$. This is where the assumption that $\tilde{M}_t$ is a martingale, hence a valid change-of-measure, comes into play. Second, because our model is infinite-horizon, $\tilde{\mathbb{P}}$ and $\mathbb{P}$ may be mutually singular asymptotically on the limiting sigma-algebra $\mathcal{F}_\infty$, even though $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent on every finite horizon. For this reason, the No-Ponzi condition (NPC-3) is written purposefully under $\tilde{\mathbb{P}}$.

First, we define a probability measure $\tilde{\mathbb{P}}$ following the recipe of Chapter 1.7 in Karatzas and Shreve (1998). Using $\tilde{M}_t$ as a change-of-measure, we set

$$\tilde{\mathbb{P}}(A) := \mathbb{E}[\tilde{M}_T \mathbf{1}_A]; \quad A \in \mathcal{F}_T, \quad 0 \leq T < \infty. \tag{A.8}$$

As proven in Chapter 1.7, Proposition 7.4 of Karatzas and Shreve (1998), the probability $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ for each $T \geq 0$ (i.e., a set in $\mathcal{F}_T$ is a $\tilde{\mathbb{P}}$-null set if and only if it is a $\mathbb{P}$-null set). Furthermore, the process

$$\tilde{Z}_t := Z_t + \int_0^t \pi_s ds$$

is a Brownian motion on under $\tilde{\mathbb{P}}$.

Consider now the process

$$H_t := e^{-\int_0^t r_s ds} n_t + \int_0^t e^{-\int_0^s r_u du} c_s ds,$$

which follows

$$dH_t = e^{-\int_0^t r_s ds} \left(q_t k_t \sigma_{R,t} \right) \cdot d\tilde{Z}_t. \tag{A.9}$$

By the non-negativity of consumption and the lower bound (NLB-3), we have that $H_t \geq -\underline{n}$, so H_t is a $\tilde{\mathbb{P}}$-super-martingale. Taking time-t risk-neutral expectations of $H_T - H_t$, we thus have

$$\tilde{\mathbb{E}}_t \left[e^{-\int_0^T r_s ds} n_T \right] + \tilde{\mathbb{E}}_t \left[\int_t^T e^{-\int_0^s r_u du} c_s ds \right] \leq e^{-\int_0^t r_s ds} n_t. \tag{A.10}$$

Because consumption is non-negative, the monotone convergence theorem implies

$$\lim_{T\to\infty} \tilde{\mathbb{E}}_t \left[\int_t^T e^{-\int_0^s r_u du} c_s ds \right] = \tilde{\mathbb{E}}_t \left[\int_t^\infty e^{-\int_0^s r_u du} c_s ds \right].$$

For the terminal wealth term, the lower bound (NLB-3) implies $(e^{-\int_0^T r_s ds} n_T)_{T\geq\infty}$ is a uniformly lower-bounded family of random variables. Because $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent on all finite horizons, the almost-sure lower-bound holds both under $\tilde{\mathbb{P}}$ and $\mathbb{P}$, so by Fatou's lemma we have

$$\liminf_{T\to\infty} \tilde{\mathbb{E}}_t \left[e^{-\int_0^T r_s ds} n_T \right] \geq \tilde{\mathbb{E}}_t \left[\liminf_{T\to\infty} e^{-\int_0^T r_s ds} n_T \right].$$

Using asymptotic No-Ponzi condition (NPC-3), the right-hand-side term is non-negative. Using these limiting results in (A.10), we have

$$\tilde{\mathbb{E}}_t\Big[\int_t^\infty e^{-\int_0^s r_u du} c_s ds\Big] \leq e^{-\int_0^t r_s ds} n_t. \tag{A.11}$$

Equation (A.11) is the usual "static" budget constraint. From (A.11), and the fact that consumption is non-negative, we immediately obtain $n_t \geq 0$. □

B Proofs for Sections 1-2

B.1 Irrelevance of type-switching for optimal behavior

The objective function with type-switching technically differs from (3), because agents understand that at a future exponentially-distributed time, they will switch occupations. Mathematically, the objective functions and indirect utilities satisfy the recursions, for each type-j (expert or household) agent

$$V_{j,t} = \sup_{c_j \geq 0, k_j \geq 0, n_j \geq 0} \mathbb{E}\Big[\int_0^{T_j} e^{-\rho_j s} \log(c_{j,t+s}) ds + e^{-\rho_j T} V_{-j,t+T_j}\Big], \quad T_j \sim \exp(\delta_j)$$

Standard homogeneity arguments imply that indirect utilities take the additively-separable form $V_{j,t} = \rho_j^{-1} \log(n_{j,t}) + \xi_{j,t}$, for processes $\xi_{j,t}$ that only depend on aggregates (i.e., not on individual net worth). Write $d\xi_{j,t} = \mu_{\xi,j,t} dt + \sigma_{\xi,j,t} \cdot dZ_t$. Then, the HJB equations associated with these equations are

$$\rho_j V_j = \max_{c,k \geq 0} \log(c) + (\partial_n V_j)[rn - c + qk(\mu_{R,j} - r)] + \frac{1}{2}(\partial_{nn} V_j)(qk)^2 |\sigma_R|^2 + \mu_{\xi,j} + \delta_e[V_{-j} - V_j]$$

where $\mu_{R,j}$ is the expected returns on capital for type j. Using the form of V_j, the HJB equations become

$$\log(n) + \rho_j \xi_j = \max_{c,k \geq 0} \log(c) + \rho_j^{-1}[r - \frac{c}{n} + \frac{qk}{n}(\mu_{R,j} - r)] - \frac{1}{2}(\frac{qk}{n})^2 |\sigma_R|^2 + \mu_{\xi,j} + \delta_e[\xi_{-j} - \xi_j]$$

Optimal choices take the familiar log-utility forms: consumptions are $c_j = \rho_j n_j$; portfolios are $\frac{qk_j}{n_j} = [\frac{\mu_{R,j} - r}{|\sigma_R|^2}]^+$. Most importantly, these choices are independent of the switching parameters δ_j. To fully verify that this is correct, we must substitute the optimality conditions back into the HJB equations and check that we recover equations for ξ_e and ξ_h that only depend on aggregate variables (e.g., capital price q, interest rate r, etc.). Doing this, we obtain

$$\rho_j \xi_j = \log(\rho_j) + \rho_j^{-1}[r - \rho_j + \frac{1}{2}(\frac{[\mu_{R,j} - r]^+}{|\sigma_R|})^2] + \mu_{\xi,j} + \delta_j[\xi_{-j} - \xi_j],$$

which verifies the conjecture, as all terms either pertain to the ξ processes or aggregate variables. □

B.2 Proof of Lemma 1

We are given η_0 and conditions (PO), (RB), (11), and (13)-(14). We need to check conditions (i)-(iii) of Definition 1. Condition (i) holds by the definition of η_0.

For condition (ii), note that standard martingale techniques can be applied to verify that individual optimality, subject to the dynamic budget constraint (2), is equivalent to the following conditions holding: $c_\ell = \rho_\ell n_\ell$; the portfolio conditions (7)-(8); and the transversality conditions in (10). We must verify that these conditions hold. Given q_t, η_t, κ_t, and individual net worths $n^i_{e,t}$ and $n^j_{h,t}$, let us set

$$c^i_{e,t} = \rho_e n^i_{e,t} \quad \text{and} \quad k^i_{e,t} = \frac{\kappa_t}{q_t \eta_t} n^i_{e,t}, \quad \text{for } i \in \mathbb{I} \tag{B.1}$$

$$c^j_{h,t} = \rho_h n^j_{h,t} \quad \text{and} \quad k^j_{h,t} = \frac{1 - \kappa_t}{q_t(1 - \eta_t)} n^j_{h,t}, \quad \text{for } j \in \mathbb{J}. \tag{B.2}$$

If we do this, then clearly the optimal consumption-wealth ratio holds. Similarly, after substituting the suggested capital holdings from (B.1)-(B.2), the optimal portfolio conditions (7)-(8) become a linear transformation of equations (RB) and (11)—i.e., equation (RB) is the difference between (7) and (8), while (11) is the sum of κ times (7) plus $1 - \kappa$ times (8). Thus, given (RB) and (11), equations (7)-(8) hold as well. Finally, after substituting the proposals in (B.1)-(B.2) into the transversality conditions in (10), we see that these hold automatically.

For condition (iii), note that $\kappa \in [0, 1]$ automatically implies capital market clearing (5). Similarly, substituting $c_\ell = \rho_\ell n_\ell$ and the definitions of κ and η into (PO), we obtain goods market clearing (4).

Thus, we have constructed an equilibrium of Definition 1. Note that (13)-(14) have not been used in this construction, but they are direct consequences (via Itô's formula) of the definition of η.

The final statement of the lemma is clearly true. Indeed, the prices (q_t, r_t) are directly involved in Definition 1, while the objects (η_t, κ_t) constitute two summary statistics of the distribution of net worth and capital $\{n^i_{e,t}, n^j_{h,t}, k^i_{e,t}, k^j_{h,t} : i \in \mathbb{I}, j \in \mathbb{J}\}$. Thus, two distinct values of $(\eta_t, q_t, \kappa_t, r_t)_{t\geq 0}$ cannot correspond to the same equilibrium of Definition 1. $\square$

B.3 Proof of Theorem 1

Step 0: Reduce the system. We will start by eliminating $(r, \kappa, \sigma_\eta, \mu_\eta)$ from the system of endogenous objects, given $(\eta, q, \sigma_q, \mu_q)$. First, price-output relation (PO) determines κ as a function of (η, q) and nothing else, given by

$$\kappa(\eta, q) := \frac{q\bar{\rho}(\eta) - a_h}{a_e - a_h}. \tag{B.3}$$

Second, substituting this result for κ, equation (11) fully determines r, given knowledge of $(\eta, q, \sigma_q, \mu_q)$. Third, equations (13)-(14), after plugging in the result for κ, fully determine (σ_η, μ_η), given knowledge of (η, q, σ_q). Thus, given (η, q), the choice of (σ_q, μ_q) needs to ensure that (RB) holds and that the dynamics of (η_t, q_t) remain inside $\mathcal{D}$ as defined by (16) in text.

The remainder of the proof is entirely devoted to addressing the boundaries of $\mathcal{D}$. Indeed, given $(\eta_t, q_t) \in \mathcal{D}^\circ$ (the interior of $\mathcal{D}$), we can set σ_q according to (B.6) below and set μ_q to any real number. This is not to suggest that the boundary points are inconsequential; on the contrary, without ensuring that the system $(\eta_t, q_t)_{t\geq 0}$ remains in $\mathcal{D}$, the solution constructed in the interior $\mathcal{D}^\circ$ would not be part of an equilibrium. Unfortunately, the choice of (σ_q, μ_q) is more delicate at the boundary $\partial\mathcal{D}$. Furthermore, verifying that $(\eta_t, q_t)_{t\geq 0}$ remains in $\mathcal{D}$ is non-trivial and requires a detailed analysis.

Step 1: Define perturbed domain. To facilitate analysis, it will be convenient to analyze a slightly modified system instead of (η, q), and on a perturbed domain. The purpose of this perturbation will be threefold. First, as q approaches the lower boundary of $\mathcal{D}$, volatility σ_q necessarily grows without bound; by perturbing this boundary slightly upward, we prevent unbounded volatilities, allowing us to use standard diffusion theory. Second, as q approaches the upper boundary of $\mathcal{D}$, there will exist a wealth level η^* such that $\kappa = 1$ cannot possibly occur on $\{\eta \leq \eta^*\}$ but can occur on $\{\eta > \eta^*\}$; by rotating this upper boundary around any wealth share above η^*, we streamline our arguments. Third, our perturbed domain will be an open set, which is easier to work with. See Figure B.1 below for a visual of the domain perturbation. By the end of this step, it will become clear that if our modified system (η, x) remains in perturbed domain $\mathcal{X}$, then the original system (η, q) remains in the original domain $\mathcal{D}$. Furthermore, after constructing an equilibrium in this perturbed domain, it will be clear that we are able to consider the limit of a sequence of such equilibria as the perturbations vanish, and so we can also construct an equilibrium on the full domain $\mathcal{D}$ (although this is not what Theorem 1 requires us to prove).

First, define the following auxiliary functions. Fix $\epsilon \in (0, \frac{a_e - a_h}{\rho_h})$. Let $\beta(\cdot)$ be a strictly increasing, continuously differentiable function such that $\beta(1) = -\beta(0) = \epsilon$, and $\beta(\eta^*_\beta) = 0$, where $\eta^*_\beta \in (\eta^*, 1)$ and

$$\eta^* := \frac{\rho_h}{\rho_e}\Big(\frac{1 - a_h/a_e}{\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\Big)^{-1}. \tag{B.4}$$

Note that $\eta^* < 1$ by Assumption 1, part (ii). Let $\alpha(\cdot)$ be an increasing, continuously differentiable function such that $\alpha(0) = 0$, $\alpha'(0) \in (0, \infty)$, and $\alpha(1) = \epsilon/2$.

Next, define the following functions,

$$\begin{aligned} q^H(\eta) &:= a_e/\bar{\rho}(\eta) \\ q^L(\eta) &:= \bar{a}(\eta)/\bar{\rho}(\eta), \end{aligned}$$

where $\bar{a}(\eta) := \eta\rho_e + (1-\eta)\rho_h$. Using (B.3), one notices that q^H corresponds to the capital price when $\kappa = 1$, whereas q^L corresponds to the capital price when $\kappa = \eta$. Put

$$\begin{aligned} q^H_\beta(\eta) &:= q^H(\eta) + \beta(\eta) \\ q^L_\alpha(\eta) &:= q^L(\eta) + \alpha(\eta). \end{aligned}$$

Using these functions, define the perturbed domain (which is an open set)

$$\mathcal{X} := \Big\{(\eta, x) \,:\, \eta \in (0,1) \quad \text{and} \quad q^L_\alpha(\eta) < x < q^H_\beta(\eta)\Big\}.$$

Note that, boundaries aside, $\mathcal{X}$ will coincide with $\mathcal{D}$ as $\epsilon \to 0$. For reference, the perturbed domain $\mathcal{X}$ is displayed in Figure B.1.

We will define a stochastic process x_t such that the capital price q coincides with x when it lies below q^H, i.e.,

$$q_t = \min\Big[x_t, \, q^H(\eta_t)\Big]. \tag{B.5}$$

By (B.5), we may analyze the dynamical system $(\eta_t, x_t)_{t\geq 0}$ rather than $(\eta_t, q_t)_{t\geq 0}$. Furthermore, to prove the claim that $(\eta_t, q_t)_{t\geq 0}$ remains in $\mathcal{D}$ almost-surely, it suffices to prove $(\eta_t, x_t)_{t\geq 0}$ remains in $\mathcal{X}$ almost-surely (Step 4 below).

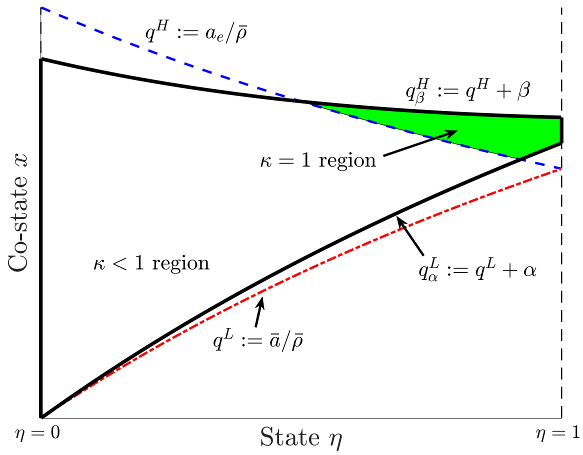

Figure B.1: The perturbed domain $\mathcal{X}$ is shown as the region surrounded by solid black lines. The original domain $\mathcal{D}$ is the region defined by the dashed lines. The perturbation functions α and β are chosen to be linear functions, with $\epsilon = 0.2$. Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$.

Step 2: Construct σ_q so that (RB) is satisfied. First consider $\{x < q^H(\eta)\}$ so that $q = x$. Note that this case corresponds to $\kappa < 1$. Let $\gamma(\eta, x) : \mathcal{X} \mapsto (0,1)$ be any C^1 function. Put

$$\sigma_q = \begin{bmatrix} \sqrt{\gamma \frac{\eta(1-\eta)}{\kappa-\eta} \frac{a_e - a_h}{q}} - \sigma \\ \sqrt{(1-\gamma)\frac{\eta(1-\eta)}{\kappa-\eta} \frac{a_e-a_h}{q}} \end{bmatrix}, \quad \text{if } x < q^H(\eta). \tag{B.6}$$

Substituting (B.6), one can verify that the second term of condition (RB) is zero. Importantly, the definitions of q^L_α and q^H_β imply that σ_q is bounded on $\mathcal{X} \cap \{x < q^H(\eta)\}$. Indeed, because of $\alpha'(0) > 0$, the slowest possible rate that $\kappa \to 0$ as $\eta \to 0$ is lower-bounded away from 1, i.e., $\liminf_{\eta \to 0, (\eta,x) \in \mathcal{X}} \kappa/\eta > 1$. And because $\alpha(1) > 0$, we have $\kappa = 1$ for all η near enough to 1; thus η is bounded away from 1 on $\{x < q^H(\eta)\}$.

Next consider $\{x \geq q^H(\eta)\}$ so that $q = q^H(\eta)$. Note that this case corresponds to $\kappa = 1$. Since q is an explicit function of η, we use Itô's formula to compute $\binom{1}{0} \cdot \sigma_q = -\sigma_\eta \bar{\rho}'/\bar{\rho}$, which after substituting equation (14) for σ_η delivers

$$\sigma_q = \begin{bmatrix} -\frac{(1-\eta)(\rho_e-\rho_h)/\bar{\rho}}{1+(1-\eta)(\rho_e-\rho_h)/\bar{\rho}}\sigma \\ 0 \end{bmatrix}, \quad \text{if } x \geq q^H(\eta). \tag{B.7}$$

Note that (B.7) will be consistent with (RB) as long as $(\eta_t, x_t)_{t \geq 0}$ remains in $\mathcal{X}$ almost-surely, which will be verified in Step 4.[24]

Note finally that σ_q defined in (B.6)-(B.7) is solely a function of (η, x), so sometimes we will write $\sigma_q(\eta, x)$. Similarly, with σ_q in hand, we now have μ_η and σ_η as functions of

[24] Plugging $q = a_e/\bar{\rho}$ into the second term of equation (RB), we require $|\sigma_R|^2 \leq \eta\bar{\rho}(\eta)(1 - a_h/a_e)$. Substituting (B.7), we obtain $|\sigma_R|^2 = \sigma^2(\bar{\rho}/\rho_e)^2$. Combining these, we require. $\eta \geq \eta^*$ when $x \geq q^H(\eta)$, where η^* is defined in (B.4). Therefore, for all $\eta < \eta^*$, we insist $x < q^H(\eta)$. As long as $(\eta, x) \in \mathcal{X}$, this will hold, because $q^H_\beta(\eta) < q^H(\eta)$ for all $\eta < \eta^*$, and $x < q^H_\beta(\eta)$ for all η.

(η, x) alone.

Step 3: Construct μ_q. Similar to σ_q, separately consider $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$. On $\{x \geq q^H(\eta)\}$, since $q = q^H(\eta)$ is an explicit function of η, we set μ_q via Itô's formula. On $\{x < q^H(\eta)\}$, we have no equilibrium considerations restricting μ_q. Thus, we will put $\mu_q = m_q$, where m_q is a function in class $\mathcal{M}$, defined as follows. A function $m : \mathcal{X} \mapsto \mathbb{R}$ is a member of $\mathcal{M}$ if m is C^1 and possesses the following boundary conditions:

$$\inf_{\eta \in (0,1)} \lim_{x \searrow q_\alpha^L(\eta)} \Big(x - q_\alpha^L(\eta)\Big) m(\eta, x) = +\infty \tag{B.8}$$

$$\sup_{\eta \in (0,1)} \lim_{x \nearrow q_\beta^H(\eta)} \Big(q_\beta^H(\eta) - x\Big) m(\eta, x) = -\infty \tag{B.9}$$

$$\text{for any } x \in (q_\alpha^L(0), q_\beta^H(0)), \quad \lim_{\eta \searrow 0} |m(\eta, x)| < +\infty \tag{B.10}$$

$$\text{for any } x \in (q_\alpha^L(1), q_\beta^H(1)), \quad \lim_{\eta \nearrow 1} |m(\eta, x)| < +\infty. \tag{B.11}$$

Collecting these results

$$\mu_q(\eta, x) = \begin{cases} m_q(\eta, x), & \text{if } x < q^H(\eta); \\ \frac{\rho_e - \rho_h}{\bar{\rho}(\eta)^2}[-\bar{\rho}(\eta)\mu_\eta(\eta, x) + |\sigma_\eta(\eta, x)|^2], & \text{if } x \geq q^H(\eta). \end{cases} \tag{B.12}$$

Step 4: Verify stationarity. We demonstrate the time-paths $(\eta_t, x_t)_{t \geq 0}$ remain in $\mathcal{X}$ almost-surely and admit a stationary distribution.

The dynamics of x_t are specified as follows. Denote its diffusion and drift coefficients by $(x\sigma_x, x\mu_x)$, where σ_x and μ_x are functions of (η, x) to be specified shortly. By (B.5), when $q_\alpha^L(\eta) < x < q^H(\eta)$, we must put $\sigma_x = \sigma_q$ and $\mu_x = \mu_q$. Outside of this region, σ_x and μ_x are unrestricted and we set them to preserve stationarity.

To this end, let $\tilde{\sigma}_x : \mathcal{X} \mapsto \mathbb{R}_+$ be any positive, bounded, C^1 function.[25] Put

$$\sigma_x(\eta, x) = \begin{cases} \sigma_q(\eta, x), & \text{if } x < q^H(\eta); \\ \tilde{\sigma}_x(\eta, x), & \text{if } x \geq q^H(\eta). \end{cases}$$

Note that σ_x is bounded (recall σ_q is bounded, and $\tilde{\sigma}_x$ is assumed bounded).

Similarly, for the drift, let $m_x : \mathcal{X} \mapsto \mathbb{R}$ be any function in class $\mathcal{M}$ defined above

[25]Note that $\tilde{\sigma}_x$ need not vanish at the boundary of $\mathcal{X}$, but if it does some of the boundary conditions on m_x, to follow, can be relaxed.

(note: m_x need not coincide with m_q above). Put

$$\mu_x(\eta, x) = \begin{cases} \mu_q(\eta, x), & \text{if } x < q^H(\eta); \\ m_x(\eta, x), & \text{if } x \geq q^H(\eta). \end{cases}$$

Thus, μ_x satisfies boundary conditions (B.8)-(B.11) on all boundaries of $\mathcal{X}$.

Corresponding to the SDEs induced by $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$, define the infinitesimal generator $\mathscr{L}$, where for any C^2 function f,

$$\mathscr{L} f = \mu_\eta \partial_\eta f + (x\mu_x)\partial_x f + \frac{1}{2}|\sigma_\eta|^2 \partial_{\eta\eta} f + \frac{1}{2}|x\sigma_x|^2 \partial_{xx} f + x\sigma_x \cdot \sigma_\eta \partial_{\eta x} f.$$

Let $\{\mathcal{X}_n\}_{n\geq 1}$ be an increasing sequence of open sets, whose closures are contained in $\mathcal{X}$, such that $\cup_{n\geq 1}\mathcal{X}_n = \mathcal{X}$. Note that $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$ are bounded on $\mathcal{X}_n$ for each n. Consequently, despite the (potential) discontinuity in $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$ at the one-dimensional subset $\{x = q^H(\eta)\}$, there exists a unique weak solution $(\tilde{\eta}^n_t, \tilde{x}^n_t)_{0\leq t\leq \tau_n}$, up to first exit time $\tau_n := \inf\{t : (\eta_t, x_t) \notin \mathcal{X}_n\}$, to the SDEs defined by the infinitesimal generator $\mathscr{L}$. See Krylov (1969, 2004) for weak existence and uniqueness in the presence of drift and diffusion discontinuities.

Letting $\tau := \lim_{n\to\infty} \tau_n$, we thus define $(\eta_t, x_t)_{0\leq t\leq \tau}$ by piecing $(\tilde{\eta}^n_t, \tilde{x}^n_t)_{0\leq t\leq \tau_n}$ together for successive n. In other words, $(\eta_t, x_t) = (\tilde{\eta}^n_t, \tilde{x}^n_t)$ for $0 \leq t \leq \tau_n$, each n. Our goal is to show (a) $\tau = +\infty$ a.s.; and (b) the resulting stochastic process $(\eta_t, x_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution on $\mathcal{X}$. These will be proved if we can obtain a function v satisfying Lemma B.1 below.

Define the positive function v by

$$v(\eta, x) := \frac{1}{\eta^{1/2}} + \frac{1}{1-\eta} + \frac{1}{x - q^L_\alpha(\eta)} + \frac{1}{q^H_{\beta,\lambda}(\eta) - x}.$$

Note that v diverges to $+\infty$ at the boundaries of $\mathcal{X}$, so assumption (i) of Lemma B.1 is proved. Next, if assumption (iii) of Lemma B.1 holds (which we will prove below), then there exists N such that $\mathscr{L}v < 0$ on $\mathcal{X}\backslash\mathcal{X}_n$ for all $n > N$. Furthermore, for each given n, $\mathscr{L}v$ is bounded on $\mathcal{X}_n$. Consequently, we can find a constant c large enough such that $\mathscr{L}v \leq cv$ on all of $\mathcal{X}$, which verifies part (ii) of Lemma B.1.

It remains to prove assumption (iii) of Lemma B.1, namely that $\mathscr{L}v \to -\infty$ as $(\eta, x) \to \partial\mathcal{X}$. We will examine the boundaries of $\mathcal{X}$ one-by-one. In the following, we use the notation $g(x) = o(f(x))$ if $g(x)/f(x) \to 0$ as $x \to 0$, and the notation $g(x) = O(f(x))$ if $g(x)/f(x) \to C$ as $x \to 0$, where C is a finite constant.

As $\eta \to 0$ (and x bounded away from $q^L_\alpha(0)$ and $q^H_\beta(0)$, such that κ is bounded away from 0 and 1, the latter due to the definition of q^H_β), we have

$$\mu_\eta = \delta_h + \frac{a_e - a_h}{x}\kappa + \eta[\rho_h - \rho_e - \delta_e - \delta_h] + o(\eta) \quad \text{and} \quad |\sigma_\eta|^2 = \eta(\kappa - \eta)\frac{a_e - a_h}{x} + o(\eta)$$
$$\mu_x = O(1) \quad \text{and} \quad |\sigma_x|^2 = O(1).$$

We used condition (B.10) to obtain μ_x bounded. Thus,

$$\mathscr{L}v = -\frac{1}{2\eta^{3/2}}[\delta_h + \frac{1}{4}\frac{a_e - a_h}{x}\kappa] + o(\eta^{-3/2}) \to -\infty,$$

irrespective of $\delta_h > 0$ or $\delta_h = 0$.

As $\eta \to 1$ (and x bounded away from $q^L_\alpha(1)$ and $q^H_\beta(1)$; note that $\kappa = 1$ at this boundary), we have

$$\mu_\eta = -\delta_e - (\rho_e - \rho_h)(1 - \eta) + o(1 - \eta) \quad \text{and} \quad |\sigma_\eta|^2 = (1 - \eta)^2\sigma^2$$
$$\mu_x = O(1) \quad \text{and} \quad |\sigma_x|^2 = O(1).$$

We used condition (B.11) to obtain μ_x bounded. Thus,

$$\mathscr{L}v = -(1 - \eta)^{-2}\delta_e - (1 - \eta)^{-1}[\rho_e - \rho_h - \sigma^2] + o((1 - \eta)^{-1}) \to -\infty,$$

irrespective of δ_e, due to Assumption 1 part (iii).

We separately calculate the limit $x \to q^L_\alpha(\eta)$ (with η bounded away from 0) in the two cases $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$, since $\kappa < 1$ in the first case, and $\kappa = 1$ in the second case. Still, we find that in both cases,

$$\mu_\eta = O(1) \quad \text{and} \quad |\sigma_\eta|^2 = O(1)$$
$$\mu_x = o((x - q^L_\alpha)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1).$$

We used condition (B.8) to obtain the order of μ_x. Thus,

$$\mathscr{L}v = -(x - q^L_\alpha)^{-2}x\mu_x + O((x - q^L_\alpha)^{-3}) \to -\infty.$$

Similarly, we separately calculate the limit $x \to q^H_\beta(\eta)$ (with η bounded away from 0) in the two cases $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$. Again, we find that in both cases,

$$\mu_\eta = O(1) \quad \text{and} \quad |\sigma_\eta|^2 = O(1)$$
$$\mu_x = (-1) \times o((q_\beta^H - x)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1).$$

We used condition (B.9) to obtain the order of μ_x. Thus,

$$\mathscr{L}v = (q_\beta^H - x)^{-2} x \mu_x + O((q_\beta^H - x)^{-3}) \to -\infty.$$

Finally, all the corners of $\mathcal{X}$ can be analyzed in a straightforward way by combining the cases above, with the exception of $(\eta, x) = (0, q_\alpha^L(0)) = (0, a_h/\rho_h)$. Approaching this corner, we must take a particular path of $x \to a_h/\rho_h$ as $\eta \to 0$. Denote this path by $\hat{x}(\eta)$ and denote the asymptotic slope by $\hat{x}'(0) \in (\frac{d}{d\eta} q_\alpha^L(0), +\infty)$, where $\frac{d}{d\eta} q_\alpha^L(0) = [\frac{a_e}{a_h} - \frac{\rho_e}{\rho_h}]\frac{a_h}{\rho_h} + \alpha'(0) > 0$, by Assumption 1, part (i), and the fact that $\alpha'(0) > 0$. Denote the associated path of κ by $\hat{\kappa}(\eta)$ and the corresponding asymptotic slope by $\hat{\kappa}'(0) = \frac{1}{a_e - a_h}[\hat{x}'(0)\rho_h + (\rho_e - \rho_h)a_h/\rho_h]$. Substituting in, we find $\hat{\kappa}'(0) \in (1 + \frac{\alpha'(0)}{a_e - a_h}, +\infty)$. When computing $\mathscr{L}v$, we will take the supremum over all possible paths, meaning over $\hat{x}'(0)$ and $\hat{\kappa}'(0)$. Using similar calculations from the initial $\eta \to 0$ case, but using these paths, we obtain

$$\mu_\eta = \delta_h + \eta[\frac{a_e - a_h}{\hat{x}}\hat{\kappa}' + \rho_h - \rho_e - \delta_e - \delta_h] + o(\eta) \quad \text{and} \quad |\sigma_\eta|^2 = \eta^2[\hat{\kappa}' - 1]\frac{a_e - a_h}{\hat{x}} + o(\eta)$$
$$\mu_x = o((\hat{x} - q_\alpha^L)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1)$$
$$\text{and} \quad \sigma_x \cdot \sigma_\eta = \eta[\frac{a_e - a_h}{\hat{x}} - \sigma(\gamma(\hat{\kappa}' - 1)\frac{a_e - a_h}{\hat{x}})^{1/2}] + o(\eta).$$

Since $\hat{x} \geq O(\eta)$ and $\hat{\kappa} \geq O(\eta)$ (in the sense that both could be $+\infty$), we may treat terms like $(\hat{x} - q_\alpha^L)^{-1}$ as smaller than η^{-1}. This identifies the dominant terms as those associated to μ_η, $|\sigma_\eta|^2$, and μ_x. Thus,

$$\mathscr{L}v = -\frac{1}{2\eta^{3/2}}\delta_h + \frac{1}{2\eta^{1/2}}[\rho_e - \rho_h + \delta_e + \delta_h - \frac{a_e - a_h}{\hat{x}} - \frac{a_e - a_h}{\hat{x}}(\hat{\kappa}' - 1)/4] + o(\eta^{-3/2})$$
$$- (\hat{x} - q_\alpha^L)^{-2} x \mu_x + O((\hat{x} - q_\alpha^L)^{-3}) \to -\infty,$$

irrespective of δ_h, because $\rho_e - \rho_h - \frac{a_e - a_h}{a_h/\rho_h} = \rho_h[\rho_e/\rho_h - a_e/a_h] < 0$ by Assumption 1, part (i), and because $\inf\{\hat{\kappa}'(0)\} > 1$.

This completes the verification that $\mathscr{L}v \to -\infty$ as $(\eta, x) \to \partial\mathcal{X}$, which proves stationarity by Lemma B.1 below. This completes the proof. □

B.4 Stochastic stability: a useful lemma

To prove the stationarity claims of Theorem 1 and Proposition 1, we need the following lemma, which is a slight generalization of Theorems 3.5 and 3.7 of Khasminskii (2011), in the sense that weaker conditions are imposed on the coefficients α and β. Indeed, any coefficients (α, β) are permissible as long as they admit existence of a weak solution to the SDE system. The other generalization is that we allow the domain to be any open domain $\mathcal{D}$ rather than $\mathbb{R}^l$ (see also Remark 3.5 and Corollary 3.1 in Khasminskii (2011)).

Lemma B.1. *Suppose $(X_t)_{0\le t\le\tau}$ is a weak solution to the SDE $dX_t = \beta(X_t)dt + \alpha(X_t)dZ_t$ in an open connected domain $\mathcal{D} \subset \mathbb{R}^l$, where Z is a d-dimensional Brownian motion and $\tau := \inf\{t : X_t \notin \mathcal{D}\}$ is the first exit time from $\mathcal{D}$. Define the infinitesimal generator $\mathscr{L}$ by (for any C^2 function f)*

$$\mathscr{L}f = \sum_{i=1}^{n} \beta_i \frac{\partial f}{\partial x_i} f + \frac{1}{2}\sum_{i,j=1}^{n} (\alpha_i \cdot \alpha_j) \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Suppose there is a non-negative C^2 function $v : \mathcal{D} \mapsto \mathbb{R}_+$ such that (i) $\liminf_{x\to\partial\mathcal{D}} v(x) = +\infty$; (ii) $\mathscr{L}v \le cv$ for some constant $c \ge 0$; and (iii) $\limsup_{x\to\partial\mathcal{D}} \mathscr{L}v(x) = -\infty$. Then,

- *(a) $\tau = +\infty$ almost-surely;*
- *(b) the distribution of X_0 can be chosen such that $(X_t)_{t\ge 0}$ is stationary.*

Proof of Lemma B.1. Let $\{\mathcal{D}_n\}_{n\ge1}$ be an increasing sequence of open sets, whose closures are contained in $\mathcal{D}$, such that $\cup_{n\ge1}\mathcal{D}_n = \mathcal{D}$. Let $\tau_n := \inf\{t : X_t \notin \mathcal{D}_n\}$, and note that $\tau = \lim_{n\to\infty} \tau_n$ is the monotone limit of these exit times. Define $w(t,x) := v(x)\exp(-ct)$, which satisfies $\mathscr{L}w \le 0$ by assumption (ii). Using Itô's formula, we have

$$\mathbb{E}[v(X_{\tau_n\wedge t})e^{-c(\tau_n\wedge t)} - v(X_0)] = \mathbb{E}\int_0^{\tau_n\wedge t} \mathscr{L}w(u, X_u)du \le 0.$$

Since $(\tau_n \wedge t) \le t$ and $v \ge 0$, we obtain

$$\mathbb{E}[v(X_{\tau_n\wedge t})] \le e^{ct}\mathbb{E}[v(X_0)].$$

Because $\mathbb{E}[v(X_{\tau_n\wedge t})] \ge \mathbb{P}[\tau_n \le t]\inf_{x\in\mathcal{D}\setminus\mathcal{D}_n} v(x)$, we thus have

$$\mathbb{P}[\tau_n \le t] \le \frac{e^{ct}\mathbb{E}[v(X_0)]}{\inf_{x\in\mathcal{D}\setminus\mathcal{D}_n} v(x)}.$$

Taking the limit $n \to \infty$, we obtain

$$\mathbb{P}[\tau \leq t] \leq \frac{e^{ct}\mathbb{E}[v(X_0)]}{\liminf_{x \to \partial \mathcal{D}} v(x)} = 0.$$

Thus, taking $t \to \infty$, we prove (a).

Next, since $\tau = +\infty$ a.s., we may consider $(X_t)_{t \geq 0}$ that is now defined for all time. Using Itô's formula,

$$\mathbb{E}[v(X_{\tau_n \wedge t}) - v(X_0)] = \mathbb{E} \int_0^{\tau_n \wedge t} \mathscr{L}v(X_u)du.$$

Note that $\min(\inf_t \mathbb{E}[v(X_t) - v(X_0)], \inf_n \mathbb{E}[v(X_{\tau_n}) - v(X_0)]) \geq b_1$ for some constant b_1, given assumption (i) and $v \geq 0$. Also note that $\sup_{x \in \mathcal{D}} \mathscr{L}v(x) \leq b_2$ for some constant b_2, given assumptions (i)-(iii) and the fact that v is C^2. (b_1 and b_2 are both independent of t and n.) Using these bounds, plus the following obvious inequality

$$\mathscr{L}v(X_u) \leq \mathbf{1}_{\{X_u \in \mathcal{D} \setminus \mathcal{D}_k\}} \sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L}v(x) + \sup_{x \in \mathcal{D}} \mathscr{L}v(x),$$

we get

$$-\sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L}v(x) \mathbb{E} \int_0^{\tau_n \wedge t} \mathbf{1}_{\{X_u \in \mathcal{D} \setminus \mathcal{D}_k\}} du \leq tb_2 - b_1.$$

Given the proof of (a), we may take the limit $n \to \infty$ (so that $\tau_n \to +\infty$), then apply Fubini's theorem, and then rearrange to obtain

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}[X_u \in \mathcal{D} \setminus \mathcal{D}_k] du \leq \frac{b_2}{-\sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L}v(x)}.$$

Taking $k \to \infty$ and using assumption (iii), we obtain

$$\lim_{k \to \infty} \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}[X_u \in \mathcal{D} \setminus \mathcal{D}_k] du \leq 0.$$

Applying Theorem 3.1 of Khasminskii (2011), there exists a stationary initial distribution for X_0. The process $(X_t)_{t \geq 0}$ augmented with this initial distribution is clearly stationary by definition. □

B.5 Proofs of Corollaries 1-3

Proof of Corollary 1. Start from the construction of S-BSE in Theorem 1, and note that we can make ϵ arbitrarily small such that the boundaries $q_\alpha^L \to \bar{a}/\bar{\rho}$ and $q_\beta^H \to a_e/\bar{\rho}$. In addition, we may take $\eta_\beta^* \to \eta^*$, its minimal possible level. Hence, an S-BSE can be constructed such that the set of prices q matches $\mathcal{Q}(\eta)$ arbitrarily closely. The result on return variance comes from using (B.6) when $\kappa < 1$ (i.e., when $\eta < \eta^*$) and using (B.7) when $\kappa = 1$ (i.e., when $\eta \geq \eta^*$ and q is at its upper bound). Using the definition of η^* provides the form of $\mathcal{V}$ with the minimum as the lower bound. □

Proof of Corollaries 2-3. These follow from the proof of Theorem 1. □

C Proofs and analysis for Section 3

C.1 Proof of Proposition 1

We provide a sketch the proof. Essentially, we want to construct an upper bound for the price based on the fundamental equilibrium, and the lower bound for the price based on a small perturbation of the worst-case price (we want to include this perturbation since volatility explodes when the price approaches its worst-case value). For notation, recall that $\bar{\rho} := \eta\rho_e + (1-\eta)\rho_h$. By analogy, define $\bar{a} := \eta a_e + (1-\eta)a_h$.

Upper and lower bounds for price. Let $(\hat{q}^0, \hat{\kappa}^0)$ be the solution to the fundamental equilibrium (which exists by assumption), and let $\eta^0 := \inf\{\eta : \hat{\kappa}^0 \geq 1\}$. By Lemma E.1 part (v), if σ is small enough then $\eta^0 < 1$, which we assume to be the case. Then, define

$$q^0(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if} \quad \eta < \eta^0; \\ \hat{q}^0(\eta) + \varphi(\eta), & \text{if} \quad \eta \geq \eta^0, \end{cases} \tag{C.1}$$

where φ is a C^2 function with the properties $\varphi(\eta^0) = 0$ and $\varphi' > (\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})'$ for all η. In words, q^0 is equal to the fundamental equilibrium price $\hat{q}^0$ whenever $\hat{\kappa}^0 \leq 1$ and above it when $\hat{\kappa}^0 = 1$. For the other extremal function, use the "worst-case" price

$$q^1(\eta) := \bar{a}(\eta)/\bar{\rho}(\eta). \tag{C.2}$$

Importantly, we have $q^0 > q^1$ for all η.

Candidate price. We proceed to combine these two extremal functions according to the following convex combination, where $\alpha \in (0,1)$ is fixed:

$$\tilde{q}(\eta,s) := (1-\alpha s)q^0(\eta) + \alpha s q^1(\eta), \quad (\eta,s) \in \mathcal{D} = (0,1) \times \mathcal{S}. \tag{C.3}$$

where $\mathcal{S} = (0,1)$ is the domain for the sunspot state s. For each $s \in \mathcal{S}$, define $\eta^*(s) := \inf\{\eta : \tilde{q}(\eta,s) \geq a_e/\bar{\rho}\}$, which can be shown is strictly increasing.[26] Put

$$q(\eta,s) := \begin{cases} \tilde{q}(\eta,s), & \text{if} \quad \eta < \eta^*(s) \\ a_e/\bar{\rho}(\eta), & \text{if} \quad \eta \geq \eta^*(s) \end{cases} \quad \text{and} \quad \kappa := \frac{\bar{\rho}q - a_h}{a_e - a_h}.$$

By construction, the pair (q,κ) satisfy equation (PO).

Volatility. Gven the fact that $\alpha < 1$ in (C.3), the resulting capital price is always bounded away from the worst-case price, except as $\eta \to 0$. Thus, the resulting equilibrium volatility will remain bounded for the exact same reasons as in the construction of Theorem 1 (which used a small perturbation of the state space to keep capital prices away from their worst-case value). We omit the construction of this return volatility $|\sigma_R|$, since it is identical to Theorem 1. Given the value of $|\sigma_R|$ and the identity $|\sigma_R|^2 = \frac{\sigma^2 + (\sigma_s \partial_s \log q)^2}{[1-(\kappa-\eta)\partial_\eta \log q]^2}$, we obtain σ_s by inverting this identity. Some technical checks are required to ensure that the resulting σ_s is real, but this can be done. (If $\sigma = 0$, this is guaranteed.)

Sunspot drift and stationarity. Having determined q, κ, and σ_s, we define μ_η and σ_η by (13)-(14). It remains to determine μ_s. We will pick $\mu_s(\eta,s) = m(\eta,s)$, where m is a C^2 function with the following properties: $\partial_s m < 0$, and for some $0 \leq s^0 < s^1 \leq 1$

[26] Indeed, note that $\tilde{q}$ is C^2 on $(\eta^0,\eta^1) \times \mathcal{S}$, which implies η^* is C^1. Then, use the fact that η^* is C^1 to differentiate $\tilde{q}(\eta^*(s),s) = a_e/\bar{\rho}(\eta^*(s))$ with respect to s, and use the fact that $\partial_s \tilde{q} = q^1 - q^0$, and finally rearrange to obtain

$$(\eta^*)'(s)\Big[\partial_\eta \tilde{q}(\eta^*(s),s) + \frac{a_e}{\bar{\rho}(\eta^*(s))}\frac{\rho_e - \rho_h}{\bar{\rho}(\eta^*(s))}\Big] = q^0(\eta^*(s)) - q^1(\eta^*(s)).$$

If at any point s, we had $(\eta^*)'(s) = 0$, we would necessarily have $q^0(\eta^*(s)) = q^1(\eta^*(s))$. But this contradicts the that $q^0 > q^1$. Thus, $(\eta^*)'(s) \neq 0$ for all s. We can also rule out $(\eta^*)'(s) < 0$ by the fact that $\eta^*(0+) = \eta^0$ and $\eta^*(s) \geq \eta^0$ for all s. Thus, $(\eta^*)'(s) > 0$ for all s.

thresholds,

$$\text{(if } s^0 > 0) \qquad \inf_{\eta \in (0,1)} \lim_{s \searrow s^0} (s - s^0) m(\eta, s) = +\infty \tag{C.4}$$

$$\text{(if } s^0 = 0) \qquad \inf_{\eta \in (0,1)} \lim_{s \searrow s^0} m(\eta, s) > 0 \tag{C.5}$$

$$\sup_{\eta \in (0,1)} \lim_{s \nearrow s^1} (s^1 - s) m(\eta, s) = -\infty. \tag{C.6}$$

Given this choice, we need to demonstrate the time-paths $(\eta_t, s_t)_{t \geq 0}$ remain in $\mathcal{D}$ almost-surely and admit a stationary distribution. This step is very similar to the stochastic stability step in Theorem 1 and is therefore omitted. We simply note that the Lyapunov function to use in this step is $v(\eta, s) := \frac{1}{\eta^{1/2}} + \frac{1}{1-\eta} + \frac{1}{1-s} + \frac{1}{s}$. □

C.2 Proofs of Propositions 2-3

PROOF OF PROPOSITION 2. Fix any $\Sigma^* > 0$. The proof is a simple consequence of the fact that σ_q must be unbounded as κ approaches η, which is as q approaches the worst-case price q^1. We fill in the technical details below.

We construct a sequence of equilibria—indexed by (α, ζ)—as follows. Recall the capital price construction in Proposition 1:

$$q = (1 - \alpha s) q^0 + \alpha s q^1, \quad \text{when} \quad \kappa < 1,$$

where $\alpha < 1$ is a parameter, q^0 is the fundamental equilibrium price, and $q^1 = \bar{a}/\bar{\rho}$ is the worst-case price. Based on the discussion in the text, we may choose μ_s such that equilibrium concentrates on any particular value of s. Thus, pick μ_s such that $s_t \geq \zeta$ almost-surely. Clearly, the choice of μ_s depends on α, but such a choice can always be made for any parameters.

Let $p_{\text{low}} > 0$, $p_{\text{high}} > 0$ be given with $p_{\text{low}} + p_{\text{high}} < 1$. First, note that there exist α^*, ζ^*, ϵ^* such that $\mathbb{P}[\eta_t \leq \epsilon \cap \kappa_t < 1] < p_{\text{low}}$ and $\mathbb{P}[\eta_t \geq 1 - \epsilon \cap \kappa_t < 1] < p_{\text{high}}$ for all $\alpha > \alpha^*$, $\zeta > \zeta^*$, and $\epsilon < \epsilon^*$. This is a consequence of the fact that in any stationary distribution, we have $\lim_{x \to 0} \mathbb{P}[\eta_t < x] = \lim_{x \to 1} \mathbb{P}[\eta_t > x] = 0$ and the fact that $\lim_{\alpha \to 1} \lim_{s \to 1} \kappa(\eta, s) < 1$ for all η.

At this point, fix such an $\epsilon < \epsilon^*$. Let a constant $M > 0$ be given satisfying

$$M \leq (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2}{\rho_e a_e / \rho_h} \frac{\epsilon(1 - \epsilon)}{\Sigma^*}. \tag{C.7}$$

Note that

$$\lim_{\alpha\to 1}\lim_{s\to 1}\sup_{\eta\in(\epsilon,1-\epsilon)}\Big|q(\eta,s)-\bar{a}(\eta)/\bar{\rho}(\eta)\Big| = 0.$$

Consequently, we may pick $\alpha > \alpha^*$ close enough to 1 and $\zeta > \zeta^*$ close enough to 1 such that

$$\sup_{s\in(\zeta,1)}\sup_{\eta\in(\epsilon,1-\epsilon)}\Big|q(\eta,s)-\bar{a}(\eta)/\bar{\rho}(\eta)\Big| \le M.$$

Finally, using equation (23) and substituting $\kappa < 1$ from (PO), we have $|\sigma\binom{1}{0}+\sigma_q|^2 = \frac{(a_e-a_h)^2}{q}\frac{\eta(1-\eta)}{\bar{\rho}q-\bar{a}}$. Note also that $q \le a_e/\rho_h$ and $\bar{\rho} \le \rho_e$ are upper bounds. Then,

$$\mathbb{E}[|\sigma\tbinom{1}{0}+\sigma_{q,t}|^2] > (1-p_{\text{low}}-p_{\text{high}})\frac{(a_e-a_h)^2}{\rho_e a_e/\rho_h}\frac{\epsilon(1-\epsilon)}{M}.$$

Using (C.7), we obtain $\mathbb{E}[|\sigma\binom{1}{0}+\sigma_{q,t}|^2] > \Sigma^*$. □

Proof of Proposition 3. First, we prove that $|\sigma_R|$ is increasing in s. From (23), we obtain $|\sigma_R|^2 = \frac{(a_e-a_h)^2}{q}\frac{\eta(1-\eta)}{\bar{\rho}q-\bar{a}}$ on $\{\kappa < 1\}$. Differentiating with respect to s, and using $\partial_s q = \alpha(q^1 - q^0) < 0$, we obtain

$$\partial_s|\sigma_R|^2 = -\eta(1-\eta)\frac{(a_e-a_h)^2}{q(\bar{\rho}q-\bar{a})}\Big[\frac{1}{q}+\frac{\bar{\rho}}{\bar{\rho}q-\bar{a}}\Big]\partial_s q > 0.$$

Next, revisiting the proof of Proposition 1, we compute on $\{\kappa < 1\}$,

$$\partial_s[(\kappa-\eta)\partial_\eta \log q] = \alpha\Big[(\kappa-\eta)\frac{(q^1)'-(q^0)'}{q}+\frac{\bar{a}(q^1-q^0)}{(a_e-a_h)q^2}\partial_\eta q\Big] < 0.$$

The inequality uses the properties of the φ function in (C.1) to say $(q^1)'-(q^0)' < 0$, along with the obvious facts $q^1 - q^0 < 0$ and $\partial_\eta q > 0$. Using $|\binom{1}{0}\cdot\sigma_R| = \frac{\sigma}{1-(\kappa-\eta)\partial_\eta \log q}$, we obtain $\partial_s|\binom{1}{0}\cdot\sigma_R| < 0$.

Using the two claims just proved, and the identity $|\sigma_R|^2 = |\binom{0}{1}\cdot\sigma_R|^2 + |\binom{1}{0}\cdot\sigma_R|^2$, we see that $|\binom{0}{1}\cdot\sigma_R|$ is increasing in s on $\{\kappa < 1\}$. For the same reason, namely $|\binom{0}{1}\cdot\sigma_R|^2$ is both smaller and increasing faster than $|\sigma_R|$, we have that $|\binom{0}{1}\cdot\sigma_R|/|\sigma_R|$ increasing in s on $\{\kappa < 1\}$. □

C.3 Model with jumps in Section 3.4

Recall that our jumps ℓ_q are assumed to occur randomly but have a known size, given observables. Therefore, optimal portfolio conditions are

$$\frac{a_e}{q} + g + \mu_q + \sigma\binom{1}{0} \cdot \sigma_q - r = \frac{\kappa}{\eta}|\sigma_R|^2 + \frac{\lambda \ell_q}{1 - \frac{\kappa}{\eta}\ell_q}$$
$$\frac{a_h}{q} + g + \mu_q + \sigma\binom{1}{0} \cdot \sigma_q - r \leq \frac{1-\kappa}{1-\eta}|\sigma_R|^2 + \frac{\lambda \ell_q}{1 - \frac{1-\kappa}{1-\eta}\ell_q}.$$

Combining these two equations, we obtain (RBJ).

We can determine the other equilibrium objects similarly to before. The riskless rate is given by

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\binom{1}{0} \cdot \sigma_q - \Big(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\Big)|\sigma_R|^2 - \lambda \ell_q \Big(\frac{\kappa}{1 - \frac{\kappa}{\eta}\ell_q} + \frac{1-\kappa}{1 - \frac{1-\kappa}{1-\eta}\ell_q}\Big).$$

The dynamics of η are now given by $d\eta_t = \mu_{\eta,t-}dt + \sigma_{\eta,t-} \cdot dZ_t - \ell_{\eta,t-}dJ_t$, where

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2 + \delta_h - (\delta_e + \delta_h)\eta + \frac{(\kappa - \eta)\lambda \ell_q}{\big(1 - \frac{\kappa}{\eta}\ell_q\big)\big(1 - \frac{1-\kappa}{1-\eta}\ell_q\big)}$$
$$\sigma_\eta = (\kappa - \eta)\sigma_R.$$

The wealth share jump ℓ_η is derived by using knowledge of the jump size in q and noting that agents' portfolios (capital and bonds) are predetermined:[27]

$$\ell_\eta = (\kappa - \eta)\frac{\ell_q}{1 - \ell_q}.$$

For a valid equilibrium, jumps cannot be so large as to send experts into bankruptcy, nor can they induce households' leverage to exceed experts' (as this would contradict (RBJ)). It turns out the no-bankruptcy condition, which says $\ell_q < \kappa/\eta$, is automatically satisfied given (RBJ) holds; intuitively, experts would never take so much risk that their wealth is wiped out. The other requirement, that jumps not send the economy into a region in

[27]The derivation is as follows. Let variables with hats, e.g., "$\hat{x}$", denote post-jump variables. Note $\hat{N}_e = \hat{q}\hat{K}\kappa - B$ and $\hat{N}_h = \hat{q}\hat{K}(1-\kappa) + B$, where B is expert borrowing (and household lending, by bond market clearing). Then, $\hat{\eta} = \hat{N}_e/(\hat{q}\hat{K}) = \kappa - B/(\hat{q}\hat{K})$ and by similar logic the pre-jump wealth share is $\eta = \kappa - B/qK$. Thus, $\ell_\eta = \eta - \hat{\eta} = B[1/(\hat{q}\hat{K}) - 1/(qK)] = qK(\kappa - \eta)[1/(\hat{q}\hat{K}) - 1/(qK)]$. Using the fact that $\hat{K} = K$ and the definition $\ell_q := 1 - \hat{q}/q$, we arrive at $\ell_\eta = (\kappa - \eta)[(1-\ell_q)^{-1} - 1]$. This derivation assumes the presumably risk-free bond price does not jump when capital prices jump. Conceptually, there is no reason why this needs to be true, but it preserves its risk-free conjecture.

which $\eta \leq \kappa$, can be stated as

$$\bar{\rho}(\hat{\eta})(1-\ell_q)q > (a_e - a_h)\hat{\eta} + a_h, \tag{C.8}$$

where $\hat{\eta} := \eta - (\kappa - \eta)\frac{\ell_q}{1-\ell_q}$ is the post-jump expert wealth share. Although it is obvious, (RBJ) implies another bound on ℓ_q that arises because of $|\sigma_R| \geq 0$, which is

$$\frac{a_e - a_h}{q} \geq \frac{\kappa - \eta}{\eta(1-\eta)}\Big[\frac{\lambda \ell_q^2}{\left(1 - \frac{\kappa}{\eta}\ell_q\right)\left(1 - \frac{1-\kappa}{1-\eta}\ell_q\right)}\Big]. \tag{C.9}$$

Condition (C.9) evaluated at equality implies that all risk is jump risk. With these equations in hand, we describe our simulation procedure.

Step 0. Given (η, q) solve for $\kappa(\eta, q)$ from (PO).

Step 1. Solve for the upper bound of $\ell_q(\eta, q)$ using (C.8)-(C.9).

Note that, fixing (η, q), the RHS of (C.9) is strictly increasing in ℓ_q when $\ell_q \in \left(0, \frac{\eta}{\kappa}\right)$ while the LHS is constant. Moreover, the inequality is satisfied for $\ell_q = 0$ and violated as $\ell_q \to \frac{\eta}{\kappa}$. Hence, this condition defines an upper bound $\ell_q^A(\eta, q)$, which can be solved by a bisection procedure.

Next, after some algebra, we can write condition (C.8) as

$$(1-\ell_q)^2 - (1-\ell_q) + \underbrace{\frac{(a_e - a_h)(\kappa - \eta)}{\bar{\rho}(\eta)q + q(\rho_e - \rho_h)(\kappa - \eta)}}_{:=\varphi(\eta,q)} > 0.$$

It is straightforward to notice that the condition holds for any $\ell_q \in (0,1)$ if $\varphi(\eta, q) \geq 1/4$. When $\varphi(\eta, q) < 1/4$, then the condition holds for $\ell_q \in (0, \ell_q^{B,low}) \cup (\ell_q^{B,high}, 1)$, where

$$1 - \ell_q^{B,high} = \frac{1}{2}\left(1 - \sqrt{1 - 4\varphi}\right) \quad \text{and} \quad 1 - \ell_q^{B,low} = \frac{1}{2}\left(1 + \sqrt{1 - 4\varphi}\right).$$

Define

$$\ell_q^B := \mathbf{1}_{\{\varphi \geq 1/4\}} + \ell_q^{B,low}\mathbf{1}_{\{\varphi < 1/4\}}.$$

Then, an upper bound that ensures all required inequalities are satisfied is

$$\ell_q^{\max}(\eta, q) := \min\{\ell_q^A(\eta, q), \ell_q^B(\eta, q)\}.$$

Step 2. Choose a sub-region within the domain $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1 - \eta) a_h < q\bar{\rho}(\eta) \leq a_e\}$ that is away from the upper and lower boundaries. For example, in our numerical exercise, we choose the sub-region $\mathcal{D}^\circ := \{(\eta, q) : \kappa < 0.98 \text{ and } \kappa > \eta + 0.02\}$. On $\mathcal{D} \backslash \mathcal{D}^\circ$, we will set $\ell_q = 0$ and choose μ_q to ensure the economy never escapes $\mathcal{D}$. In fact, we can choose μ_q in a way that the boundary of $\mathcal{D}^\circ$ acts arbitrarily close to a reflecting boundary, which is what we have done for Figure 8. Pick an arbitrary function $\ell_q(\eta, q) \in [0, \ell_q^{\max}(\eta, q))$ and an arbitrary μ_q for the set $\mathcal{D}^\circ$.

Step 3. Use risk-balance condition (RBJ) to solve for $|\sigma_R|^2$. For each (η, q), assign $\gamma(\eta, q)$ fraction of the variance to the fundamental Brownian shock, and $1 - \gamma(\eta, q)$ to the sunspot Brownian shock. In constructing Figure 8, we set $\gamma \equiv 1$. Then, solve for other equilibrium objects from the equations above.

Online Appendix 2 (not for publication):

Rational Sentiments and Financial Frictions

Paymon Khorrami and Fernando Mendo

June 19, 2024

D Model extensions and further analyses

D.1 Partial equity issuance

We extend the model to allow some equity issuance by capital holders, subject to a constraint. In particular, at any point of time, agents managing capital can issue some equity to the market, but the issuer must keep at least $\chi \in [0,1]$ fraction of their capital risk—this is a so-called "skin-in-the-game" constraint. In other words, if experts and households retain χ_e and χ_h of their capital risk, respectively, it must be the case that

$$\chi_{\ell,t} \geq \chi, \quad \ell \in \{e,h\}. \tag{D.1}$$

Thus, the frictionless model corresponds to $\chi = 0$, while our baseline model corresponds to $\chi = 1$. Outside equity contracts are risky, having risk exposure σ_R (the endogenous capital return volatility), so they must promise an excess return $\sigma_R \cdot \pi$, where π is the equilibrium risk price vector that applies to securities tradable by both experts and households.

Agents' dynamic budget constraints are now given by

$$\begin{aligned} dn_{\ell,t} = \Big[(n_{\ell,t} - q_t k_{\ell,t}) r_t - c_{\ell,t} + a_\ell k_{\ell,t}\Big] dt + d(q_t k_{\ell,t}) \\ + [\theta_{\ell,t} n_{\ell,t} - (1-\chi_{\ell,t}) q_t k_{\ell,t}] \sigma_{R,t} \cdot (\pi_t dt + dZ_t). \end{aligned} \tag{D.2}$$

The second line of (D.2) contains the new terms pertaining to equity-issuance: $\theta_{\ell,t} \geq 0$ denotes purchases of equity contracts in the market, per unit of wealth, while $\chi_{\ell,t}$ denotes the fraction of capital risk. Notice that it will be without loss of generality to assume $\chi_{\ell,t} = \chi$ at all times and for all agents, because the purchase variable $\theta_{\ell,t}$ is available as a control. For example, an agent with a slack equity-issuance constraint ($\chi_\ell > \chi$) could issue equity to the constraint (D.1) and then buy back such equity by increasing their θ_ℓ control. Going forward, we simply assume $\chi_{e,t} = \chi_{h,t} = \chi$. The presence of a public equity market implies an additional market clearing condition for equity securities, namely

$$\theta_{e,t}N_{e,t} + \theta_{h,t}N_{h,t} = (1-\chi)q_t K_t. \tag{D.3}$$

At this point, we may solve for equilibrium.

Model solution. The introduction of equity issuance changes nothing about optimal consumption choices, so the price-output relation (PO) still holds.

Optimal portfolio choice now implies the following four FOCs:

$$\mu_{R,e} - (1-\chi)\sigma_R \cdot \pi - r = \chi\Big(\frac{\chi q k_e}{n_e} + \theta_e\Big)|\sigma_R|^2 \tag{D.4}$$

$$\mu_{R,h} - (1-\chi)\sigma_R \cdot \pi - r \leq \chi\Big(\frac{\chi q k_h}{n_h} + \theta_h\Big)|\sigma_R|^2, \quad \text{with equality if } k_h > 0 \tag{D.5}$$

$$\Big(\frac{\chi q k_e}{n_e} + \theta_e\Big)|\sigma_R|^2 \geq \sigma_R \cdot \pi, \quad \text{with equality if } \theta_e > 0 \tag{D.6}$$

$$\Big(\frac{\chi q k_h}{n_h} + \theta_h\Big)|\sigma_R|^2 \geq \sigma_R \cdot \pi, \quad \text{with equality if } \theta_h > 0 \tag{D.7}$$

where $\mu_{R,\ell} := \frac{a_\ell}{q} + g + \mu_q + \sigma\sigma_q \cdot \binom{1}{0}$ is the expected return on capital for agent ℓ. Equations (D.4)-(D.5) are the FOCs for capital holdings, and (D.6)-(D.7) are the FOCs for equity purchases. Note that the equality in (D.4) assumes $k_e > 0$, which is easy to verify must always be the case in equilibrium, exactly as in the baseline model.

We can derive a new "risk-balance" condition, analogously to the baseline model. If in addition to $k_e > 0$ we have $k_h > 0$, then we cannot simultaneously have $\theta_e > 0$, as this would contradict $\mu_{R,e} > \mu_{R,h}$. Thus, $\theta_e = 0$ whenever $k_h > 0$, and so we may difference (D.4)-(D.5) and use the market clearing condition (D.3) to substitute $\theta_h = \frac{1-\chi}{1-\eta}$, which leads to

$$0 = \min\Big[1-\kappa,\ \frac{a_e - a_h}{q} - \chi\frac{\chi\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2\Big]. \tag{RBE}$$

In addition to (RBE), equation (D.7) must hold with equality and (D.6) with inequality when $\kappa < 1$. By (D.7) and the derived expression $\theta_h = \frac{1-\chi}{1-\eta}$, we have $\sigma_R \cdot \pi = \frac{1-\chi\kappa}{1-\eta}|\sigma_R|^2$, for which a viable solution is

$$\pi = \frac{1-\chi\kappa}{1-\eta}\sigma_R, \quad \text{if } \kappa < 1. \tag{D.8}$$

Using this expression for π, (D.6) requires $\chi\kappa \geq \eta$, which holds by equation (RBE).

By contrast, when $k_h = 0$ (so $\kappa = 1$), equations (D.6)-(D.7) imply

$$\pi = \min\left(1, \frac{1-\chi}{1-\eta}\right)\sigma_R, \quad \text{if } \kappa = 1. \tag{D.9}$$

To prove this, combine the two possible cases:

(i) Suppose $\theta_e > 0$. Note that $\theta_h = 0$ cannot occur, as $\theta_e > 0$ implies $\sigma_R \cdot \pi > 0$ while $k_h = \theta_h = 0$ implies the opposite. Thus, we may combine (D.6)-(D.7), both evaluated under equality, to obtain $\theta_h = \theta_e + \frac{\chi}{\eta}$. Plugging this result into market clearing (D.3) yields $\theta_e = 1 - \chi/\eta$ and $\theta_h = 1$. Using $\theta_h = 1$ back in (D.7), we obtain $\sigma_R \cdot \pi = |\sigma_R|^2$, for which a viable solution is $\pi = \sigma_R$. Note that $\theta_e = 1 - \chi/\eta > 0$ if and only if $\eta > \chi$.

(ii) Suppose $\theta_e = 0$. Note that market clearing (D.3) implies $\theta_h = \frac{1-\chi}{1-\eta} > 0$ in this case. By (D.7), we have $\sigma_R \cdot \pi = \frac{1-\chi}{1-\eta}|\sigma_R|^2$, for which a viable solution is $\pi = \frac{1-\chi}{1-\eta}\sigma_R$. Using the expression for π, (D.6) requires $\eta \leq \chi$.

Putting the results of (D.8)-(D.9) together, we have that

$$\pi = \min\left(1, \frac{1-\chi\kappa}{1-\eta}\right)\sigma_R. \tag{D.10}$$

Finally, the riskless interest rate can be derived as always, by summing a $(\kappa, 1-\kappa)$-weighted-average of equations (D.4)-(D.5) to get

$$\begin{aligned} r = {} & \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - (1-\chi)\sigma_R \cdot \pi \\ & - \chi\left[\kappa\left(\frac{\chi\kappa}{\eta} + \theta_e\right) + (1-\kappa)\left(\frac{\chi(1-\kappa)}{1-\eta} + \theta_h\right)\right]|\sigma_R|^2. \end{aligned} \tag{D.11}$$

We can simplify this equation using the following facts. First, from the discussion above, $\theta_h > 0$ always holds, so that (D.7) holds with equality, hence $\theta_h = \frac{\sigma_R \cdot \pi}{|\sigma_R|^2} - \frac{\chi(1-\kappa)}{1-\eta}$. Next, we may use the market clearing condition (D.3) to obtain $\theta_e = \frac{1-\chi}{\eta} - \frac{1-\eta}{\eta}\theta_h$. We use these two facts to eliminate θ_e and θ_h from (D.11), then we substitute the solution for π from (D.10), and finally we simplify the result to obtain

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - |\sigma_R|^2 - \left(\frac{\chi\kappa}{\eta} - 1\right)\max\left(0, \frac{\chi\kappa - \eta}{1-\eta}\right). \tag{D.12}$$

This completes the derivation of equilibrium.

Properties of equilibrium. For any $\chi > 0$, we can construct S-BSEs using a similar procedure as the baseline model, i.e., by solving equation (PO) for κ as a function of (η, q), and then substituting this into (RBE) to also solve for $|\sigma_R|$ as a function of (η, q). Importantly, any solution to equation (RBE) requires $\chi\kappa \geq \eta$, and so the effect of lower equity issuance frictions (lower χ) is to reduce the range of possible fluctuations of κ, hence q, for any given η. This effect is depicted in Figure D.1, which shows that the range of possible fluctuations for price q is unambiguously shrinking as χ falls.

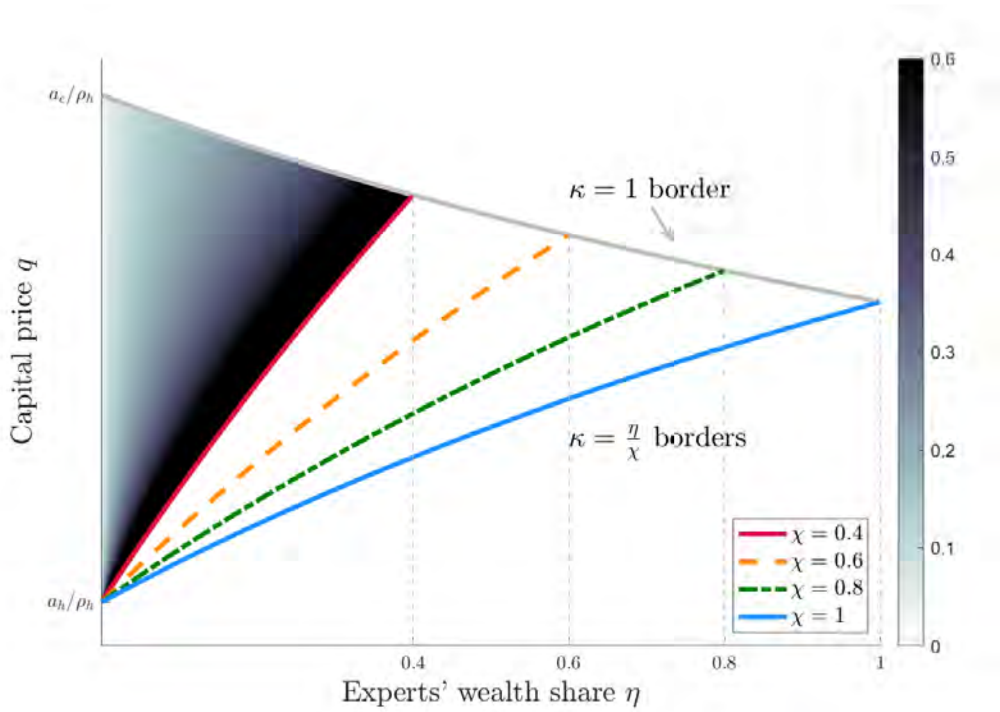

Figure D.1: Colormap of volatility $|\sigma_R|$ as a function of (η, q), in the region $\mathcal{D} := \{(\eta, q) : \eta \in (0,1)$ and $(\eta/\chi)a_e + (1 - \eta/\chi)a_h < q\bar{\rho}(\eta) \leq a_e\}$. Volatility is truncated for aesthetic purposes (because $|\sigma_R| \to \infty$ as $\kappa \to \eta/\chi$). Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$.

In particular, as $\chi \to 0$, no sunspot equilibrium can exist. This is very easy to see—a solution $\kappa < 1$ to (RBE) requires $\chi\kappa \geq \eta$, but as $\chi \to 0$ this becomes impossible for any $\eta > 0$. Thus, as $\chi \to 0$, capital misallocation converges to zero, and capital prices converge to their maximum $a_e/\bar{\rho}(\eta)$. Relatedly, taking $\chi \to 0$ in equation (D.10), we see that $\pi \to \sigma_R$ for each $\eta > 0$. This is the complete-markets risk price with log utility agents. Thus, as $\chi \to 0$, risk allocations converge to the frictionless solution. As both capital and risk are allocated frictionlessly in the limit, the First Welfare Theorem obtains.

Proposition D.1. *As $\chi \to 0$, the set of equilibria converges to a singleton, namely the non-stochastic Fundamental Equilibrium with $\kappa_t = 1$ and $q_t = a_e/\bar{\rho}(\eta_t)$.*

D.2 Idiosyncratic uncertainty

Here, we add idiosyncratic risk to capital. Doing so raises two substantive points: (i) small idiosyncratic uncertainty can provide some equilibrium refinement, by selecting equilibria with the property $\lim_{\eta \to 0} \kappa = 0$; (ii) the addition of idiosyncratic uncertainty allows us to study, in a non-trivial way, the stability properties of the "deterministic steady state" of our model.

Setting. In addition to the model assumptions listed in Section 1, individual capital now evolves as

$$dk_{i,t} = k_{i,t}[gdt + \tilde{\sigma} d\tilde{B}_{i,t}], \tag{D.13}$$

where $(\tilde{B}_i)_{i \in [0,1]}$ is a continuum of independent Brownian motions. Agents with indexes $i \in [0, \nu]$ are experts, and those with $i \in [\nu, 1]$ are households. As in Section 1, the aggregate stock of capital $K_t := \int_0^1 k_{i,t} di$ grows as $dK_t = K_t[gdt + \sigma dZ_t^{(1)}]$. Also as before, the second shock $Z^{(2)}$ is the sunspot shock, independent of $Z^{(1)}$. The idiosyncratic Brownian motions are independent of Z. Besides this addition of idiosyncratic uncertainty, the definition of equilibrium is the same as Definition 1. Conjecture $dq_t = q_t[\mu_{q,t} dt + \sigma_{q,t} \cdot dZ_t]$.

Small uncertainty as equilibrium refinement. The first result in this environment is that *any* equilibrium must feature full deleveraging by experts, as they become poor, simply as a consequence of portfolio optimality. To see this, note that risk-balance condition (RB), the combination of expert and household capital FOCs, is now modified to read

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)} (\tilde{\sigma}^2 + |\sigma_R|^2) \right], \tag{D.14}$$

where $\sigma_R := \sigma(\begin{smallmatrix}1\\0\end{smallmatrix}) + \sigma_q$ is the aggregate diffusion in capital returns, as before. Note that $a_e - a_h > 0$ and $\tilde{\sigma}^2 + |\sigma_R|^2 > 0$. Thus, as $\eta \to 0$, we must have $\kappa \to 0$, a property that holds for any arbitrarily small $\tilde{\sigma}$.

Lemma D.1. *Any equilibrium with $\tilde{\sigma} > 0$ has the property $\lim_{\eta \to 0} \kappa = 0$.*

Intuitively, idiosyncratic risk gives experts an additional motive to sell capital. This motive is magnified as experts become relatively poorer, because the risk is embedded in the capital stock, which is then amplified by leverage in affecting experts' net worth. In fact, the selling motive is magnified infinitely, because experts that do not sell capital will see their leverage grow unboundedly as their wealth shrinks. Thus, even a small amount idiosyncratic risk is enough to force coordination on maximal selling in response to negative shocks.

Steady state instability. In an attempt to differentiate ourselves from the literature, here we examine the traditional stability properties of this model. The addition of idiosyncratic risk provides a convenient environment for stability analysis, for the following reason. Stability properties are typically studied around the "steady state" of a deterministic equilibrium. In our model with $\tilde{\sigma} = 0$, studying a deterministic equilibrium instead puts us in the FE, which trivially has $\kappa = 1$ always. With idiosyncratic risk, we can study a fundamental equilibrium in which capital prices evolve deterministically, even though $\kappa < 1$ in steady state. To do this, we set aggregate fundamental risk to zero, $\sigma = 0$, and study the properties of the non-stochastic equilibrium having $\sigma_q = 0$.

The crucial feature this model, as we show below, is that capital prices are determined by a function q such that $q_t = q(\eta_t)$. Supposing that to be true, a steady state is fully characterized by the value $\eta = \eta^{ss}$ such that all non-growing variables are constant over time. This steady state is thus determined by the equation $\dot{\eta} = 0$, where

$$\dot{\eta} = \eta(1-\eta)\Big[\rho_h - \rho_e + \tilde{\sigma}^2\Big((\frac{\kappa}{\eta})^2 - (\frac{1-\kappa}{1-\eta})^2\Big)\Big] + \delta_h - (\delta_e + \delta_h)\eta.$$

It is straightforward to show that equilibrium features stable state variable dynamics, in the sense that $\frac{\partial \dot{\eta}}{\partial \eta}|_{\eta=\eta^{ss}} < 0$. However, because the "co-state" q is determined explicitly as a function of η, the steady state is not "stable" in the usual sense required by the multiplicity literature. Technically, there is only one stable eigenvalue of the dynamical system (η_t, q_t) near steady state (η^{ss}, q^{ss}).

Lemma D.2. *The steady state of the model with idiosyncratic risk is saddle path stable.*

Proof of Lemma D.2. First, we show that q is a function of η, i.e., $q_t = q(\eta_t)$. Goods market clearing is still characterized by the price-output relation (PO). With idiosyncratic risk, the risk-balance condition (RB) is now (D.14). The solution to the system (PO) and (D.14) can be computed explicitly. Indeed, define

$$\eta^* := \sup\{\eta : (a_e - a_h)\eta\bar{\rho}(\eta) = a_e\tilde{\sigma}^2\}.$$

Then, $\kappa = 1$ for all $\eta \in (\eta^*, 1)$. For $\eta \in (0, \eta^*)$, we compute $\kappa < 1$ as the positive root $\tilde{\kappa}$ from

$$0 = (a_e - a_h)\tilde{\kappa}^2 + [a_h - \eta(a_e - a_h)]\tilde{\kappa} - \eta a_h - \frac{\eta(1-\eta)(a_e - a_h)\bar{\rho}(\eta)}{\tilde{\sigma}^2}.$$

After determining κ for all values of η, capital price q can be computed from (PO), as an explicit function of η.

Given $q_t = q(\eta_t)$, the dynamics of q_t are given by $\dot{q}_t = q'(\eta_t)\dot{\eta}_t$, which only depends on η and not q (notice that $\dot{\eta}_t$ also only depends on η and not q). Consequently, the linearized system near steady state takes the form

$$\begin{bmatrix} \dot{\eta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ m_2 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ q \end{bmatrix}$$

for $m_1, m_2 \neq 0$. The eigenvalues of this system are $m_1 < 0$ and 0. □

As a result of Lemma D.2, there is a unique transition path $(\eta_t, q_t)_{t\geq 0}$ to steady state, given an initial condition η_0. In other words, q_0 is pinned down uniquely. Our sunspot equilibria are not constructed by randomizing over a multiplicity of transition paths that arise due to steady state stability, which is the usual approach (Azariadis, 1981; Cass and Shell, 1983).

D.3 Limited commitment as equilibrium refinement

Here, we add a small limited commitment friction, in the spirit of Gertler and Kiyotaki (2010). The result: only equilibria with the property $\lim_{\eta\to 0} \kappa = 0$ survive, similarly to equilibria with a small amount of idiosyncratic risk (Appendix D.2).

Suppose capital holders can abscond with a fraction $\lambda^{-1} \in (0,1)$ of their assets and renege on repayment of their short-term bonds. After doing this diversion, the capital holder would have net worth $\tilde{n}_{j,t} := \lambda^{-1} q_t k_{j,t}$.

To prevent diversion, bondholders will impose some limitation on borrowing. To see this, note that diversion delivers utility $\log(\tilde{n}_{j,t}) + \xi_t$, where ξ_t is an aggregate process (independent of the identity j of the diverter). This is the form of indirect utility for a log utility investor in our model, as discussed in Appendix B.1. For diversion to be sub-optimal, it must be the case that $\log(\tilde{n}_{j,t}) + \xi_t \leq \log(n_{j,t}) + \xi_t$. As a result, bondholders impose the following leverage constraint to ensure non-diversion is incentive compatible:

$$\frac{q_t k_{j,t}}{n_{j,t}} \leq \lambda. \tag{D.15}$$

We will study the equilibrium with constraint (D.15) additionally imposed, and then we will take $\lambda \to \infty$ so that the limited commitment friction is vanishingly small.

Risk-balance condition (RB) is now replaced by

$$0 = \min\left[1-\kappa,\, \lambda\eta - q\kappa,\, \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}|\sigma_R|^2\right]. \tag{D.16}$$

The most important feature of equation (D.16) is that leverage constrained experts ($\lambda\eta = q\kappa$) must hold less than the full capital stock ($\kappa < 1$).

Condition (D.16) implies that there exists a threshold $\eta^{\lambda} := \inf\{\eta : \lambda\eta > q\kappa\}$ below which experts' leverage constraints bind. By combining $\lambda\eta = q\kappa$ with condition (PO) for κ, we obtain an explicit formula for the capital price in this region:

$$q = \frac{1}{2}\left[\frac{a_h}{\bar{\rho}} + \sqrt{(a_h/\bar{\rho})^2 + 4\lambda\eta(a_e - a_h)/\bar{\rho}}\right], \quad \text{if} \quad \eta \leq \eta^{\lambda}. \tag{D.17}$$

Taking the limit $\eta \to 0$ in equation (D.17) shows that $q \to a_h/\rho_h$ and thus $\kappa \to 0$. This proves that there is no flexibility for coordination on a worst-case capital price, unlike the leverage-unconstrained economy. The equilibrium worst-case capital price must coincide with $\kappa_0 = 0$.

As the limited commitment problem vanishes ($\lambda \to \infty$), the leverage constraint becomes non-binding at all times (formally $\eta^{\lambda} \to 0$).[28] But along the sequence, $\kappa_0 = 0$ is uniformly required. We collect these results.

Lemma D.3. *Among all equilibria, only those with the property* $\lim_{\eta\to 0}\kappa = 0$ *survive a vanishingly-small limited commitment friction.*

Intuitively, the leverage constraint gives experts an additional motive to sell capital, which forces coordination on maximal selling in response to negative shocks. Said differently: due to the prospect of violating the leverage constraint, losses incurred from retaining capital when others are selling is larger than losses incurred from selling capital when others are retaining it. This property is reminiscent of "risk dominant" equilibria being selected by strategic uncertainty (Harsanyi and Selten, 1988; Frankel et al., 2003), but the exact modeling is different here.

D.4 Correlation between sentiment and fundamentals

What happens if sentiment shocks are correlated with fundamental shocks? To model this, we allow

$$ds_t = \mu_{s,t}dt + \sigma_{s,t}^{(1)}dZ_t^{(1)} + \sigma_{s,t}^{(2)}dZ_t^{(2)}.$$

[28] This intuitive property can be shown easily by taking $\lambda \to \infty$ in (D.17). For any fixed $\eta \in (0,1)$, taking this limit implies $q \to \infty$, which is ruled out by price-output relation (PO).

In Section 3.1, we restricted attention to $\sigma_{s,t}^{(1)} = 0$. Without this assumption, equations (23) and (22) are modified to read:

$$0 = \min \Big[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)} \Big(\frac{(\sigma + \sigma_s^{(1)} \partial_s \log q)^2 + (\sigma_s^{(2)} \partial_s \log q)^2}{(1-(\kappa-\eta)\partial_\eta \log q)^2} \Big) \Big]$$
$$\sigma_q = \frac{\binom{1}{0}(\kappa-\eta)\sigma \partial_\eta \log q + \sigma_s \partial_s \log q}{1-(\kappa-\eta)\partial_\eta \log q}.$$

The rest of the equilibrium restrictions are identical.

For the present illustration, we additionally assume that $\sigma_{s,t}^{(2)} = 0$, i.e., sentiment shocks *only* load on fundamental shocks. What emerges is the possibility that sentiment shocks "hedge" fundamental shocks: we can have $\sigma_s^{(1)} \partial_s \log q < 0$, which lowers return volatility and raises asset prices. In one extreme, if $\sigma_s^{(1)} \partial_s \log q \to -\sigma$, the price function converges to that of a Fundamental Equilibrium with vanishing fundamental risk $\sigma \to 0$; call this FE(0). At the other end, if $\sigma_s^{(1)} \partial_s \log q \to 0$, the economy resembles the Fundamental Equilibrium with positive fundamental shocks; call this FE(σ). Thus, by constructing our conjectured capital price function as a convex combination of FE(0) and FE(σ), with weights $1-s$ and s, we can ensure that $\sigma_s^{(1)} \partial_s \log q$ endogenously emerges negative. Figure D.2 displays the equilibrium constructed this way.

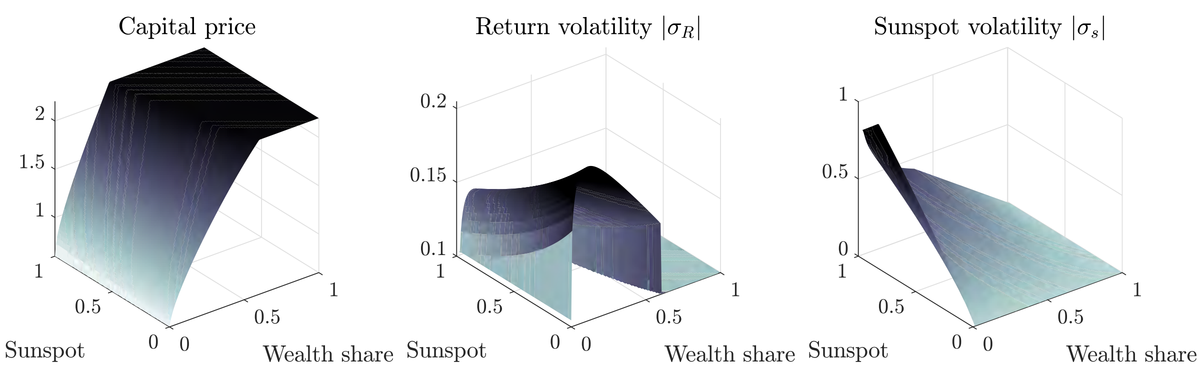

Figure D.2: Capital price q, volatility of capital returns $|\sigma_R|$, and sunspot shock volatility $|\sigma_s|$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.10$.

D.5 Exogenous sunspot dynamics

In Section 3.1, we solved for a Markov S-BSE that featured endogenous sunspot dynamics, i.e., (σ_s, μ_s) could potentially depend on η. Here, we show that sunspot equilibria can be built on top of *exogenous* sunspot dynamics as well. As we will show, this construction can be naturally viewed as the limit of equilibria in which the variable s has a vanishing contribution to fundamentals. With that in mind, we actually start from a more general setting in which s can impact fundamental volatility, and then we take the limit as this impact becomes vanishingly small.

Consider the following stochastic volatility model:

$$\frac{dK_t}{K_t} = gdt + \sigma\sqrt{1+\omega s_t}dZ_t$$
$$ds_t = \mu_s(s_t)dt + \vartheta\sqrt{1+\omega s_t}dZ_t$$

where $\vartheta > 0$ is an exogenous parameter and $\omega \in \mathbb{R}$ measures the impact of s_t on capital growth volatility. Thus, the diffusion of s_t, namely $\sigma_s(s) := \vartheta\sqrt{1+\omega s}$, is specified exogenously. Also, $\mu_s(s)$ is an exogenous function that is specified to ensure that $s_t \in (s_{\min}, s_{\max})$, for some pre-specified interval satisfying $s_{\min} \geq 0$ and $\omega s_{\max} > -1$. Such a choice can always be made, e.g., by putting $\mu_s(s) = -(s_{\max} - s)^{-(1+\beta)} + (s - s_{\min})^{-(1+\beta)}$. Note that s_t becomes a sunspot when $\omega = 0$. When $\omega < 0$, the state s_t is an inverse measure of capital's volatility.

For simplicity, we assume there is a single aggregate shock, i.e., Z is a one-dimensional Brownian motion; this can easily be generalized to multiple shocks. Also for simplicity of expressions, we assume here that $\rho_e = \rho_h = \rho$. Then, an equilibrium capital price function $q(\eta, s)$ must satisfy the PDE defined by the following system

$$\rho q = \kappa a_e + (1-\kappa)a_h$$
$$0 = \min\left[1-\kappa,\ \frac{a_e - a_h}{q} - \frac{(\kappa-\eta)(1+\omega s)}{\eta(1-\eta)}\left(\frac{\sigma + \vartheta\partial_s \log q}{1-(\kappa-\eta)\partial_\eta \log q}\right)^2\right].$$

Technically, the multiplicity arises from the selection of the boundary conditions on $q(\eta, s_{\min})$ and $q(\eta, s_{\max})$, which are not pinned down by any equilibrium restriction.

We perform two exercises. First, we show that there are multiple equilibria for a given set of parameters. We use $\omega < 0$ here, along with $s_{\min} = 0$ and $s_{\max} = 2$. In this case, the "natural" and intuitive solution is for q to increase with s, because volatility decreases. In Figure D.3, we pick a "low" boundary condition for $q(\eta, 0)$ and the solution follows this intuition.[29]

However, agents could equally well coordinate on a "high" boundary condition, which results in the solution of Figure D.4.[30] Notice the capital price and return volatil-

[29]This "low" boundary condition is a weighted average between the solution with infinite volatility and the fundamental equilibrium solution. The fundamental equilibrium, which is the capital price solution that keeps $s = 0$ fixed forever, is discussed in Online Appendix E. The infinite-volatility solution has $\kappa = \eta$, hence $q = (\eta a_e + (1-\eta)a_h)/\bar{\rho}(\eta)$.

[30]This "high" boundary condition is a weighted average between $\lim_{v\to 0}$ FE(v) and FE(σ), where FE(σ) denotes the Fundamental Equilibrium solution with exogenous risk σ.

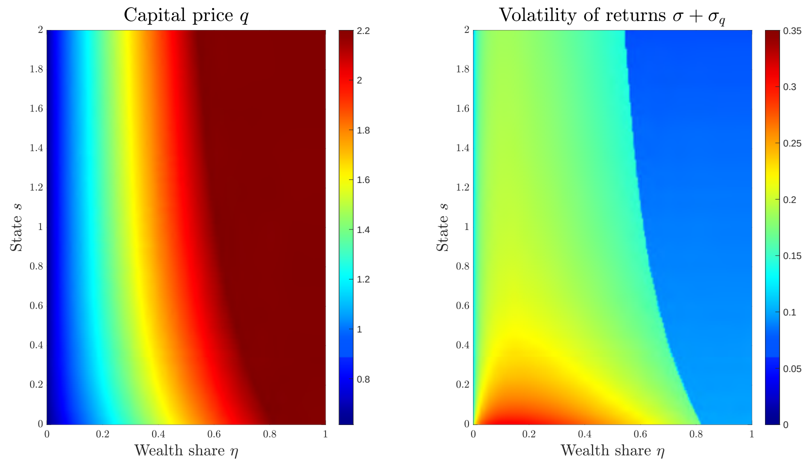

Figure D.3: Equilibrium with $\omega = -0.25$, and the "low" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and the infinite-volatility equilibrium. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

ity exhibit a non-monotonicity in s. At low values of s, q is decreasing in s, while return volatility increases. The very different behavior in Figures D.3 and D.4 is made possible by coordination on the different boundary conditions.

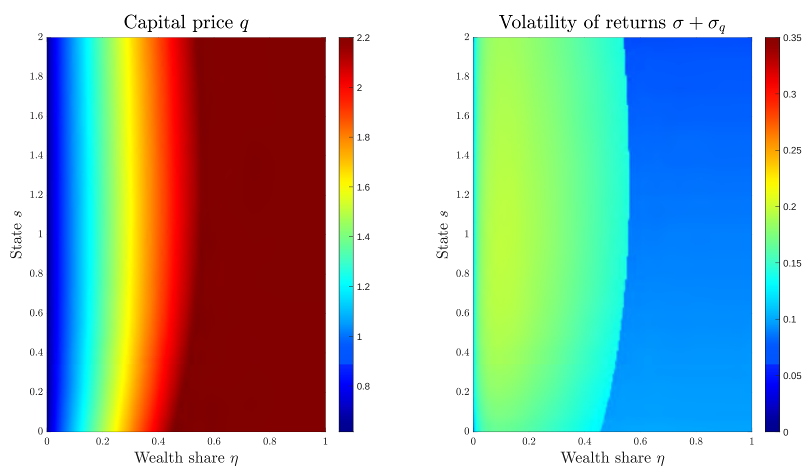

Figure D.4: Equilibrium with $\omega = -0.25$, and the "high" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of FE(σ) and $\lim_{v \to 0}$ FE(v), where FE(σ) denotes the fundamental equilibrium solution with fundamental risk σ. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

Our second exercise considers the limit $\omega \to 0$. Figure D.5 shows the solution for $\omega = -10^{-6}$, again equipped with the "low" boundary condition for $q(\eta, 0)$. There remains a tremendous amount of variation in the equilibrium as s varies, illustrating convergence to a sunspot equilibrium. Thus, as promised, we are able to construct sunspot equilibria even if the dynamics (σ_s, μ_s) are specified exogenously. In fact, it appears that the amount of price volatility is relatively insensitive to the real effects s has (i.e., the size of ω), which is reminiscent of the "volatility paradox" of Brunnermeier and Sannikov (2014) but one level deeper. Their paradox is that total volatility is only modestly sensitive to exogenous fundamental volatility; our paradox is that total volatility is only modestly sensitive to the *exogenous impact of s on fundamental volatility*.

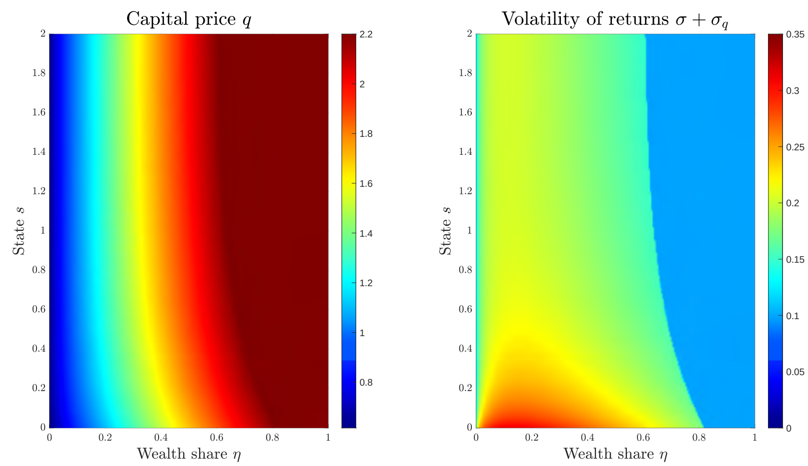

Figure D.5: Equilibrium with near-sunspot $\omega = -10^{-6}$ and the "low" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of FE(σ) and the infinite-volatility equilibrium (which has $\kappa = \eta$). Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

E Fundamental Equilibria

In this section, we investigate properties of equilibria where sunspot shocks $Z^{(2)}$ are irrelevant and experts' wealth share η serves as the only state variable, i.e., fundamental equilibria. We illustrate previously undocumented multiplicity along two dimensions: the disaster belief κ_0 and the sign of the sensitivity of capital returns to fundamental shocks $\sigma + \sigma_q$. The key equations describing FEs are:

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h \tag{E.1}$$

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2\right]. \tag{E.2}$$

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}\sigma. \tag{E.3}$$

Equation (E.1) just restates (PO). Equation (E.2) is the risk-balance condition (RB) when there is only the fundamental shock $Z^{(1)}$. Equation (E.3) comes from resolving the two-way feedback between wealth share volatility $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$ and asset-price volatility $\sigma_q = \frac{q'}{q}\sigma_\eta$, which arises from Itô's formula. Finally, wealth share dynamics are given in (13)-(14), restated here for convenience:

$$\mu_\eta = -\eta(1-\eta)(\rho_e - \rho_h) + \mathbf{1}_{\{\kappa<1\}}(\kappa - 2\kappa\eta + \eta^2)\frac{a_e - a_h}{q} + \delta_h - (\delta_e + \delta_h)\eta \tag{E.4}$$

$$\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q). \tag{E.5}$$

We define a fundamental equilibrium as follows, analogously to Lemma 1.

Definition 4. Given $\eta_0 \in (0,1)$, a *Markov fundamental equilibrium* consists of adapted processes $(\eta_t, q_t, \kappa_t, r_t)_{t\geq 0}$ such that (E.1)-(E.3) and (11) hold, and (E.4)-(E.5) describe dynamics of η_t.

Note that the interest rate r_t can be simply set from (11), given the other variables, and it affects no other equilibrium equation. Similarly, the dynamics of η_t are set from (E.4)-(E.5), and they affect none of (E.1)-(E.3). Hence, below, we will often refer to a fundamental equilibrium simply by reference to (q, κ).

E.1 Properties

We describe here some properties of fundamental equilibria, where we additionally impose the full-deleveraging condition $\kappa(0) = 0$.

Lemma E.1. *Assuming it exists, suppose* (q, κ) *is a fundamental equilibrium in* η *in the sense of Definition 4. Assume* $\kappa(0+) = 0$*. Define* $\eta^* := \inf\{\eta : \kappa = 1\}$*. Then, the following hold:*

(i) $(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}$*, for all* $\eta \in (0, \eta^*)$*.*

(ii) $\eta a_e + (1-\eta)a_h < \bar{\rho}q < a_e$*, for all* $\eta \in (0, \eta^*)$*.*

(iii) $\frac{q'(0+)}{q(0+)} = \frac{a_e}{a_h} - \frac{\rho_e}{\rho_h} + \rho_h\big(\frac{a_e - a_h}{\sigma a_h}\big)^2$*.*

(iv) If σ *is sufficiently small, then* $q' > \frac{a_e - a_h}{\bar{\rho}}$*, for* $\eta \in (0, \eta^*)$*.*

(v) If σ *is sufficiently small, then* $\frac{\rho_h}{\rho_e}\big(\frac{1 - a_h/a_e}{\sigma^2} - 1 + \frac{\rho_h}{\rho_e}\big)^{-1} < \eta^* < 1$*.*

(vi) On $\eta \in (0, \eta^*)$*, the solution* q *is infinitely-differentiable.*

Proof of Lemma E.1. Since a fundamental equilibrium is assumed to exist, we make use of equations (E.1) and (E.2). Recall that $\bar{\rho} := \eta\rho_e + (1-\eta)\rho_h$. By analogy, let $\bar{a} := \eta a_e + (1-\eta)a_h$.

(i) Start from equation (E.2), and rearrange to obtain the result, where we have implicitly selected the solution with $1 > (\kappa - \eta)\frac{q'}{q}$.

(ii) The first inequality, which is equivalent to $\kappa > \eta$, is a direct implication of equation (E.2). The second inequality, equivalent to $\kappa < 1$, is a restatement of the definition of η^*.

(iii) Start from equation (E.2). Taking the limit $\eta \to 0$, and using $\kappa(0+) = 0$, delivers an equation for $\kappa'(0+)$. Differentiating (E.1), we may then substitute $\kappa'(0+) = \frac{\rho_h q'(0+) + (\rho_e - \rho_h)q(0+)}{a_e - a_h}$. Rearranging, we obtain the desired result.

(iv) By part (iii), there exists $\eta^\circ > 0$ and $\bar{\sigma} > 0$ such that uniformly for all $\sigma < \bar{\sigma}$, we have $q' > \frac{a_e - a_h}{\bar{\rho}}$ on the set $\{\eta < \eta^\circ\}$. On the set $\{\eta^\circ \leq \eta < \eta^*\}$, we know that $\kappa - \eta$ is bounded away from zero, uniformly for all $\sigma < \bar{\sigma}$. Using the expression in part (i), the fact that q is bounded by $a_e/\bar{\rho}$ uniformly for all σ, and the previous fact about $\kappa - \eta = \bar{\rho}q - \bar{a}$, we can write

$$q' = \frac{a_e - a_h}{\bar{\rho}q - \bar{a}}q - o(\sigma), \quad \eta \in (\eta^\circ, \eta^*).$$

Therefore,

$$q' + o(\sigma) = \frac{a_e - a_h}{\bar{\rho}q - \bar{a}}q = \frac{a_e - a_h}{\bar{\rho}}\frac{q}{q - \bar{a}/\bar{\rho}} > \frac{a_e - a_h}{\bar{\rho}}, \quad \eta \in (\eta^\circ, \eta^*),$$

where the last inequality is due to $\bar{\rho}q > \bar{a}$ [part (ii)]. Taking σ is small enough implies the result on (η°, η^*), which we combine with the result on $(0, \eta^\circ)$ to conclude.

(v) Consider the function $\tilde{q} := \bar{a}/\bar{\rho}$, whose derivative is $\tilde{q}' = \frac{a_e - a_h}{\bar{\rho}} - \frac{\bar{a}}{\bar{\rho}}\frac{\rho_e - \rho_h}{\bar{\rho}} < \frac{a_e - a_h}{\bar{\rho}}$. Combining this result with part (iv), we obtain $q' > \tilde{q}'$. If $\tilde{q}$ was the capital price, then equation (E.1) implies the associated capital share $\tilde{\kappa} = \eta$. On the other hand, the fact that $q' > \tilde{q}'$ implies $\kappa' > \tilde{\kappa}' = 1$, which implies $\eta^* < 1$.

Next, consider $\eta \in (\eta^*, 1)$ so that $\kappa = 1$. By equation (E.2), with $q = a_e/\bar{\rho}$, we must have

$$\sigma^2 \leq \eta\bar{\rho}\frac{a_e - a_h}{a_e}\Big(1 + (1-\eta)\frac{\rho_e - \rho_h}{\bar{\rho}}\Big)^2, \quad \eta \geq \eta^*.$$

This is equivalent to

$$1 \leq \eta\frac{\rho_e}{\rho_h}\Big(\frac{a_e - a_h}{a_e\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\Big), \quad \eta \geq \eta^*.$$

Substituting $\eta = \eta^*$, and rearranging, we obtain the first inequality. There is no contradiction with $\eta^* < 1$, due to the assumption that σ can be made small enough.

(vi) Note that $F(\eta, q) := q[\frac{a_e - a_h}{\bar{\rho}(\eta)q - \bar{a}(\eta)} - \sigma(\frac{\eta(1-\eta)(\bar{\rho}(\eta)q - \bar{a}(\eta))}{q})]$ is infinitely differentiable in both arguments on $\{(\eta, q) : \eta \in (0,1),\, \bar{\rho}(\eta)q > \bar{a}(\eta)\}$. Thus, the result is a simple consequence of differentiating part (i), noting that by part (ii) we have $\bar{\rho}(\eta)q(\eta) > \bar{a}(\eta)$, and then using induction. □

E.2 "Disaster beliefs" and deleveraging

The existing literature always imposes $\kappa(0+) = 0$, i.e., experts fully deleverage as their wealth vanishes.[31] This is actually not a necessary feature of a fundamental equilibrium. Let $\kappa_0 \in [0, 1)$ and suppose $\kappa(0+) = \kappa_0$. We will call κ_0 the *disaster belief* about experts' deleveraging. Existence of an equilibrium with such disaster belief boils down simply to

[31] Brunnermeier and Sannikov (2014) justify $\kappa_0 = 0$ in their online appendix: "because in the event that η_t drops to 0, experts are pushed to the solvency constraint and must liquidate any capital holdings to households." This is technically not needed; as shown in Lemma E.2 below, the dynamics of η_t will not allow it to ever reach 0, so there is no contradiction to equilibrium with both $\kappa_0 > 0$ and $\sigma > 0$. Although we do not prove an existence result, Appendix E.1 presents several numerical examples. The continuum of fundamental equilibria, indexed by κ_0, may be of independent theoretical interest.

In some sense, the literature has picked the worst possible fundamental equilibrium (minimal-price, maximal-volatility) by imposing $\kappa_0 = 0$. This can be partly justified by the refinement results of Sections D.2 and D.3, i.e., only the belief $\kappa_0 = 0$ survives vanishingly-small idiosyncratic risk or a vanishingly-small limited commitment friction.

existence of a solution to a first-order ODE with a given boundary condition $\kappa(0) = \kappa_0$. If solutions exist for a variety of different choices for κ_0, then a variety of fundamental equilibria could exist, and indeed we provide a numerical example after the following lemma and proof.

Lemma E.2. *A fundamental equilibrium with disaster belief $\kappa_0 \in [0, 1)$ exists if the free boundary problem*

$$(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}, \quad on \quad \eta \in (0, \eta^*), \tag{E.6}$$

$$subject\ to \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} \quad and \quad q(\eta^*) = \frac{a_e}{\bar{\rho}(\eta^*)}, \tag{E.7}$$

has a solution.

Proof of Lemma E.2. A fundamental equilibrium in state variable η exists if and only if equations (E.1), (E.2), and (E.3) hold, and if the time-paths $(\eta_t)_{t\geq 0}$ induced by dynamics (σ_η, μ_η) avoid $\eta = 0$ almost-surely. We will demonstrate these conditions.

Suppose (E.6)-(E.7) has a solution (q, η^*) corresponding to $\kappa_0 \in [0, 1)$. If there are multiple solutions, we pick the one such that $q(\eta) < a_e/\bar{\rho}(\eta)$ for all $\eta \in (0, \eta^*)$, which is always possible because the boundary conditions (E.7) imply $\bar{\rho}(0)q(0) < \bar{\rho}(\eta^*)q(\eta^*)$. Set $q(\eta) = a_e/\bar{\rho}(\eta)$ for all $\eta \geq \eta^*$. Define $\kappa = \frac{\bar{\rho}q - a_h}{a_e - a_h}$. Note that (E.1) is automatically satisfied. Note that (E.3) is also satisfied automatically, by applying Itô's formula to the solution $q(\eta)$ and using $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$.

We show (E.2) holds separately on $(0, \eta^*)$ and $[\eta^*, 1)$. Using (E.1) and (E.3) in the ODE (E.6) and rearranging, we show that (E.2) holds when $\kappa < 1$. The boundary condition $q(\eta^*) = a_e/\bar{\rho}(\eta^*)$ is equivalent to $\kappa(\eta^*) = 1$, which shows that $\kappa(\eta) < 1$ for all $\eta < \eta^*$. Therefore, we have proven that (E.2) holds on $(0, \eta^*)$.

If $\eta^* = 1$, then we are done verifying (E.2); otherwise, we need to verify (E.2) on $[\eta^*, 1)$. On this set, $\kappa = 1$, so we need to verify

$$\eta\frac{a_e - a_h}{q} \geq (\sigma + \sigma_q)^2 \quad \text{for all} \quad \eta \geq \eta^*. \tag{E.8}$$

First, we show that it suffices to verify this condition exactly at η^*. Indeed, on $(\eta^*, 1)$, we have $\kappa = 1$ and $q = a_e/\bar{\rho}$. Substituting these and (E.3) into (E.8), we obtain

$$\text{(E.8) holds} \Leftrightarrow \Big(\frac{a_e - a_h}{a_e\sigma^2}\rho_e - \frac{\rho_e - \rho_h}{\rho_e}\Big)\eta \geq \frac{\rho_h}{\rho_e} \quad \text{for all} \quad \eta \geq \eta^*.$$

But since the left-hand-side is increasing in η, if it holds at $\eta = \eta^*$, it holds for all $\eta > \eta^*$.

Now, writing (E.8) at η^*, using (E.3) to replace σ_q, and using ODE (E.6) to replace $\eta^* \frac{a_e - a_h}{q(\eta^*)} = \sigma[1 - (1-\eta^*)q'(\eta^*-)/q(\eta^*)]^{-1}$, we need to verify

$$\text{(E.8) holds} \;\Leftrightarrow\; \frac{\sigma}{1 - (1-\eta^*)q'(\eta^*-)/q(\eta^*)} \geq \frac{\sigma}{1 - (1-\eta)q'(\eta^*+)/q(\eta^*)} \;\Leftrightarrow\; q'(\eta^*-) \geq q'(\eta^*+).$$

We clearly have $q'(\eta^*-) \geq q'(\eta^*+)$ by the simple fact that $q < a_e/\bar{\rho}$ for $\eta < \eta^*$ and $q = a_e/\bar{\rho}$ for $\eta \geq \eta^*$.

Finally, it remains to very that η_t almost-surely never reaches the boundary 0. Near $\eta = 0$, the dynamics in (E.4)-(E.5) become

$$\mu_\eta(\eta) = \kappa_0 \frac{a_e - a_h}{q(0+)} + \delta_h + o(\eta)$$
$$\sigma_\eta^2(\eta) = \kappa_0 \frac{a_e - a_h}{q(0+)}\eta + o(\eta).$$

By the same analysis as in Theorem 1, the boundary 0 is unattainable. □

What happens in an equilibrium of Lemma E.2 in which $\kappa_0 > 0$? Behavior at the boundary $\eta = 0$ is substantially different than the $\kappa_0 = 0$ case, because equation (E.2) can only hold there if $\sigma_q \to -\sigma$ as $\eta \to 0$. Capital prices "hedge" fundamental shocks to capital, in a brief region of the state space $(0, \eta^{\text{hedge}})$. Said differently, given the formula (E.3), the fact that $\sigma_q(0+) = -\sigma$ implies $q'(0+) = -\infty$, so that prices rise as experts lose wealth in a region of the state space. The hedging region is exactly what incentivizes experts to take so much leverage (indeed, expert leverage κ/η blows up near 0). For $\eta > \eta^{\text{hedge}}$, this behavior reverses, and the equilibrium behaves very much like the equilibrium with $\kappa_0 = 0$. Overall, there is no inconsistency with equilibrium even though $q' < 0$ in the region $(0, \eta^{\text{hedge}})$.[32]

Figure E.1 displays several examples of equilibria with different choices of $\kappa_0 > 0$. The solid black lines, which are equilibrium outcomes with $\kappa_0 = 0.001$, corresponds approximately to the equilibrium choice made by Brunnermeier and Sannikov (2014). The

[32]One may think that $q'(0+) = -\infty$, and more generally that $q' < 0$ in some region of the state space, could imply that κ hits η at some point. However, this cannot happen. Indeed, since $\kappa_0 > 0$, we have that $q(0+) > \tilde{q}(0+)$, where $\tilde{q}(\eta) := ((a_e - a_h)\eta + a_h)/\bar{\rho}$ is the price function consistent with $\kappa = \eta$. Now, assume there is an $\hat{\eta} \in (0,1)$ such that $\kappa(\hat{\eta}) = \hat{\eta}$ (or equivalently, $q(\hat{\eta}) = \tilde{q}(\hat{\eta})$). If there is more than one, consider the minimum among them, so $q(\eta) > \tilde{q}(\eta)$ for all $\eta \in (0, \hat{\eta})$. From the $\tilde{q}(\eta)$ definition, we have $\tilde{q}'(\eta) = (a_e - a_h)/\bar{\rho} - ((a_e - a_h)\hat{\eta} + a_h)(\rho_e - \rho_h)/\bar{\rho}^2 < \infty$, while from (E.6) it must be that $q'(\hat{\eta}-) \to \infty$. But this implies that q crosses $\tilde{q}$ from below, contradicting $q(\eta) > \tilde{q}(\eta)$ on $\eta \in (0, \hat{\eta})$.

other curves, with higher disaster beliefs κ_0, are new to the literature. More optimistic disaster beliefs raise capital prices and reduce capital price volatility.

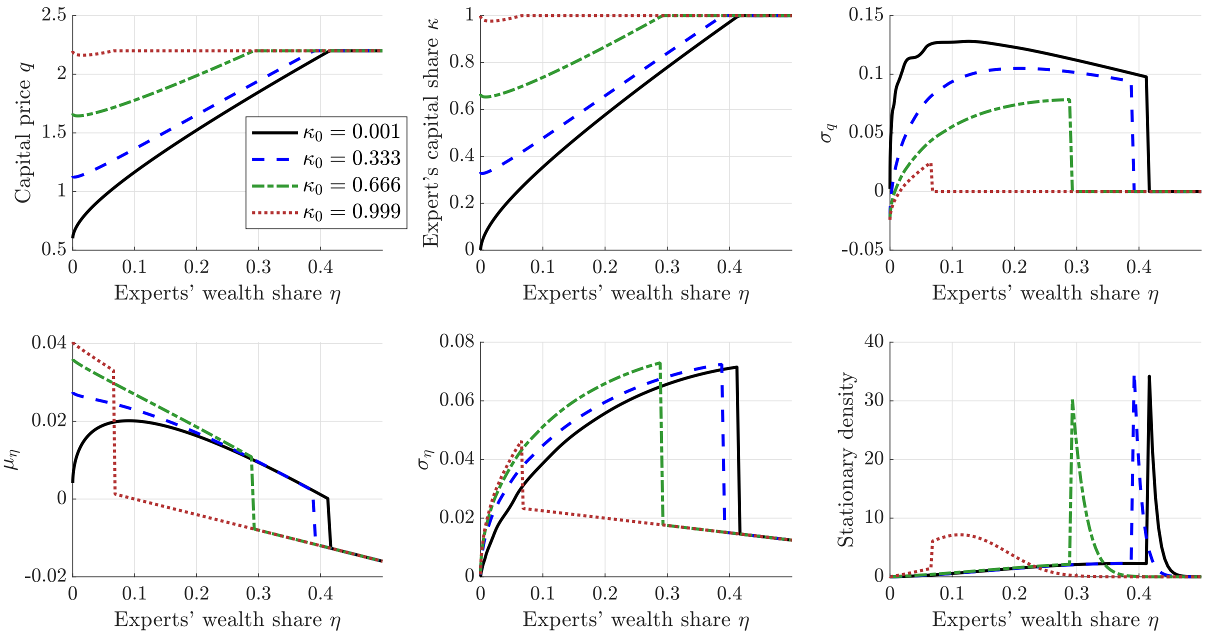

Figure E.1: Fundamental equilibria with different disaster beliefs κ_0. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$.

E.3 The "hedging" equilibrium

The equilibria described above are "normal" in the sense that a positive exogenous shock increases asset prices and experts' wealth share.[33] But technically, agents do not care about the direction prices move when they make their portfolio choices. They only care about risk which is measured in return variance; this can be seen in the optimality condition (E.2) where $(\sigma + \sigma_q)^2$ appears. An immediate implication is that two types of equilibria are possible: the "normal" one has $\sigma + \sigma_q > 0$; an alternative equilibrium has $\sigma + \sigma_q < 0$. For a conjecture of this specific type of indeterminacy, see footnote 16 of Kiyotaki and Moore (1997).

We term this latter equilibrium the "hedging" equilibrium because asset price movements move oppositely to exogenous shocks. In fact, asset price responses are so strong in opposition that experts actually gain in wealth share upon a negative fundamental

[33]Except, as we have discussed above, if full-deleveraging does not hold ($\kappa_0 > 0$), then there is a (small) region $\eta \in (0, \eta^{\text{hedge}})$ on which $q' < 0$, so negative shocks reduce experts' wealth share but increase asset prices. But broadly speaking, especially if accounting for the stationary distribution, the equilibria feature $q' > 0$ and thus the "normal" behavior (see Figure E.1).

shock. This can only happen because of coordination: experts and households simply believe negative shocks are good news for asset prices, so they rush to purchase capital, which percolates through equilibrium relationships to justify beliefs about price increases. Such coordination stands in contrast to the normal equilibrium, in which negative shocks beget fire sales that push down asset prices.

Mathematically, we need only solve a slightly different capital price ODE. Whereas ODE (E.6) holds in the normal equilibrium, the hedging equilibrium requires

$$(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h + \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}, \quad \text{on} \quad \eta \in (0, \eta^*). \tag{E.9}$$

The difference between (E.9) and (E.6) is merely the sign in front of σ, which ensures different signs for σ_q. Finally, note that just like the normal equilibria, hedging equilibria could exist for $\kappa_0 \neq 0$. Figure E.2 compares a normal equilibrium to a hedging equilibrium.

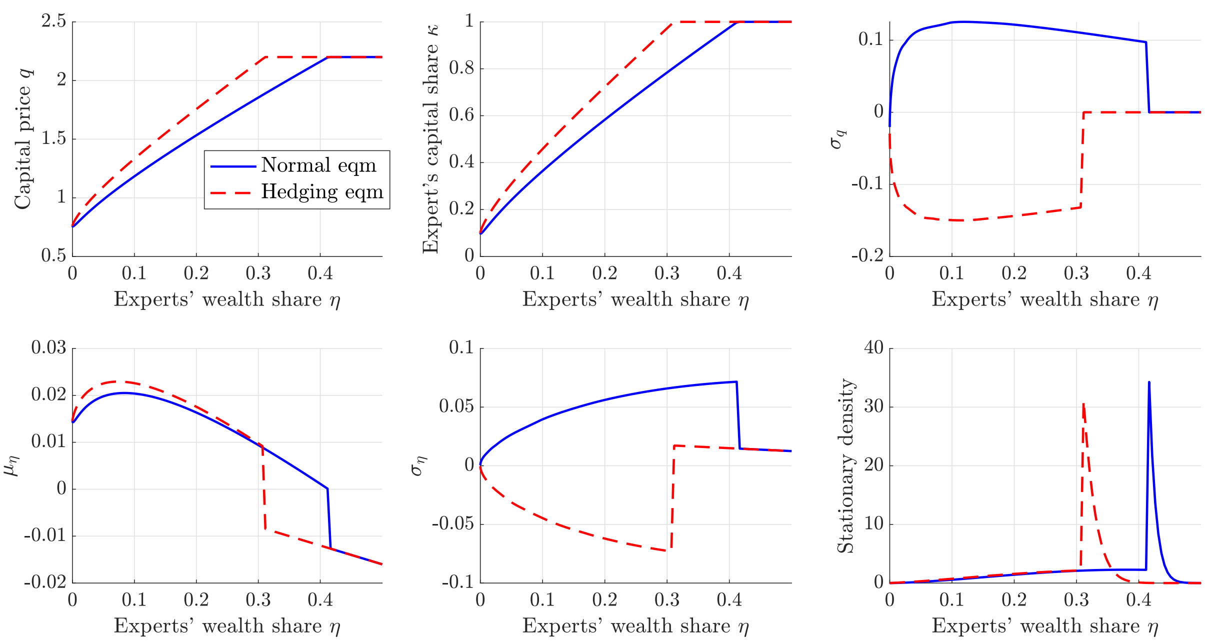

Figure E.2: Two equilibria (normal versus hedging) both with disaster belief $\kappa_0 = 0.1$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$.

F Discrete-time model

The following discrete-time model is exactly analogous to our continuous-time model. We model each decision on a time-step of Δ (it will turn out that the decision interval Δ cannot be arbitrarily large).

Technology. For simplicity, we assume that aggregate capital K is fixed, i.e., there is no fundamental uncertainty. Note nevertheless that individual positions on capital are not predetermined since agents can trade capital.

Individual agent problem. An individual can hold two assets, riskless bonds b_t and capital k_t, and decides consumption c_t. The individual net worth, just before consuming, is $n_t = b_t + q_t k_t$, where q_t is the market price of capital. The one-period return on bonds is $R_t^f = 1 + r_t\Delta$, and the return-on-capital is $R_{t+\Delta}^k := \frac{a\Delta}{q_t} + \frac{q_{t+\Delta}}{q_t}$, where a is the agent's productivity per unit of time while holding capital. Then, the agent's dynamic budget constraint is[34]

$$n_{t+\Delta} = q_t k_t (R_{t+\Delta}^k - R_f^f) + (n_t - c_t) R_t^f. \tag{F.1}$$

Each agent takes q_t, R_t^f, and $R_{t+\Delta}^k$ as given and chooses (c, k, n) to maximize

$$\mathbb{E}\left[\sum_{i=0}^{\infty}\left(\frac{1}{1+\rho\Delta}\right)^i \log(c_{i\Delta})\right], \tag{F.2}$$

subject to (F.1), subject to the no-shorting constraint $k_t \geq 0$, and subject to the solvency constraint $n_t \geq 0$.

The first-order optimality conditions are the standard Euler equations

$$1 = \frac{1}{1+\rho\Delta} R_t^f \mathbb{E}_t\left[\frac{c_t}{c_{t+\Delta}}\right] \tag{F.3}$$

$$0 \geq \frac{1}{1+\rho\Delta} \mathbb{E}_t\left[\frac{c_t}{c_{t+\Delta}}(R_{t+\Delta}^k - R_t^f)\right], \tag{F.4}$$

where (F.4) holds with equality when $k_t > 0$ is chosen.

[34]To derive (F.1), proceed as follows. First, note that the bond market account next period, before adjusting the portfolio of bonds and capital, will have value $b'_{t+\Delta} = R_f^f(b_t - c_t) + ak_t\Delta$—that is, after consumption expenditures are made, the residual earns the interest rate, and the cash flows from holding capital are also added at the end of the period. Second, the capital holdings k_t will have value $q_{t+\Delta}k_t$ next period. Adding these two quantities must equal tomorrow's net worth $n_{t+\Delta}$. Hence, $n_{t+\Delta} = R_f^f(b_t - c_t) + ak_t\Delta + q_{t+\Delta}k_t$. Using the definition $n_t = b_t + q_t k_t$ gives the result (F.1).

In addition, it is straightforward to show that optimal consumption satisfies the standard log utility formula[35]

$$c_t = \frac{\rho\Delta}{1+\rho\Delta} n_t. \tag{F.5}$$

Using this fact, plus the budget constraint (F.1) in (F.3)-(F.4), we obtain

$$1 = \frac{1}{1+\rho\Delta} R_t^f \mathbb{E}_t \left[\frac{1}{\theta_t (R_{t+\Delta}^k - R_t^f) + (1+\rho\Delta)^{-1} R_t^f} \right] \tag{F.6}$$

$$0 \geq \frac{1}{1+\rho\Delta} \mathbb{E}_t \left[\frac{R_{t+\Delta}^k - R_t^f}{\theta_t (R_{t+\Delta}^k - R_t^f) + (1+\rho\Delta)^{-1} R_t^f} \right], \quad \text{with equality if } \theta_t > 0 \tag{F.7}$$

where $\theta_t := \frac{q_t k_t}{n_t}$ is the share of wealth allocated to capital. At this point, one can prove that (F.6) holds automatically if (F.7) holds.[36] Therefore, we can drop the bond Euler equation (F.6) from the remainder of the analysis, i.e., (F.5) and (F.7) fully characterize the agent's optimal choices.

Aggregation and equilibrium conditions. As in the main text, we assume there are two types of agents: experts have productivity a_e and discount rate ρ_e, while households have productivity $a_h < a_e$ and discount rate $\rho_h \leq \rho_e$. Clearly, then, experts have a higher return-on-capital than households: $R_{e,t+\Delta}^k > R_{h,t+\Delta}^k$.

We now aggregate. The market clearing condition for goods, capital, and bonds are given by, respectively,

$$c_{e,t} + c_{h,t} = (a_e k_{e,t} + a_h k_{h,t})\Delta \tag{F.8}$$

$$k_{e,t} + k_{h,t} = K \tag{F.9}$$

$$b_{e,t} + b_{h,t} = c_{e,t} + c_{h,t}. \tag{F.10}$$

Equation (F.10) says that bondholdings just after consuming (which is $b_t - c_t$) sum to

[35]This can be showed by writing out the Bellman equation and guessing-and-verifying that the value function takes the form $v_t = (1-\beta)^{-1}\log(n_t) + f(\Omega_t)$ for $\beta = (1+\rho\Delta)^{-1}$ and some function f that only depends on aggregate states Ω_t. Then, the envelope condition says $c_t^{-1} = \frac{\partial}{\partial n} v_t = (1-\beta)^{-1} n_t^{-1}$, which is the consumption formula.

[36]Indeed, if $\theta_t = 0$ it is obvious that (F.6) holds. If $\theta_t > 0$, then (F.7) holds with equality, so we then have

$$0 = \mathbb{E}_t \Big[\frac{\theta_t (R_{t+\Delta}^k - R_t^f)}{\theta_t (R_{t+\Delta}^k - R_t^f) + (1+\rho\Delta)^{-1} R_t^f} \Big]$$

Adding this expression to equation (F.6), we obtain the identity $1 = 1$.

the zero net supply. By combining (F.10) with the individual net worth definition $n_t = b_t + q_t k_t$, we obtain an alternative statement of bond market clearing that we will use:

$$n_{e,t} + n_{h,t} = q_t K + c_{e,t} + c_{h,t}. \tag{F.11}$$

Definition 5. An *equilibrium* is a collection of stochastic processes for allocations $(k_{j,t\Delta}, n_{j,t\Delta}, c_{j,t\Delta})_{t=0}^{\infty}$ for $j \in \{e, h\}$ with $k_{e,0}$ and $k_{h,0}$ given, and for prices $(q_{t\Delta}, R^f_{t\Delta})_{t=0}^{\infty}$ such that (i) given prices, allocations solve each agent type's problem, and (ii) markets clear.

F.1 Equilibrium characterization

We have already characterized optimal decisions and market clearing conditions. In particular, a collection of stochastic processes for allocations and prices constitute an equilibrium if they satisfy (F.1), (F.5), and (F.7) for each agent type (experts and households), along with equations (F.8), (F.9), and (F.11) at the aggregate level.

We further tighten this characterization and reduce it to four stochastic processes satisfying a set of conditions, exactly as in our continuous-time model. First, to keep track of the distribution of wealth and capital, let $\eta_t := (1 + \rho_e \Delta)^{-1} n_{e,t}/q_t K$ and $\kappa_t := k_{e,t}/K$ denote expert's wealth and capital shares.[37] Whereas κ_t is a "jumpy" variable because it is linked to agent's capital choices, η_t is a "state" variable because it is determined via agent's slow-moving wealths. Using the budget constraint (F.1), we can obtain the dynamics of η_t as

$$\eta_{t+\Delta} = \frac{1}{1 + \rho_e \Delta} \left(\frac{\kappa_t (R^k_{e,t+\Delta} - R^f_t) + \eta_t R^f_t}{q_{t+\Delta}/q_t} \right). \tag{F.12}$$

Next, we aggregate the consumption decisions across these two types. To do this, plug the consumption rules from (F.5) into the goods and bond market clearing conditions (F.8) and (F.11), and combine the results to obtain

$$q_t \bar{\rho}(\eta_t) = \kappa_t a_e + (1 - \kappa_t) a_h, \tag{F.13}$$

where $\bar{\rho}(\eta) := \eta \rho_e + (1 - \eta) \rho_h$ is a wealth-weighted average discount rate. Identical to our continuous-time model, equation (F.13) is a *price-output relation* that links asset values q_t to the efficiency of the capital distribution κ_t. Finally, we aggregate the Euler

[37] Note that the wealth share is defined just after consumption choices are made, i.e., $\eta_t = (n_{e,t} - c_{e,t})/(n_{e,t} + n_{h,t} - c_{e,t} - c_{h,t})$ is the definition we are using.

equations (F.7) within the two types using the fact that experts will always be on the margin (i.e., since $R^k_{e,t+\Delta} > R^k_{h,t+\Delta}$, we have $k_{e,t} > 0$ at all times). We also use the fact that $\theta_{e,t} = \frac{q_t k_{e,t}}{n_{e,t}} = \frac{1}{1+\rho_e\Delta}\frac{\kappa_t}{\eta_t}$ and $\theta_{h,t} = \frac{q_t k_{h,t}}{n_{h,t}} = \frac{1}{1+\rho_h\Delta}\frac{1-\kappa_t}{1-\eta_t}$ to write the results in a more convenient way. The results are

$$0 = \mathbb{E}_t\left[\frac{q_{t+\Delta} + a_e\Delta - R^f_t q_t}{\frac{\kappa_t}{\eta_t}\left(q_{t+\Delta} + a_e\Delta - R^f_t q_t\right) + R^f_t q_t}\right] \tag{F.14}$$

$$0 \geq \mathbb{E}_t\left[\frac{q_{t+\Delta} + a_h\Delta - R^f_t q_t}{\frac{1-\kappa_t}{1-\eta_t}\left(q_{t+\Delta} + a_h\Delta - R^f_t q_t\right) + R^f_t q_t}\right] \tag{F.15}$$

where the latter holds as an equality when households hold capital, i.e., when $\kappa_t < 1$.

Thus, an equilibrium is fully characterized by the collection of stochastic processes $(\eta_{t\Delta}, \kappa_{t\Delta}, q_{t\Delta}, R^f_{t\Delta})^\infty_{t=0}$, with $\eta_0 = k_{e,0}/K$ given, such that the two optimality conditions (F.14)-(F.15) hold; the price-output relation (F.13) holds; and the law of motion for η_t is given by (F.12). To establish the analog to our continuous-time model, we also state this characterization as a lemma—notice that the verbiage is almost identical to Lemma 1.

Lemma F.1. *Given $\eta_0 \in (0,1)$, consider stochastic processes $\{\eta_{t\Delta}, q_{t\Delta}, \kappa_{t\Delta}, R^f_{t\Delta}\}^\infty_{t=0}$ such that η_t evolution is described by (F.12). If $\eta_t \in [0,1]$, $\kappa_t \in [0,1]$, and equations (F.13), (F.14), and (F.15) hold for all $t \geq 0$, then $\{\eta_{t\Delta}, q_{t\Delta}, \kappa_{t\Delta}, R^f_{t\Delta}\}^\infty_{t=0}$ corresponds to an equilibrium.*

Notice from Lemma F.1 that we have as many equations as unknown non-state variables (q_t, κ_t, R^f_t). However, Euler equations (F.14)-(F.15) also depend on the probability distribution of the future asset price $q_{t+\Delta}$, in order to determine the asset price q_t and riskless rate R^f_t today. This will be the key reason why the set of equilibrium conditions above is not enough to pin down q_t uniquely. In the continuous-time model, the distribution of future asset prices was summarized by the drift and the volatility (μ_q, σ_q). Here, the distribution of $q_{t+\Delta}$ could be more general, but we present a binomial example below. We now proceed to analysis of the two types of equilibria: fundamental and non-fundamental.

F.2 Fundamental equilibrium

A *fundamental equilibrium* has $\kappa_t = 1$ for all periods. In such an equilibrium, (F.13) says that the capital price should be

$$q_t = \frac{a_e}{\bar{\rho}(\eta_t)}, \quad \text{if} \quad \kappa_t = 1. \tag{F.16}$$

Substituting this result into the state dynamics (F.12), we have

$$\eta_{t+\Delta} = \frac{1}{1+\rho_e\Delta}\Big[1+\bar{\rho}(\eta_{t+\Delta}) - \frac{\bar{\rho}(\eta_{t+\Delta})}{\bar{\rho}(\eta_t)}(1-\eta_t)R_t^f\Big], \quad \text{if} \quad \kappa_t = \kappa_{t+\Delta} = 1. \tag{F.17}$$

As the only $(t+\Delta)$-measurable object in (F.17), $\eta_{t+\Delta}$ evolves deterministically in a fundamental equilibrium. Because q_t is solely a function of η_t in (F.16), $q_{t+\Delta}$ is also known as of time t. As a result, experts' return-on-capital must coincide with the riskless rate, i.e., $R_t^f = \frac{a_e\Delta}{q_t} + \frac{q_{t+\Delta}}{q_t}$, or

$$R_t^f = \bar{\rho}(\eta_t) + \frac{\bar{\rho}(\eta_t)}{\bar{\rho}(\eta_{t+\Delta})}, \quad \text{if} \quad \kappa_t = \kappa_{t+\Delta} = 1. \tag{F.18}$$

Combining (F.17) and (F.18), we obtain the solved dynamics

$$\eta_{t+\Delta} = \frac{\eta_t(1+\rho_e\Delta)^{-1}}{\eta_t(1+\rho_e\Delta)^{-1} + (1-\eta_t)(1+\rho_h\Delta)^{-1}}, \quad \text{if} \quad \kappa_t = \kappa_{t+\Delta} = 1. \tag{F.19}$$

Thus, expert's wealth share asymptotically tends toward zero. Intuitively, they earn zero excess capital returns and consume at a higher rate than households.

F.3 Non-fundamental equilibrium

A *non-fundamental equilibrium* has $\kappa_t < 1$ for some t. We proceed with a simple binomial tree example to show that non-fundamental equilibria exist, although more complicated information structures are also likely possible. We conjecture an equilibrium with

$$q_{t+\Delta} = \begin{cases} u_t q_t, & \text{with probability } 1-\pi_t; \\ d_t q_t, & \text{with probability } \pi_t. \end{cases} \tag{F.20}$$

The "up" and "down" returns u_t and $d_t \in (0, u_t)$ may be state dependent, as may the probability of a price drop π_t. As in our baseline model, we will take the "state space" to be the set of possible (η_t, q_t), or equivalently (η_t, κ_t). In other words, (u_t, d_t, π_t) will be functions of (η_t, κ_t), as will the interest rate r_t. The rest of this appendix constructs an example equilibrium under the binomial scheme (F.20). In particular, we will prove the following by construction:

Proposition F.1. *For all Δ sufficiently small, a non-fundamental equilibrium exists.*

To start, we may solve for the optimal portfolios explicitly in this binomial environment. Using (F.12) and (F.20) in the expert Euler equation (F.14), we have

$$\frac{\kappa_t}{\eta_t} = -R_t^f \frac{(1-\pi_t)u_t + \pi_t d_t + \frac{a_e\Delta}{q_t} - R_t^f}{(u_t + \frac{a_e\Delta}{q_t} - R_t^f)(d_t + \frac{a_e\Delta}{q_t} - R_t^f)}. \tag{F.21}$$

Doing the same for the household Euler equation (F.15), we have

$$\frac{1-\kappa_t}{1-\eta_t} = -R_t^f \min\Big(0, \frac{(1-\pi_t)u_t + \pi_t d_t + \frac{a_h\Delta}{q_t} - R_t^f}{(u_t + \frac{a_h\Delta}{q_t} - R_t^f)(d_t + \frac{a_h\Delta}{q_t} - R_t^f)}\Big). \tag{F.22}$$

Next, note that the price-output relation (F.13) and state dynamics (F.12) are unchanged by the binomial setup, and we repeat them here for convenience:

$$\bar{\rho}(\eta_t) = \frac{\kappa_t a_e + (1-\kappa_t)a_h}{q_t} \tag{F.23}$$

$$\eta_{t+\Delta} = \frac{1}{1+\rho_e\Delta} \frac{\kappa_t(\frac{a_e\Delta}{q_t} + \frac{q_{t+\Delta}}{q_t} - R_t^f) + \eta_t R_t^f}{q_{t+\Delta}/q_t}. \tag{F.24}$$

As mentioned in Lemma F.1, to find an equilibrium we only need to check that we can pick (u_t, d_t, π_t) to satisfy (F.21)-(F.24) at every point in the state space and that the resulting equilibrium dynamics do not cause the dynamical system to "exit the feasible region." To this end, we immediately note that $\eta_t \in (0,1)$ on any equilibrium path, which can be verified by checking the state dynamics (F.24).[38]

To continue, we will specialize below to a particular choice of u and d. Our construction will correspond to an approximation of Brownian motion in the "interior" of the

[38]Examine the state dynamics (F.24) in the down state and substitute (F.21) to obtain

$$d_t \frac{\eta_{t+\Delta}^d}{\eta_t} = \frac{1}{1+\rho_e\Delta} R_t^f \Big(1 - \frac{(1-\pi_t)u_t + \pi_t d_t + \frac{a_e\Delta}{q_t} - R_t^f}{u_t + \frac{a_e\Delta}{q_t} - R_t^f}\Big) > 0.$$

Similarly, mirroring (F.24), the symmetric condition for household's net worth share dynamics is

$$1-\eta_{t+\Delta} = \frac{1}{1+\rho_e\Delta} \frac{(1-\kappa_t)(\frac{a_h\Delta}{q_t} + \frac{q_{t+\Delta}}{q_t} - R_t^f) + (1-\eta_t)R_t^f}{q_{t+\Delta}/q_t}$$

Examining this condition in the up state and substituting (F.22), we obtain

$$u_t \frac{1-\eta_{t+\Delta}^u}{1-\eta_t} = \frac{1}{1+\rho_h\Delta} R_t^f \Big(1 - \min\Big(0, \frac{(1-\pi_t)u_t + \pi_t d_t + \frac{a_h\Delta}{q_t} - R_t^f}{d_t + \frac{a_h\Delta}{q_t} - R_t^f}\Big)\Big) > 0.$$

Thus, the requirement to keep $\eta_t \in (0,1)$ is automatically satisfied.

state space, with special considerations imposed at the "boundaries" of this state space. More specifically, we define the following regions. First, we have the entire feasible state space

$$\mathcal{D} := \Big\{(\eta,\kappa) : \eta \in (0,1),\, \kappa \in (\eta,1]\Big\}.$$

The reason why $\kappa > \eta$ is required is because $\kappa \leq \eta$ is inconsistent with the expert and household Euler equations (F.21)-(F.22), since $a_e > a_h$. Next, there will be a region near the top of $\mathcal{D}$, where κ is close to 1, such that positive shocks will just take the economy to the border:

$$\mathcal{D}_{high} := \Big\{(\eta,\kappa) \in \mathcal{D} : \kappa < 1,\, f(\kappa,\eta) < 0\Big\}.$$

for some function f to be defined endogenously below. At the other ends, let us pick some $\epsilon > 0$ and define the lower boundary region:

$$\mathcal{D}^{\epsilon}_{low} := \Big\{(\eta,\kappa) \in \mathcal{D}\backslash\mathcal{D}_{high} : \kappa \leq (1+\epsilon)\eta\Big\}.$$

For reasons that will become clear at the end of the construction, we will impose

$$\epsilon > \frac{a_h \rho_e}{(a_e - a_h)\rho_h}. \tag{F.25}$$

Finally, we will detail a separate method to deal with the top border region

$$\mathcal{D}_1 := \Big\{(\eta,\kappa) \in \mathcal{D} : \kappa = 1\Big\}.$$

The "interior" region is defined by subtracting these boundary regions:

$$\mathcal{D}^{\circ} := \mathcal{D}\backslash(\mathcal{D}_{high} \cup \mathcal{D}^{\epsilon}_{low} \cup \mathcal{D}_1).$$

We explain our construction in each of these regions in sequence.

Brownian approximation in the interior. In the interior region $\mathcal{D}^{\circ}$, we construct a non-fundamental equilibrium by explicitly specifying (u_t, d_t, π_t) to take a form that approximates Brownian motion in the $\Delta \to 0$ limit. In particular, we set

$$u_t = 1 + v_t\sqrt{\Delta} \tag{F.26}$$

$$d_t = 1 - v_t\sqrt{\Delta} \tag{F.27}$$

$$\pi_t = \frac{v_t - m_t\sqrt{\Delta}}{2v_t}, \tag{F.28}$$

for some endogenous variables m_t and v_t. Note that $\pi_t \in (0,1)$ requires $m_t\sqrt{\Delta} \in (-v_t, v_t)$. Of course, we also require $v_t \leq 1/\sqrt{\Delta}$. These constraints on m_t and v_t become arbitrarily loose as $\Delta \to 0$.

One can verify that (F.26)-(F.28) imply that

$$\mathbb{E}_t[\frac{q_{t+\Delta} - q_t}{q_t}] = m_t\Delta.$$

Thus, the interpretation of the variable m_t introduced is as the drift of percentage price changes. Also, we may compute

$$\mathbb{E}_t[(\frac{q_{t+\Delta} - q_t}{q_t})^2] = v_t^2\Delta,$$

so that v_t corresponds roughly to the instantaneous volatility of percentage price changes. Notice that any higher moments of price changes are of order $o(\Delta)$. Similarly, substituting the specification (F.26)-(F.28) into (F.24), one can verify that the state dynamics converge as $\Delta \to 0$ to the continuous-time model. Indeed, examine the conditional mean and second moment of $\eta_{t+\Delta} - \eta_t$:

$$\begin{aligned}
\mathbb{E}_t[\eta_{t+\Delta} - \eta_t] &= \Big(\kappa_t\frac{a_e}{q_t} - \eta_t\rho_e + (\kappa_t - \eta_t)(m_t - r_t - v_t^2)\Big)\Delta + o(\Delta) \\
\mathbb{E}_t[(\eta_{t+\Delta} - \eta_t)^2] &= (\kappa_t - \eta_t)^2 v_t^2\Delta + o(\Delta).
\end{aligned}$$

Dividing by Δ and taking $\Delta \to 0$, it becomes clear that these moments coincide with those of the continuous-time model.

Now, we determine what m_t and v_t must be to satisfy agents' optimality conditions. In this Brownian approximation, the expert and household Euler equations (F.21)-(F.22) become

$$\frac{\kappa_t}{\eta_t} = (1 + r_t\Delta)\frac{\frac{a_e}{q_t} + m_t - r_t}{v_t^2 - (\frac{a_e}{q_t} - r_t)^2\Delta} \tag{F.29}$$

$$\frac{1 - \kappa_t}{1 - \eta_t} = (1 + r_t\Delta)\max\Big\{0, \frac{\frac{a_h}{q_t} + m_t - r_t}{v_t^2 - (\frac{a_h}{q_t} - r_t)^2\Delta}\Big\}. \tag{F.30}$$

As $\Delta \to 0$, these two specialized Euler equations (F.29)-(F.30) coincide with the familiar mean-variance portfolio choice. However, to recover the same equations as in our continuous-time model, let us take the difference between (F.29)-(F.30) to get

$$0 = \min\left\{1-\kappa_t,\ (1+r_t\Delta)\Big[\frac{\frac{a_e}{q_t}+m_t-r_t}{v_t^2-(\frac{a_e}{q_t}-r_t)^2\Delta}-\frac{\frac{a_h}{q_t}+m_t-r_t}{v_t^2-(\frac{a_h}{q_t}-r_t)^2\Delta}\Big]-\frac{\kappa_t-\eta_t}{\eta_t(1-\eta_t)}\right\}. \quad \text{(F.31)}$$

Equation (F.31) clearly coincides with our baseline risk-balance condition as $\Delta \to 0$. Then, summing (F.29)-(F.30), weighted by κ_t and $1-\kappa_t$ respectively, we have

$$\frac{\kappa_t^2}{\eta_t}+\frac{(1-\kappa_t)^2}{1-\eta_t} = (1+r_t\Delta)\Big[\kappa_t\frac{\frac{a_e}{q_t}+m_t-r_t}{v_t^2-(\frac{a_e}{q_t}-r_t)^2\Delta}+(1-\kappa_t)\frac{\frac{a_h}{q_t}+m_t-r_t}{v_t^2-(\frac{a_h}{q_t}-r_t)^2\Delta}\Big]. \quad \text{(F.32)}$$

Again, this coincides with the equation for μ_q in the continuous-time model as $\Delta \to 0$.

To solve the model, first we use the expert Euler equation to solve for v_t^2:

$$v_t^2 = (1+r_t\Delta)\Big[\frac{a_e}{q_t}+m_t-r_t\Big]\frac{\eta_t}{\kappa_t}+(\frac{a_e}{q_t}-r_t)^2\Delta.$$

Then, we use the household Euler equation, when $\kappa_t < 1$, to also solve for v_t^2:

$$v_t^2 = (1+r_t\Delta)\Big[\frac{a_h}{q_t}+m_t-r_t\Big]\frac{1-\eta_t}{1-\kappa_t}+(\frac{a_h}{q_t}-r_t)^2\Delta.$$

Setting these expressions equal gives an equation for m_t, which is

$$m_t = r_t+\frac{(1-\kappa_t)\eta_t}{\kappa_t-\eta_t}\frac{a_e}{q_t}-\frac{\kappa_t(1-\eta_t)}{\kappa_t-\eta_t}\frac{a_h}{q_t}+\frac{\kappa_t(1-\kappa_t)\big[(\frac{a_e}{q_t}-r_t)^2-(\frac{a_h}{q_t}-r_t)^2\big]}{(1+r_t\Delta)(\kappa_t-\eta_t)}\Delta. \quad \text{(F.33)}$$

Substituting back into the equations for v_t^2, we solve for

$$v_t^2 = (1+r_t\Delta)\frac{\eta_t(1-\eta_t)}{\kappa_t-\eta_t}\frac{a_e-a_h}{q_t}+\frac{\kappa_t(1-\eta_t)(\frac{a_e}{q_t}-r_t)^2-\eta_t(1-\kappa_t)(\frac{a_h}{q_t}-r_t)^2}{\kappa_t-\eta_t}\Delta. \quad \text{(F.34)}$$

Given a choice for r_t, we can obtain m_t and v_t^2 from equations (F.33)-(F.34), for any point in the interior of the state space. The only restriction is that we choose r_t so that $m_t\sqrt{\Delta} \in (-v_t, v_t)$ and hence that $\pi_t \in (0,1)$, which leaves a wide range of choices. To be explicit, we will choose r_t such that $m_t = O(\Delta)$, in particular we set

$$r_t = \frac{\kappa_t(1-\eta_t)}{\kappa_t-\eta_t}\frac{a_h}{q_t}-\frac{(1-\kappa_t)\eta_t}{\kappa_t-\eta_t}\frac{a_e}{q_t}. \quad \text{(F.35)}$$

This choice makes it automatic that $m_t\sqrt{\Delta} \in (-v_t, v_t)$ if Δ is also chosen small enough.

As an aside, note that these equations, in the $\Delta \to 0$ limit, are identical to the continuous-time versions (when there is zero fundamental risk and zero growth). Indeed, equation (F.34) says

$$v_t^2 = \frac{\eta_t(1-\eta_t)}{\kappa_t - \eta_t}\frac{a_e - a_h}{q_t} + O(\Delta).$$

Next, by doing some algebra on (F.33), it reads

$$m_t = r_t - \bar{\rho}(\eta_t) + \Big(\frac{\kappa_t^2}{\eta_t} + \frac{(1-\kappa_t)^2}{1-\eta_t}\Big)v_t^2 + O(\Delta).$$

Consequently, m_t and v_t are indeed the discrete-time counterparts to $\mu_{q,t}$ and $\sigma_{q,t}$.

Reflection approximation near the lower boundary. In the lower region $\mathcal{D}^{\epsilon}_{low}$, we proceed with a different construction that ensures the economy never exits $\mathcal{D}$ through its lower border. Luckily, in everything so far, r_t was indeterminate, and this flexibility is what allows us to construct such an equilibrium. In particular, to ensure we always have $\kappa_t \in (\eta_t, 1)$, we impose some rules similar to our "boundary conditions" in continuous time.

In $\mathcal{D}^{\epsilon}_{low}$, we will use the binomial specification

$$u_t = 1 + v_t^2/m_t \tag{F.36}$$

$$d_t = 1 \tag{F.37}$$

$$\pi_t = \frac{v_t^2 - m_t^2\Delta}{v_t^2} \tag{F.38}$$

Equations (F.36)-(F.38) preserve the desired moment properties that $\mathbb{E}_t[\frac{q_{t+\Delta}-q_t}{q_t}] = m_t\Delta$ and $\mathbb{E}_t[(\frac{q_{t+\Delta}-q_t}{q_t})^2] = v_t^2\Delta$. Again, we must have probabilities in between zero and one, so we always require $m_t\sqrt{\Delta} \in (-v_t, v_t)$.

With this specification, the Euler equations become

$$\frac{\kappa_t}{\eta_t} = (1+r_t\Delta)\frac{\frac{a_e}{q_t} + m_t - r_t}{\frac{v_t^2}{m_t}(r_t - \frac{a_e}{q_t}) - (\frac{a_e}{q_t} - r_t)^2\Delta} \tag{F.39}$$

$$\frac{1-\kappa_t}{1-\eta_t} = (1+r_t\Delta)\frac{\frac{a_h}{q_t} + m_t - r_t}{\frac{v_t^2}{m_t}(r_t - \frac{a_h}{q_t}) - (\frac{a_h}{q_t} - r_t)^2\Delta}. \tag{F.40}$$

As before, we may use these two equations to solve for m_t and v_t^2:

$$m_t = r_t + \frac{(1+r_t\Delta)\big[\frac{\eta_t}{\kappa_t}(r_t - \frac{a_h}{q_t})\frac{a_e}{q_t} - \frac{1-\eta_t}{1-\kappa_t}(r_t - \frac{a_e}{q_t})\frac{a_h}{q_t}\big] - (\frac{a_e - a_h}{q_t})(r_t - \frac{a_e}{q_t})(r_t - \frac{a_h}{q_t})\Delta}{(1+r_t\Delta)\big[\frac{1-\eta_t}{1-\kappa_t}(r_t - \frac{a_e}{q_t}) - \frac{\eta_t}{\kappa_t}(r_t - \frac{a_h}{q_t})\big]} \tag{F.41}$$

$$v_t^2 = m_t\Big[\frac{(1+r_t\Delta)\frac{\eta_t}{\kappa_t}(\frac{a_e}{q_t} + m_t - r_t)}{r_t - \frac{a_e}{q_t}} + (r_t - \frac{a_e}{q_t})\Delta\Big] \tag{F.42}$$

Given that the Euler equations hold for this choice of (m_t, v_t^2), we have an equilibrium as long as $m_t\sqrt{\Delta} \in (-v_t, v_t)$ and $\kappa_t > \eta_t$ in all periods.

The condition that $\kappa_t > \eta_t$ is the more complex and restrictive condition. The key issue is that (η_t, κ_t) can jump from $\mathcal{D}^{\epsilon}_{low}$ to a point outside of the feasible region $\mathcal{D}$.[39] Resolving this issue requires us to make particular choices for r_t such that the dynamics of (η_t, κ_t) "point toward the interior" of the state space, i.e., the dynamics starting from $\mathcal{D}^{\epsilon}_{low}$ are such that $(\eta_{t+\Delta}, \kappa_{t+\Delta})$ moves closer to $\mathcal{D}^{\circ}$. Sufficient conditions for this are that $\eta_{t+\Delta} \le \eta_t$ when $(\eta_t, \kappa_t) \in \mathcal{D}^{\epsilon}_{low}$. Indeed, if $\eta_{t+\Delta} \le \eta_t$, then the dynamics of q_t are such that $\kappa_{t+\Delta} \ge \kappa_t$. Since the lower-boundary of $\mathcal{D}$ is upward-sloping in (η, κ)-space, the combination of $\eta_{t+\Delta} \le \eta_t$ and $\kappa_{t+\Delta} \ge \kappa_t$ implies that the new point is further away from exiting $\mathcal{D}$.

Ensuring that $\eta_{t+\Delta} \le \eta_t$ translates to the following condition on the risk-free rate:

$$r_t \ge \tilde{r}_t, \quad \text{whenever} \quad (\eta_t, \kappa_t) \in \mathcal{D}^{\epsilon}_{low}, \tag{F.43}$$

$$\text{where} \quad \tilde{r}_t := \max\Big[\frac{\kappa_t a_e - \rho_e \eta_t q_t}{q_t(\kappa_t - \eta_t)}, \ \frac{\kappa_t a_e - \rho_e \eta_t q_t(1 + v_t^2/m_t)}{q_t(\kappa_t - \eta_t)} + \frac{v_t^2}{m_t\Delta}\Big].$$

Now, the equilibrium values of v_t and m_t in (F.41)-(F.42) depend on r_t, so the comparison between r_t and $\tilde{r}_t$ is not explicit. However, we can show that a valid solution to (F.43) exists if Δ is made small enough.

To see this, let us set

$$r_t = \frac{\kappa_t a_e - \rho_e \eta_t q_t}{q_t(\kappa_t - \eta_t)} + \frac{\alpha_t}{\Delta} + C_r \tag{F.44}$$

for some $\alpha_t > 0$ small enough and some constant C_r. Using equations (F.44) and (F.41)-(F.42), one may conjecture and verify that, as $\Delta \to 0$, the variables (r_t, m_t, v_t^2) obey the

[39] Another potential issue is that (η_t, κ_t) can jump from the interior $\mathcal{D}^{\circ}$ to a point outside of the feasible region $\mathcal{D}$ This issue is removed by choosing small enough Δ, because the step sizes in the interior are proportional to $\sqrt{\Delta}$.

following asymptotic relationships

$$r_t\Delta \to \alpha_t$$
$$m_t\Delta \to \alpha_t$$
$$v_t^2/m_t \to \alpha_t.$$

In that case, we have that $r_t - \tilde{r}_t \sim \frac{\rho_e \eta_t q_t \alpha_t}{q_t(\kappa_t-\eta_t)} + \frac{\alpha_t - v_t^2/m_t}{\Delta} + C_r$ as $\Delta \to 0$. Thus, if we pick $C_r = -\lim_{\Delta\to 0} \Delta^{-1}(\alpha_t - v_t^2/m_t)$, the inequality $r_t \geq \tilde{r}_t$ holds for all small enough Δ. It is easy to see that $\Delta^{-1}(v_t^2/m_t - \alpha_t) = O(1)$ as $\Delta \to 0$ so that C_r will be a finite constant. Furthermore, given that α_t is a free parameter, it may be chosen small enough so that upward percentage step size v_t^2/m_t is small enough. Given that the choice (F.44) is continuous in Δ, and equations (F.41)-(F.42) are continuous in r_t, it follows that for all small enough Δ, a valid r_t exists satisfying (F.43).

The final question is whether or not this choice also satisfies $m_t\sqrt{\Delta} \in (-v_t, v_t)$, such that the probabilities of up- and down-moves are within zero and one. To answer this, we can study

$$\frac{v_t^2}{m_t^2\Delta} = 1 + \frac{\frac{a_e}{q_t} + m_t - r_t}{m_t}\Big[\frac{(1+r_t\Delta)\frac{\eta_t}{\kappa_t}}{r_t\Delta - \frac{a_e\Delta}{q_t}} - 1\Big]. \tag{F.45}$$

We can see from equation (F.41) that as $\Delta \to 0$, we have

$$\frac{a_e}{q_t} + m_t - r_t \to \frac{1}{1+\alpha_t}\frac{\kappa_t(1-\kappa_t)}{\kappa_t - \eta_t}\frac{a_e - a_h}{q_t}\Big[\alpha_t - (1+\alpha_t)\frac{1-\eta_t}{1-\kappa_t}\Big] > 0.$$

In addition, the term in square brackets in equation (F.45) is positive in the $\Delta \to 0$ limit if and only if $\kappa_t/\eta_t < (1+\alpha_t)/\alpha_t$. Therefore, by picking α_t small enough, we ensure that the expression in (F.45) is strictly larger than 1 for all Δ small enough. This shows that $m_t\sqrt{\Delta} \in (-v_t, v_t)$ by choosing α_t and Δ small enough.

Jumps to efficiency. At some points when κ_t is sufficiently close to 1, the Brownian approximation above could potentially make κ_t jump above 1, which is inconsistent with equilibrium. At these points, we must instead design the shocks so that κ_t jumps to 1. Such points will constitute the region earlier denoted by $\mathcal{D}_{high}$, whose border with $\mathcal{D}^\circ$ was previously left unspecified and which we will now make explicit.

First, let us define the binomial scheme by

$$u_t = \frac{a_e}{q_t \bar{\rho}(\eta_t^{max})} \tag{F.46}$$

$$d_t = \text{free parameter} \tag{F.47}$$

$$\pi_t = \frac{u_t - 1 - m_t \Delta}{u_t - d_t}, \tag{F.48}$$

where

$$\eta_t^{max} := \frac{\kappa_t a_e(1+\rho_e\Delta) - (\kappa_t - \eta_t) q_t \rho_h (1 + r_t\Delta)}{a_e[1 + \rho_e\Delta - \kappa_t(\rho_e - \rho_h)\Delta] + (\kappa_t - \eta_t) q_t (1 + r_t\Delta)(\rho_e - \rho_h)} \tag{F.49}$$

is the net worth share that would arise if κ jumps to 1.[40] It is straightforward to check that for Δ small enough, we have $\eta_t^{max} < \kappa_t < 1$, so that η_t^{max} is a valid wealth share. Note also that the setup in (F.46)-(F.48) by construction preserves specification of m_t as the local mean $\mathbb{E}_t[\frac{q_{t+\Delta} - q_t}{q_t}] = m_t \Delta$.

The Euler equations become

$$\frac{\kappa_t}{\eta_t} = -(1 + r_t\Delta) \frac{(m_t + \frac{a_e}{q_t} - r_t)\Delta}{(u_t + \frac{a_e\Delta}{q_t} - (1 + r_t\Delta))(d_t + \frac{a_e\Delta}{q_t} - (1 + r_t\Delta))} \tag{F.50}$$

$$\frac{1 - \kappa_t}{1 - \eta_t} = -(1 + r_t\Delta) \frac{(m_t + \frac{a_h}{q_t} - r_t)\Delta}{(u_t + \frac{a_h\Delta}{q_t} - (1 + r_t\Delta))(d_t + \frac{a_h\Delta}{q_t} - (1 + r_t\Delta))}. \tag{F.51}$$

We can use the two Euler equations to solve for m_t and d_t as

$$\begin{aligned} m_t = r_t + \frac{1}{1 + r_t\Delta} & \frac{\kappa_t(1-\kappa_t)\frac{a_e - a_h}{q_t}(u_t + \frac{a_h\Delta}{q_t} - (1 + r_t\Delta))(u_t + \frac{a_e\Delta}{q_t} - (1 + r_t\Delta))}{(\kappa_t - \eta_t)(u_t - (1 + r_t\Delta)) + \kappa_t(1 - \eta_t)\frac{a_e\Delta}{q_t} - \eta_t(1 - \kappa_t)\frac{a_h\Delta}{q_t}} \\ & - \frac{(\kappa_t - \eta_t)\frac{a_e a_h \Delta}{q_t^2} + [\kappa_t(1 - \eta_t)\frac{a_h}{q_t} - \eta_t(1 - \kappa_t)\frac{a_e}{q_t}](u_t - (1 + r_t\Delta))}{(\kappa_t - \eta_t)(u_t - (1 + r_t\Delta)) + \kappa_t(1 - \eta_t)\frac{a_e\Delta}{q_t} - \eta_t(1 - \kappa_t)\frac{a_h\Delta}{q_t}} \end{aligned} \tag{F.52}$$

[40] In particular, if κ_t jumps to $\kappa_{t+\Delta} = 1$, then from (F.23) q_t jumps to $q_{t+\Delta} = a_e / \bar{\rho}(\eta_{t+\Delta})$. But the dynamics of η from (F.24) must also hold, which means that $\eta_{t+\Delta}$ solves

$$\eta_{t+\Delta} = \frac{1}{1 + \rho_e\Delta} \frac{\kappa_t\left[\frac{a_e\Delta}{q_t} + \frac{a_e}{q_t \bar{\rho}(\eta_{t+\Delta})} - (1 + r_t\Delta)\right] + \eta_t(1 + r_t\Delta)}{a_e / (q_t \bar{\rho}(\eta_{t+\Delta}))}.$$

We denote the solution by η_t^{max}, given in (F.49).

and

$$d_t = (1 + r_t\Delta)\Big[1 - \frac{\eta_t}{\kappa_t}\frac{(m_t + \frac{a_e}{q_t} - r_t)\Delta}{u_t + \frac{a_e\Delta}{q_t} - (1 + r_t\Delta)}\Big] - \frac{a_e\Delta}{q_t}. \tag{F.53}$$

To guarantee that this constitutes an equilibrium, we must verify $\pi_t \in (0, 1)$ along with $0 < d_t < 1 < u_t$.

To check these conditions explicitly, let us pick $r_t = 0$, and let us consider Δ small. As it will turn out (which we will verify below), when Δ is small the region $\mathcal{D}_{high}$ will be associated with $\kappa_t = 1 - O(\sqrt{\Delta})$, so that our choice implies $m_t = -a_h/q_t + O(\sqrt{\Delta})$ from equation (F.52). Substituting this result into equation (F.53), we see that $0 < d_t < 1$ if Δ is small enough. It is easy to check that $u_t > 1$ holds as long as $\rho_e - \rho_h$ is not too large, which we implicitly assume. Lastly, given these results just discussed, we have $\pi_t \in (0, 1)$ automatically when Δ is small enough. This shows that, if Δ is small enough, then $r_t = 0$ is a valid choice, and the other equilibrium conditions all hold.

Finally, we need to specify the boundary between $\mathcal{D}_{high}$ and the interior region $\mathcal{D}^\circ$. The procedure will be to compute v_t associated to $\mathcal{D}^\circ$—from equation (F.34)—and then compare $1 + v_t\sqrt{\Delta}$ to $a_e/(q_t\bar{\rho}(\eta_t^{max}))$. If $1 + v_t\sqrt{\Delta} > a_e/(q_t\bar{\rho}(\eta_t^{max}))$ at a given point $(\eta_t, \kappa_t) \in \mathcal{D}$, then we allocate that point to set $\mathcal{D}_{high}$. Otherwise, the given point (η_t, κ_t) is considered to be part of $\mathcal{D}^\circ$. This proves the result used above that $u_t - 1 = O(\sqrt{\Delta})$, and hence $1 - \kappa_t = O(\sqrt{\Delta})$.

Analysis at $\kappa = 1$ border. Finally, given that $\kappa_t = 1$ sometimes, we must describe how the economy exits this region and re-enters the interior $\mathcal{D}^\circ$. We specify a particularly simple approach that always works, although it is unnecessarily restrictive in general.

We will consider a binomial scheme that either maintains $\kappa_{t+\Delta} = 1$ with some probability and otherwise has $\eta_{t+\Delta} \approx 0$ (i.e., expert near-bankruptcy) with the residual probability. This scheme is

$$u_t = 1 \tag{F.54}$$

$$d_t = 1 - \frac{(\eta_t - \omega_t)(1 + \rho_e\Delta)}{1 - \omega_t(1 + \rho_e\Delta)} \tag{F.55}$$

$$\pi_t = \text{free parameter}, \tag{F.56}$$

along with a particular choice for the riskless rate:

$$r_t = \rho_h. \tag{F.57}$$

Using (F.54), (F.55), and (F.57) in the state dynamics (F.24), one can verify that

$$\eta^u_{t+\Delta} = \eta_t$$
$$\eta^d_{t+\Delta} = \omega_t.$$

In other words, a positive shock keeps (η_t, q_t) in place, while a negative shock drives η down to ω_t.

For this to be a valid construction, we require that $q^d_{t+\Delta} = d_t q_t$ is larger than the minimum possible price at the new wealth share, which is $q_{min}(\eta^d_{t+\Delta}) = q_{min}(\omega_t) = (\omega_t a_e + (1-\omega_t)a_h)/\bar{\rho}(\omega_t)$. Using the fact that $q_t = a_e/\bar{\rho}(\eta_t)$, this validity condition is equivalent to

$$\bar{\rho}(\omega_t)\left[1-\eta_t - \left(\bar{\rho}(\eta_t) - (1-\eta_t)\rho_h\right)\Delta\right]a_e > \bar{\rho}(\eta_t)\left[1-\omega_t(1+\rho_e\Delta)\right]\left(\omega_t a_e + (1-\omega_t)a_h\right).$$

As $\Delta \to 0$, this condition becomes

$$\bar{\rho}(\omega_t)(1-\eta_t)a_e > \bar{\rho}(\eta_t)(1-\omega_t)\left(\omega_t a_e + (1-\omega_t)a_h\right).$$

Taking $\omega_t \to 0$ as well, we have the condition

$$\rho_h(1-\eta_t)a_e > \bar{\rho}(\eta_t)a_h \;\Leftrightarrow\; \eta_t < \frac{(a_e-a_h)\rho_h}{(a_e-a_h)\rho_h + a_h\rho_e} := \eta_{top}.$$

Finally, we use the choice of ϵ in (F.25), which implies that the line $\kappa = (1+\epsilon)\eta$ intersects the horizontal line $\kappa = 1$ at a point $\eta < \eta_{top}$. Consequently, if Δ is chosen small enough, equilibrium paths with $\kappa_t = 1$ in period t will have $\eta_t < \eta_{top}$ in the same period. This implies that if Δ and ω_t are chosen small enough, then we can ensure that $q^d_{t+\Delta} > q_{min}(\eta^d_{t+\Delta})$.

Given that $\kappa_t = 1$ at these points, the household Euler inequality (F.22) must hold with strict inequality. A sufficient condition is that households make negative excess returns when capital price remains constant, i.e.,

$$0 > \frac{a_h\Delta + q_{t+\Delta}}{q_t} - R^f_t = \left[\frac{a_h}{a_e}\bar{\rho}(\eta_t) - \rho_h\right]\Delta$$

which always holds since $\rho_e > \bar{\rho}(\eta)$ and $a_e/\rho_e > a_h/\rho_h$.

It remains to verify that the expert Euler equation (F.21) holds. However, this is guaranteed if the remaining free parameter π_t takes the particular value

$$\pi_t = \frac{(\bar{\rho}(\eta_t) - \rho_h)\Delta}{1 - d_t} + \frac{(\bar{\rho}(\eta_t) - \rho_h)(d_t - 1 + (\bar{\rho}(\eta_t) - \rho_h)\Delta)\Delta}{\eta_t(1 + \rho_h\Delta)(1 - d_t)}.$$

Plugging in d_t from (F.55), we have

$$\pi_t = \frac{1 - \omega_t(1 + \rho_e\Delta)}{(\eta_t - \omega_t)(1 + \rho_e\Delta)}\left[1 + \frac{(\bar{\rho}(\eta_t) - \rho_h)\Delta - \frac{(\eta_t - \omega_t)(1 + \rho_e\Delta)}{1 - \omega_t(1 + \rho_e\Delta)}}{\eta_t(1 + \rho_h\Delta)}\right](\bar{\rho}(\eta_t) - \rho_h)\Delta.$$

Note that $\eta_t > \frac{(\eta_t - \omega_t)}{1 - \omega_t}$, so that $\pi_t > 0$ for all Δ small enough. In addition, note that $\pi_t \to 0$ as $\Delta \to 0$. Therefore, for all Δ small enough, we are guaranteed to have $\pi_t \in (0, 1)$.