A Sticky-Dispersed Information Phillips Curve:*
A model with partial and delayed information

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Abstract

This paper puts to test **?** conjecture that a small incidence of informational stickiness can lead to a large amount of persistence in aggregate prices in a world of differential information. We do so by assuming that firms receive private noisy signals about the state in an otherwise standard model of price setting with sticky-information. We prove there exists a unique equilibrium of the incomplete information game induced by the firms' pricing decisions and derive the resulting Sticky-Dispersed Information (SDI) Phillips curve. The main effect of dispersion is to substantially magnify the *immediate* impact of a given shock when the degree of stickiness is small. Its effect for persistence, however, is minor: even when information is largely dispersed, a substantial amount of informational stickiness is needed to generate persistence in aggregate prices and inflation.

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1 Introduction

? sticky information model achieves two important goals at once: (i) explains why prices fail to respond quickly to nominal shocks and (ii) reconciles the backward-looking behavior needed to generate the observed persistence in aggregate prices with the assumption that agents are fully rational. Nevertheless, in a world in which information becomes more frequently available, the assumption that information is sticky is less plausible. Consequently, one would expect that the power of a sticky-information model in explaining the observed degree of persistence in prices to diminish. **?**, however, conjecture that only a small incidence of price stickiness is needed to generate substantial persistence in aggregate prices in a world of *differential information*. This paper puts such conjecture to test.[1]

We do so by studying how individual firms set prices when information is both sticky *and* dispersed, and analyze the resulting dynamics for aggregate prices and inflation rates. In our model, a firm's optimal price is a convex combination of the current state of the economy and the aggregate price level. Nevertheless, as firms do not observe the current state nor other firm's princing decisions, they have to use the available information to infer the optimal price. As in **?**, only a fraction of firms update their information set at each period. Those who update receive two sources of information: the first piece is the value of all previous periods states, while the second piece is a noisy, idiosyncratic, private signal about the current state of the economy. Since noisy signals are idiosyncratic, the firms that update their information set will have heterogenous information about the state (as in **?** and **?**). Hence, in our model, *heterogenous* information disseminates slowly in the economy.

Firms must not only form beliefs about the current state but also form beliefs about the other firms' beliefs about the current state, and so on, so that higher-order beliefs play a key role in our model. Hence, the pricing decisions by firms induce an incomplete information game among them.

In our main result, we prove that there exists a unique equilibrium of such game. The uniqueness of the equilibrium allows us to unequivocally speak about the sticky-dispersed-information (henceforth, SDI) aggregate price level and Phillips curve. The SDI aggregate price level we derive depends on all the current and past states of the economy. This is so for two reasons. First, there are firms in the economy for which the information set has been last updated in the far past. This is a *direct* effect of sticky information. Second, firms that have just received new information will behave, at least partly, as if they were backward-looking. This happens because of an *strategic* effect: their optimal relative price depends on how they believe all other firms (including those that have outdated information sets) in the economy are setting prices.

From aggregate prices, we are able to derive the SDI Phillips curve and show that inflation also

[1] **?** last sentence is "The incidence of sticky information or rational inattention that is necessary to account for the observed degree of persistence may be quite small when embedded in a world of differential information. "

depends on all the current and past states of the economy. This result is linked to the one obtained in **?**, in which inflation depends on past expectations of current economic conditions, due to the fact that firms compute expectations based on outdated information. This is an implication of the stickiness of information in our model and was already present in **?**. In our model, however, in addition to being sticky, information is also noisy and dispersed. The fact that information is noisy leads a firm that has its information set updated in t to find it optimal to place positive weight on the states from periods $t - j$, $j > 0$, to predict the state in period t. Hence, in comparison to **?**, the adjustment of prices to shocks will be slower in an economy with noisy information. Through the complementarities in price setting, the fact that, on top of being noisy, information is dispersed magnifies such effect.

Hence, we establish analytically that dispersion magnifies the effects of stickiness on persistence. To evaluate the quantitative importance of dispersion in increasing the effect of stickiness, we perform some numerical simulations. These simulations suggest that, on the one hand, in settings with small incidence of sticky information, dispersion plays at most a minor role in generating persistence. However, for such cases, dispersion substantially magnifies the immediate impact of a given shock. Without a substantial amount of price stickiness, although much larger than what would have been if there were no dispersion, the effect of a shock tends to quickly vanish.

On the other hand, when there is a large amount of price stickiness, the effects of dispersion on both the immediate impact of a given shock and its persistence in the economy tend to be very small. Therefore, in a model of sticky-dispersed information, what is key for the generation of persistence is a substantial amount of price stickiness.

Related Literature. In addition to the papers already mentioned, our work follows a large number of papers that sheds new light into the tradition that dates back to **?** and **?** of considering the effects of imperfect information on price-setting decisions. **?** provide the most recent survey on the impact of informational frictions on pricing decisions, comparing a partial (dispersed) information model with a delayed (sticky) information model, and deriving their common implications.[2] In turn, **?** introduce dispersed information (and explicitly discuss the role of higher order beliefs) in an otherwise standard setting with sticky prices *à la* **?**. **?** covers a myriad of topics related to informational asymmetries and information acquisition in macroeconomics and finance. Our paper connects to this broad literature through two specific strands. In our model, (i) information is sticky, as in **?** and others, and (ii) following **?** and **?**, among others, information is dispersed.

The paper that is the closest to ours is **?**. The authors combine in a single model both dispersed information and informational stickiness to study the signaling role of policy actions. The aspects that differenciate our paper to **?** pertain to two issues: assumptions and focous. Concerning as-

[2]The theories of "rational inattention" proposed by ? (?, ?) and "inattentiveness" proposed by ? (?, ?) have been used to justify models of dispersed information and sticky information.

sumptions, we explicit incorporate dynamics in the relevant fundamental of our model economy, in contrast to **?** that considers that the fundamental is repetidaly drawn from a normal distribution. Dynamics changes considerably the way agents form their beliefs in the sense they can make predictions even when they have a outdated information set. Therefore, we have dispersed information even among those agents who has outdated information. Dynamics also alows us to obtain impulse responses to both structural and informational shocks. Concerning focous, our framework enfasizes the interaction between stickness and dispersion and allows us to investigate how these informational frictions interchanges. To the best of our knowledge, we are the first to offer a dynamic model with an integrated approach to study the interaction of dispersion and stickiness on pricing decisions. By focusing on informational stickiness (rather than price stickiness), we complement the analysis of **?**.

Organization. The paper is organized as follows. In section 2, the set-up of the model is described. In section 3, we derive the unique equilibrium of the pricing game played by the firms, and derive the implied aggregate prices and inflation rates. Section 5 compares our SDI Phillips curve with three benchmarks: the complete information, the sticky-information and the dispersed information Phillips curves. In section 4, we define a measure of persistence in our setting, and then analytically derive the effects of the parameters that capture stickiness and dispersion in the model on our measure of dispersion, and perform some numerical simulations to evaluate the quantitative importance of each of these factors. Section 6 draws the concluding remarks. All derivations that are not in the text can be found in the Appendix.

2 The Model

The model is a variation of **?** sticky information model.[3] There is a continuum of firms, indexed by $i \in [0,1]$, that set prices at every period $t \in \{1,2,...\}$.

Although prices can be re-set at no cost at each period, information regarding the state of the economy is made available to the firms infrequently. At period t, only a fraction λ of firms is selected to update their information sets about the current state. For simplicity, the probability of being selected to adjust information sets is the same across firms and independent of history.

We depart from a standard sticky-information model by allowing information to be *heterogeneous* and *dispersed*: a firm that updates its information set receives public information regarding the past states of the economy as well as a *private* signal about the current state.

Pricing Decisions Under complete information, any given firm $z \in [0,1]$ set its (log-linear) price $p_t(z)$ equal to the optimal price decision p_t^* given by

[3] Subsequent refinements of the sticky information models can be found in **?** (?, ?, ?) and **?** (?, ?, ?).

$$p_t^* \equiv rP_t + (1-r)\,\theta_t, \tag{1}$$

where $P_t \equiv \int_0^1 p_t(z)\,dz$ is the aggregate price level, and θ_t is the nominal aggregate demand, the current state of the economy. This pricing rule is standard, and, although we don't do it explicitly, can be derived from a firm's profit maximization problem in a model of monopolistic competition in the spirit of **?**.

Information The state θ_t follows a random walk

$$\theta_t = \theta_{t-1} + \epsilon_t, \tag{2}$$

with $\epsilon_t \sim N\left(0, \alpha^{-1}\right)$.

If firm $z \in [0,1]$ is selected to update its information set in period t, it observes all *previous* periods realizations of the state, $\{\theta_{t-j}, j \geq 1\}$. Moreover, it obtains a noisy private signal about the current state. Denoting such signal by $x_t(z)$, we follow the literature and assume:

$$x_t(z) = \theta_t + \xi_t(z), \tag{3}$$

where $\xi_t(z) \sim N\left(0, \beta^{-1}\right)$, β is the precision of $x_t(z)$, and the error term $\xi_t(z)$ is independent of ϵ_t for all z, t.

As a result, if one defines

$$\Theta_{t-j} = \{\theta_{t-k}\}_{k=j}^{\infty}, \tag{4}$$

at period t, the information set of a firm z that was selected to update its information j periods ago is

$$I_{t-j}(z) = \{x_{t-j}(z), \Theta_{t-j-1}\}. \tag{5}$$

3 Equilibrium

Using (1), the best response for a firm z that was selected to update its information j periods ago – and, therefore, has $I_{t-j}(z)$ as its information set – is its forecast of p_t^*, given the available information $I_{t-j}(z)$:

$$p_t(z) = E\left[p_t^* \mid I_{t-j}(z)\right]. \tag{6}$$

Denoting by Λ_{t-j} the set of firms that last updated its information set at period $t-j$, we can express the aggregate price level P_t as

$$\begin{aligned} P_t &= \int_{\cup_{j=0}^{\infty}\Lambda_{t-j}} p_t(z)\, dz \\ &= \sum_{j=0}^{\infty}\int_{\Lambda_{t-j}} E\left[p_t^* \mid I_{t-j}(z)\right] dz. \end{aligned} \tag{7}$$

Since the optimal price p_t^* is a convex combination of the state, θ_t, and the aggregate price level, firm z needs to forecast the state of the economy *and* the pricing behavior of the other firms in the economy. The pricing behavior of each of these firms, in turn, depends on their own forecast of the other firms' aggregate behavior. It follows that firm z must not only forecast the state of the economy but also, to predict the behavior of the other firms in the economy, must make forecasts of these firms' forecasts about the state, forecasts about the forecasts of these firms forecasts about the state, and so on and so forth. In other words, higher order beliefs will play a key role in the derivation of an equilibrium in our model.

Indeed, if one defines the average k-th order belief about the current state recursively as follows:

$$\bar{E}^k[\theta_t] = \begin{cases} \theta_t, & k = 0 \\ \sum_{j=0}^{\infty}\int_{\Lambda_{t-j}} E\left[\bar{E}^{k-1}[\theta_t] \mid I_{t-j}(z)\right] dz, & k \geq 1 \end{cases}, \tag{8}$$

we have that, in equilibrium, the aggregate price level is

$$P_t = (1-r)\sum_{k=1}^{\infty} r^{k-1}\bar{E}^k[\theta_t]. \tag{9}$$

3.1 Computing the Equilibrium

In this section, we derive the unique equilibrium of the pricing game played by the firms. Following **?**, we do this in two steps. We first derive an equilibrium for which the aggregate price level is a linear function of fundamentals. We then establish, using (9), that this linear equilibrium is the unique equilibrium of our game.

3.1.1 Posterior Distribution

In the Appendix, we show that, given the distribution of the private signals and the process $\{\theta_t\}$ implied by (2), a firm z that updated its information set in period $t-j$ makes use of the variables $x_{t-j}(z) = \theta_{t-j} + \xi_{t-j}(z)$ and $\theta_{t-j-1} = \theta_{t-j} - \epsilon_{t-j}$, to form the following belief about the current state θ_{t-j}:

$$\theta_{t-j} \mid I_{t-j}(z) \sim N\left((1-\delta)\, x_{t-j}(z) + \delta\theta_{t-j-1}, (\alpha+\beta)^{-1}\right), \tag{10}$$

where

$$\delta \equiv \frac{\alpha}{\alpha+\beta} \in (0,1). \tag{11}$$

Hence, a firm that updated its information set in $t-j$ expects the current state to be a convex combination of the private signal $x_{t-j}(z)$ and a (semi) public signal θ_{t-j-1} – the only relevant piece of information that comes from learning all previous states $\{\theta_{t-j-k}\}_{k\geq 1}$.[4] The relative weights given to $x_{t-j}(z)$ and θ_{t-j-1} when the firm computes the expected value of state θ_{t-j} depend on the precision of such signals.

Using (2), one has that, for $m \leq j$,

$$\theta_{t-m} = \theta_{t-j} + \textstyle\sum_{k=0}^{j-m-1} \epsilon_{t-m-k}. \tag{12}$$

Thus, the expectation of a firm z that last updated its information set at $t-j$ about θ is

$$E\left[\theta_{t-m} \mid I_{t-j}(z)\right] = \begin{cases} E\left[\theta_{t-j} \mid I_{t-j}(z)\right] = (1-\delta)\, x_{t-j}(z) + \delta\theta_{t-j-1} & : m \leq j \\ \theta_{t-m} & : m > j \end{cases}. \tag{13}$$

In words, a firm that last updated its information set in period $t-j$ expects that all future values of the fundamental θ will be the same as the expected value of the fundamental at the period $t-j$. Moreover, since at the moment it adjusts its information set the firm observes all previous states, the firm will know for sure the value of θ_{t-m} for $m > j$.

3.1.2 Beliefs

We establish that there is a unique linear equilibrium in the game by computing the aggregate price level in period t as an weighed average of all (average) higher order beliefs about the state θ_t, as stated in (9).

First Order Beliefs: Using (13), we are able to compute (8) for the case $k=1$.

$$\bar{E}^1\left[\theta_t\right] = \lambda \textstyle\sum_{j=0}^{\infty} (1-\lambda)^j \left[(1-\delta)\,\theta_{t-j} + \delta\theta_{t-j-1}\right]. \tag{14}$$

Higher Order Beliefs: In the Appendix, we use (14) and the recursion (8) to derive the following useful result:

Lemma 1 *The average k-th order forecast of the state is given by*

$$\bar{E}^k\left[\theta_t\right] = \lambda \textstyle\sum_{m=0}^{\infty} (1-\lambda)^m \left[\kappa_{m,k}\theta_{t-m} + \delta_{m,k}\theta_{t-m-1}\right], \tag{15}$$

[4] θ_{t-j-1} is the only piece of information in $\Theta_{t-j} = \{\theta_{t-j-k}\}_{k=1}^{\infty}$ the firm needs to use because the state's process is Markovian.

with the weights $(\kappa_{m,k}, \delta_{m,k})$ are recursive defined for $k \geq 1$

$$\begin{bmatrix} \kappa_{m,k+1} \\ \delta_{m,k+1} \end{bmatrix} = [1-(1-\lambda)^m]^k \begin{bmatrix} (1-\delta) \\ \delta \end{bmatrix} + A_m \begin{bmatrix} \kappa_{m,k} \\ \delta_{m,k} \end{bmatrix},$$

where the matrix A_m is given by

$$A_m \equiv \begin{bmatrix} \left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta\left[1-(1-\lambda)^m\right]\right] & 0 \\ \delta\left[\left[1-(1-\lambda)^{m+1}\right] - \left[1-(1-\lambda)^m\right]\right] & \left[1-(1-\lambda)^{m+1}\right] \end{bmatrix},$$

and the initial weights are $(\kappa_{m,1}, \delta_{m,1}) \equiv (1-\delta, \delta)$.

3.1.3 Equilibrium Price Level and SDI Phillips curve

We obtain the the equilibrium aggregate price level by plugging (15) into the expression for P_t, (9) and the SDI Phillips curve by taking the first difference.

Proposition 1 *In an economy in which information is sticky and dispersed, and the state follows (2), there is a unique equilibrium in the pricing game played by the firms. In such equilibrium, the aggregate price level is given by*

$$P_t = \sum_{m=0}^{\infty} K_m \left[(1-\Delta_m)\,\theta_{t-m} + \Delta_m \theta_{t-m-1}\right], \tag{16}$$

and the SDI Phillips curve is given by

$$\pi_t = \sum_{m=0}^{\infty} K_m \left[(1-\Delta_m)\left(\theta_{t-m} - \theta_{t-m-1}\right) + \Delta_m \left(\theta_{t-m-1} - \theta_{t-m-2}\right)\right], \tag{17}$$

where

$$K_m \equiv \frac{(1-r)\,\lambda\,(1-\lambda)^m}{\left(1-r\left[1-(1-\lambda)^m\right]\right)\left(1-r\left[1-(1-\lambda)^{m+1}\right]\right)}, \tag{18}$$

$$\Delta_m \equiv \frac{\delta\left[1-r\left[1-(1-\lambda)^m\right]\right]}{1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta\left[1-(1-\lambda)^m\right]\right]}. \tag{19}$$

Note that the current aggregate price level P_t depends on current and past states of the economy. This is so for two reasons. First, there are firms in the economy for which the information set has been last updated in the far past. This is a *direct* effect of sticky information, captured by the term K_m. Second, even firms that have just adjusted their information set will be, at least partly, backward-looking. This happens because of an *strategic* effect: their optimal relative price depends on how they believe all other firms (including those that have outdated information sets) in the economy are setting prices. This strategic effects is captured by the terms Δ_m. While they depend on δ, they reflect a non trivial interaction between dispersion and stickiness due to strategic complementarity in pricing decisions. As the weight firms attach to other firms' behavior vanishes, $r \to 0$, the effects of dispersion and stickness are completely disentangled in the resulting weights $\Delta_m \to \delta$ and $K_m \to \lambda (1-\lambda)^m$.

From aggregate prices, It is immediate to show that inflation also depends on the current and all past states of the economy. This result is linked to the one obtained in **?**, in which inflation depends on past expectations of current economic conditions, due to the fact that in our model, as shown in (9), individual expectations about the current state *are* functions of the past states os the economy.

In our model, however, on top of being sticky, information is also *dispersed*. The effect of dispersion is captured by the positive weight given to the state in period θ_{t-m-1} by a firm that has its information set updated in $t-m$. If, instead of having a private signal of θ_{t-m}, firms knew the state, they would ignore the information given by θ_{t-m-1}. But, as the private signal the firm observes is noisy, it is always optimal to place some weight on past states to forecast the current state. Hence, in comparison to an economy à la **?**, the adjustment of prices to shocks will be slower in an economy with disperse information. This result, in line with **?**, shows that the introduction of differential information in an otherwise standard sticky information model tends to magnify the effect of stickiness on the persistence of aggregate prices and inflation rates. In the next section, we aim to quantify the potential contribution of dispesion for persistence.

As a side remark, we point out that the introduction of dispersion in an sticky information model leads to price and inflation inertia irrespective of assumptions regarding the firms' capacity to predict equilibrium outcomes. Indeed, although they may not have their information sets up to date, the firms in our model correctly predict the equilibrium behavior of their opponents. In spite of correctly predicting the *strategies* (i.e., contingent plans) adopted by the opponents in equilibrium, a firm cannot infer what is the actual price set by them (i.e., the action taken), since it cannot observe its opponents' private signals. Hence, a firm that hasn't updated its information set cannot infer the current state from the behavior of its opponents .This is in contrast to **?** who, in order to obtain price and information inertia in a model with sticky but non-dispersed information, (implicitly) assume that agents cannot condition on equilibrium behavior from the opponents. In fact, in their main experiment, there is a (single) nominal shock that only a fraction of the firms observe. Trivially, the prices set by those firms (as well as aggregate prices) will reflect such change in the fundamental. Hence, a firm that hasn't observed the shock but can predict the equilibrium behavior of the opponents will be able to infer the fundamental from such behavior.[5] It follows that *all* firms will adjust prices in response.

[5] The argument here is similar to the one in Rational Expectations Equilibrium models à la Grosmann (1981).

4 The Relative of Importance of Stickiness and Dispersion of Information for Persistence

This section evaluates the importance of dispersion in a setting in which information is sticky. We first define a measure of persistence in our setting. We then (i) analytically derive the effects of the parameters that capture stickiness and dispersion in the model on our measure of dispersion and (ii) perform some numerical simulations to evaluate the quantitative importance of each of these factors.

4.1 The Effects of Dispersion and Stickiness on Persistence:

Note that we can rewrite (17) as

$$\pi_t = \textstyle\sum_{m=0}^{\infty} c_m \pi_{C,t-m} \tag{20}$$

where the coefficients c_m, $m \geq 0$, are given by

$$c_m \equiv \begin{cases} \left(\frac{1-r}{r}\right)\left[\frac{1}{1-r(1-\rho)} - 1\right] & \text{if } m = 0 \\ \left(\frac{1-r}{r}\right)\left[\frac{1}{1-r[1-\rho(1-\lambda)^m]} - \frac{1}{1-r\left[1-\rho(1-\lambda)^{m-1}\right]}\right] & \text{if } m \geq 1, \end{cases} \tag{21}$$

and $\rho = 1 - \lambda(1-\delta)$.

Therefore, using equations (2), that describes the evolution of θ, and (17), one can write the SDI inflation as

$$\pi_t = \textstyle\sum_{k=0}^{\infty} c_k \varepsilon_{t-k},$$

The *direct* impact of a period t shock, ε_t, on the inflation rate in period $t+m$, π_{t+m}, is c_m. Hence, the *cumulative* impact of a shock ε_t on period π_{t+m} is:

$$\begin{aligned} C_m &= \textstyle\sum_{k=0}^{m} c_k \\ &= \left(\frac{1-r}{r}\right)\left[\frac{1}{1-r+r\rho(1-\lambda)^m} - 1\right]. \end{aligned}$$

Since

$$C_\infty = \lim_{m\to\infty} \textstyle\sum_{k=0}^{m} c_k = 1,$$

$1 - C_m$ can be interpreted as the impact of a shock ε_t that still remains to be propagated after m periods. Hence, we take $1 - C_m$ as the measure of persistence.of a shock in our setting. We now evaluate how dispersion and stickiness affect persistence.

Not surprisingly, when information becomes less sticky, persistence falls:

$$\begin{aligned}\frac{\partial(1-C_m)}{\partial\lambda} &= \left[\frac{1-r}{[1-r+r\rho(1-\lambda)^m]^2}\right]\frac{\partial\rho(1-\lambda)^m}{\partial\lambda} \\ &= -\left(\frac{1-r}{r}\right)\left[\frac{1}{[1-r+r\rho(1-\lambda)^m]^2}\right]\left[(1-\lambda)^m(1-\delta)+\rho m(1-\lambda)^{m-1}\right] < 0.\end{aligned}$$

It is natural to take δ, the weight placed by an agent on the previous values of the state vis à vis his private information, as a meaure of information dispersion.[6] The higher δ, the more dispersed is the information. Notice that,

$$\frac{\partial(1-C_m)}{\partial\delta} = -\frac{\partial C_m}{\partial\delta} = \frac{(1-r)\lambda(1-\lambda)^m}{[1-r+r\rho(1-\lambda)^m]^2} > 0.$$

Hence, as expected, whenever $\lambda \in (0,1)$, dispersion increases the persistence of a shock.

We are also interested in understanding how the effect of stickiness on persistence varies with the degree of dispersion as captured by δ. Notice that

$$\frac{\partial^2(1-C_m)}{\partial\lambda\partial\delta} = \left(\frac{1-r}{r}\right)\left[\frac{(1-\lambda)^m}{[1-r+r\rho(1-\lambda)^m]^2}\right] > 0$$

so that the effect of sticky information on persistence is higher in a world of differential information.

We summarize this discussion in the following result.

Proposition 2 *For a given degree of dispersion (stickiness), the higher the degree of stickiness (dispersion), the more persistent in the effect of a shock on inflation rates. Moreover, the effect of stickiness on persistence increases with the amount of dispersion in the economy.*

4.2 Numerical Simulations

Proposition 2 shows that dispersion magnifies the effects of stickiness. Through numerical simulations, we now aim to evaluate whether this effect is quantitatively relevant. In particular, we want to verify wether a small incidence of price stickiness can lead to a large amount of persistence in aggregate prices in a world of differential information.

A first hint of an answer comes from the fact that both $\frac{\partial(1-C_m)}{\partial\delta}$ and $\frac{\partial^2(1-C_m)}{\partial\lambda\partial\delta}$ are very close to zero when λ, the fraction of agent getting new information, is close to one. Our simulations confirm such impression.

Figure 1 deals with the case in which, starting from $\lambda = 0.95$, one raises the amount of dispersion by moving from $\delta = 0.01$ to $\delta = 0.09$. The effect on persistence is minimal. While the shock fully

[6] The weight δ depends on the relative precision of an agent's private information, β.

vanishes after 3 periods when $\delta = 0.01$, it takes only an extra period to vanish when $\delta = 0.09$. It seems that, when the degree of price stickiness is low, the main effect of dispersion (in *any* degree) is to substantially magnify the immediate impact of a given shock. Indeed, when compared to a setting in which $\lambda = 0.05$, the price increase after a shock when $\lambda = 0.95$ is larger by a factor of 40 (see figure 2).

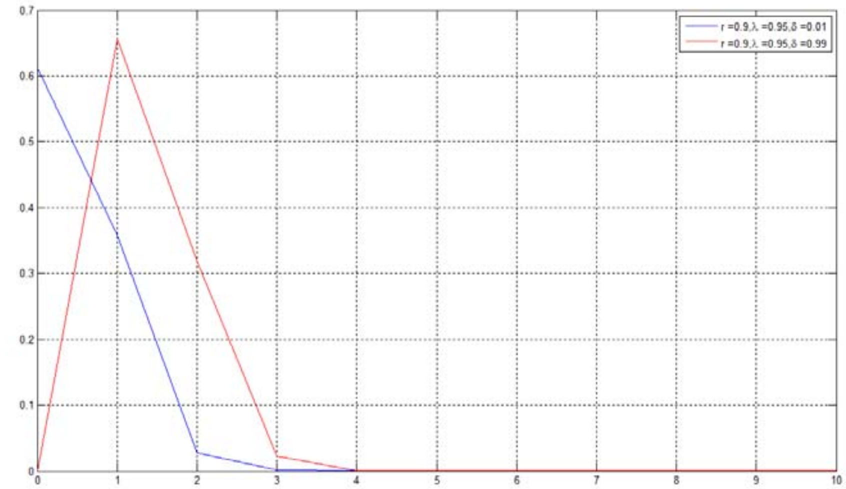
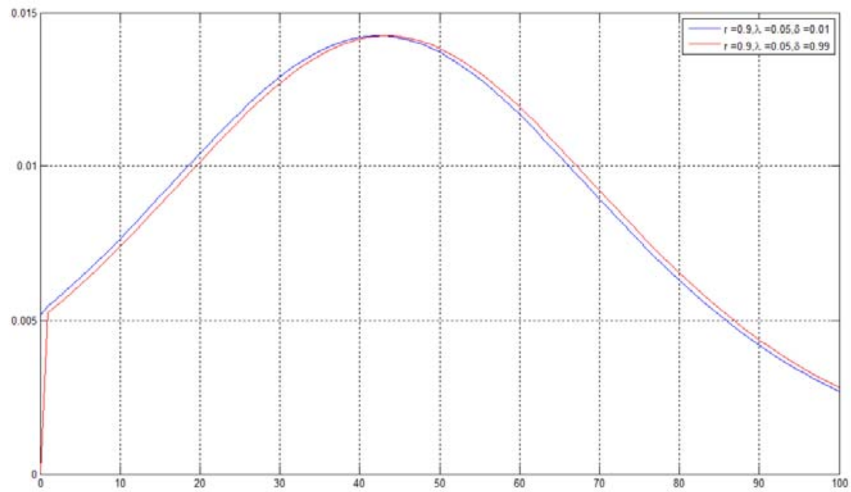


Figure 2 analyzes the effect of dispersion when $\lambda = 0.05$. It is worth noticing that the effect of a substantial increase in the amount of disperion (moving from $\delta = 0.01$ to $\delta = 0.09$) is minor both in terms of the size of the impact of a shock as well as in terms of its persistence.



This is true throughout all simulations we have performed: whenever the degree of price stickiness is large, the quantitative contribution of dispersion tends to be small.

5 Benchmarks for the SDI Phillips Curve

Finally, we show that our model nests the dispersed information model ($\lambda = 1$) and the sticky information model ($\beta^{-1} \to 0$) as special cases. In order to understand the properties of the SDI Phillips curve, in what follows, we compare it to those two benchmarks as well as to the Phillips curve implied by the complete information case.

5.1 Benchmark 1: Complete-information Inflation

If the state θ_t is common knowledge, the price of any firm z is

$$p_t(z) = p_t^* \equiv rP_t + (1-r)\,\theta_t.$$

Since firms are identical, they all set the same price. As a result

$$P_t = rP_t + (1-r)\,\theta_t \Rightarrow P_t = \theta_t.$$

Hence, under complete information, the equilibrium entails an inflation rate $\pi_{C,t}$ – that we call the *complete-information inflation* – that is equal to the change of states:

$$\pi_{C,t} = \theta_t - \theta_{t-1}. \tag{22}$$

5.2 Benchmark 2: Dispersed-information Inflation

If stickiness vanishes ($\lambda = 1$), our results converge to the ones obtained by **?** and **?**. Denoting the inflation rate for the economy without stickiness by $\pi_{D,t}$ (the *dispersed-information inflation*), we have:

$$\pi_{D,t} = (1-\Delta)\,\pi_{C,t} + \Delta\pi_{C,t-1}, \tag{23}$$

so that the inflation rate in period t is a convex combination of complete-information inflations of period t and $t-1$, with the weight on period $t-1$ complete-information inflation given by

$$\Delta = c_1 \equiv \frac{\delta}{1-r\,(1-\delta)}. \tag{24}$$

Alternatively, as in **?**, we can say that inflation in t is a convex combination of the "state/fundamental", $\pi_{C,t}$, and the "public signal", $\pi_{C,t-1}$.

When compared to the full information case, the inflation rate that prevails with dispersed information displays more inertia. Moreover, note that

$$E\left[\pi_{D,t} \mid I_t(z)\right] = (1-\Delta)\, E\left[\pi_{C,t} \mid I_t(z)\right] + \Delta \pi_{C,t-1}.$$

Hence, when information is dispersed, the forecast error

$$\pi_{D,t} - E\left[\pi_{D,t} \mid I_t(z)\right] = (1-\Delta)\left[\pi_{C,t} - E\left[\pi_{C,t} \mid I_t(z)\right]\right]$$

is proportional to the forecast error of the complete-information inflation $\pi_{C,t}$.

5.3 Benchmark 3: Sticky-information Inflation

The other polar case occurs when information is sticky but not dispersed ($\delta = 0$). In such case, the Phillips curve we obtain resembles the one in **?**. Denoting the *sticky-information inflation* by $\pi_{S,t}$, we have

$$\pi_{S,t} = \textstyle\sum_{m=0}^{\infty} K_m \pi_{C,t-m}. \tag{25}$$

where inflation is also a function of current and past complete-information inflation, but with the weights K_m in (18) replacing the coefficients c_m defined in (21). Note that, for $m = 0$

$$c_0 \equiv \frac{(1-r)\,\lambda\,(1-\delta)}{1-r\lambda\,(1-\delta)} < \frac{(1-r)\,\lambda}{1-r\lambda} \equiv K_0$$

because

$$\frac{\partial c_0}{\partial \delta} \equiv \frac{-(1-r)\,\lambda}{\left[1-r\lambda\,(1-\delta)\right]^2} < 0.$$

Considering that $\sum_{m=0}^{\infty} K_m = \sum_{m=0}^{\infty} c_m = 1$, we obtain that dispersion redistribute the weights from current to past complete-information inflations.

5.4 Benchmark contribution to SDI inflation

We can rewrite our SDI Phillips curve as a combination of the inflation rates that prevail under the three benchmarks cases discussed above. First, note that the SDI inflation π is a function of complete-information inflations π_C of current and previous periods. Indeed, using (20) or (16), we obtain

$$\begin{aligned} \pi_t &= \textstyle\sum_{j=0}^{\infty} c_j \pi_{C,t-j} \\ &= \textstyle\sum_{m=0}^{\infty} K_m \left[(1-\Delta_m)\,\pi_{C,t-m} + \Delta_m \pi_{C,t-m-1}\right]. \end{aligned} \tag{26}$$

Using (17) and (25), we can also relate the SDI inflation to the sticky-information inflation π_S as follows:

$$\pi_t = \pi_{S,t} - \sum_{m=0}^{\infty} K_m \Delta_m \left(\pi_{C,t-m} - \pi_{C,t-m-1}\right).$$

Finally, if we combine this last equation with (23), we obtain a decomposition of SDI inflation that includes all the mentioned benchmarks

$$\pi_t = \pi_{S,t} + \sum_{m=0}^{\infty} K_m \left(\frac{\Delta_m}{\Delta}\right) \left[\pi_{D,t-m} - \pi_{C,t-m}\right]. \tag{27}$$

Thus, compared to the case in which information is sticky, inflation under sticky *and* dispersed information will be higher if, and only if, dispersed-information inflation, $\pi_{D,t-m}$, is on "average" higher than the complete-information inflation $\pi_{C,t-m}$.

6 Conclusion

In this paper, we have considered the impact of sticky and dispersed information on individual price setting decisions, and the resulting effect on the aggregate price level and the inflation rate. We also evaluated **?** conjecture that, in a world of dispersed information, a small incidence of information stickiness can lead to large amounts of persistence in aggregate prices. Contrary to their conjecture, we show that, with a small incidence of price stickiness, while dispersion substantially magnifies the immediate impact of a shock, its effect on persistence tends to be minor.

The model we put forth nests the dispersed information model and the sticky information model as special cases, and can be extended in many interesting directions. One could, for instance, consider the case in which agents receive public information from, say, a central banker about the state of the economy. In such a setting, it would be natural to evaluate what is the best disclosure policy for a benevolent central banker. One could also use our model to analyze how communication interacts with other policy instruments (e.g., interest rates) available to a central banker. We believe these extensions/applications of the model we have developed are interesting avenues for future research.

7 Appendix

7.1 Posterior Distribution

At this appendix, we calculate the distribution of the fundamental θ_{t-j} given that the firm updated its information set at period $t-j$. We can compute $f\left(\theta_{t-j} \mid \Theta_{t-j-1}, x_{t-j}\right)$ as

$$\begin{aligned}
f\left(\theta_{t-j} \mid \theta_{t-j-1}, x_{t-j}\right) &= \frac{f\left(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}\right)}{\int_{-\infty}^{\infty} f\left(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}\right) d\theta_{t-j}} \\
&= \frac{f\left(\theta_{t-j-1}, x_{t-j} \mid \theta_{t-j}\right) f\left(\theta_{t-j}\right)}{\int_{-\infty}^{\infty} f\left(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}\right) d\theta_{t-j}} \\
&= \frac{f\left(\theta_{t-j-1} \mid \theta_{t-j}\right) f\left(x_{t-j} \mid \theta_{t-j}\right) f\left(\theta_{t-j}\right)}{\int_{-\infty}^{\infty} f\left(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}\right) d\theta_{t-j}}
\end{aligned}$$

where the last equality holds due to the independence of $\xi_t(z)$ and ϵ_{t-j}. As

$$\begin{aligned}
x_{t-j}(z) &= \theta_{t-j} + \xi_{t-j}(z) \\
\theta_{t-j-1} &= \theta_{t-j} - \epsilon_{t-j}.
\end{aligned}$$

where $\xi_t(z) \sim N\left(0, \beta^{-1}\right)$ and $\epsilon_{t-j} \sim N\left(0, \alpha^{-1}\right)$, we have that $f\left(x_{t-j} \mid \theta_{t-j}\right) = N\left(\theta_{t-j}, \beta^{-1}\right)$ and $f\left(\theta_{t-j-1} \mid \theta_{t-j}\right) = N\left(\theta_{t-j}, \alpha^{-1}\right)$. If the dynamics of θ_t was

$$\theta_{t-j-1} = \rho\theta_{t-j} - \epsilon_{t-j}.$$

we would have

$$\begin{aligned}
E\left[\theta_{t-j}\right] &= E\left[\theta_t\right] = \frac{E\left[\epsilon_t\right]}{1-\rho} = 0 \\
Var\left[\theta_{t-j}\right] &= Var\left[\theta_t\right] = \frac{Var\left[\epsilon_t\right]}{1-\rho^2} = \frac{\alpha^{-1}}{1-\rho^2}.
\end{aligned}$$

Therefore, the distribution of θ_{t-j} would be given by $f\left(\theta_{t-j}\right) = N\left(0, \Psi^{-1}\right)$ where $\Psi =.\alpha\left(1-\rho^2\right)$. Thus, we would obtain

$$\begin{aligned}
f\left(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}\right) &= c\exp\left\{-\frac{1}{2}\left[\frac{\left(x_{t-j}(z) - \theta_{t-j}\right)^2}{\beta^{-1}} + \frac{\left(\theta_{t-j-1} - \rho^{-1}\theta_{t-j}\right)^2}{\left(\rho^2\alpha\right)^{-1}} + \frac{\theta_{t-j}^2}{\Psi^{-1}}\right]\right\} \\
&= c\exp\left\{-\frac{1}{2}\left[\left(\beta + \alpha + \Psi\right)\theta_{t-j}^2 - 2\left(\beta x_{t-j}(z) + \alpha\rho\theta_{t-j-1}\right)\theta_{t-j}\right]\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left[\beta x_{t-j}^2(z) + \alpha\rho^2\theta_{t-j-1}^2\right]\right\} \\
&= cd\frac{1}{\sqrt{2\pi}\sigma\Sigma}\exp\left\{-\frac{1}{2}\frac{\left(\theta_{t-j} - \mu\right)^2}{\Sigma^2}\right\}
\end{aligned}$$

where

$$
\begin{array}{ll}
c = (2\pi)^{-3/2} (\beta\alpha\Psi)^{1/2} & d = \sqrt{2\pi}\sigma \exp\left\{-\frac{1}{2}\left[-\mu^2\Sigma^{-2} + \beta x_{t-j}^2(z) + \alpha\rho^2\theta_{t-j-1}^2\right]\right\} \\
\mu = [\Delta x_{t-j}(z) + (1-\Delta) z_{t-j-1}] & \Delta = \beta(\beta+\alpha+\Psi)^{-1} \\
z_{t-j-1} = \Lambda\rho\theta_{t-j-1} & \Lambda = \alpha(\beta+\alpha)^{-1} \\
\Sigma^2 = (\beta+\alpha+\Psi)^{-1} &
\end{array}
$$

As $\rho \to 1$, we have $\Psi \to 0$, $\Delta \to \delta$, and $\Sigma^2 \to (\beta+\alpha)^{-1}$. Thus $f(\theta_{t-j} \mid \theta_{t-j-1}, x_{t-j}) = N(\mu, \sigma^2)$ where $\mu = [\delta x_{t-j}(z) + (1-\delta)\theta_{t-j-1}]$, and $\sigma^2 = (\beta+\alpha)^{-1}$.

7.2 Higher Order Beliefs

In this appendix we derive the general formula of the k-th order average expectation

$$\bar{E}^k[\theta_t] = \lambda \textstyle\sum_{m=0}^{\infty} (1-\lambda)^m [\kappa_{m,k}\theta_{t-m} + \delta_{m,k}\theta_{t-m-1}]$$

with the weights $(\kappa_{m,k}, \delta_{m,k})$ are recursive defined for $k \geq 1$

$$\begin{bmatrix}\kappa_{m,k+1} \\ \delta_{m,k+1}\end{bmatrix} = \begin{bmatrix}(1-\delta) \\ \delta\end{bmatrix} [1-(1-\lambda)^m]^k + A_m \begin{bmatrix}\kappa_{m,k} \\ \delta_{m,k}\end{bmatrix},$$

where the matrix A_m is given by

$$A_m \equiv \begin{bmatrix} \left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta[1-(1-\lambda)^m]\right] & 0 \\ \delta\left[\left[1-(1-\lambda)^{m+1}\right] - [1-(1-\lambda)^m]\right] & \left[1-(1-\lambda)^{m+1}\right] \end{bmatrix},$$

and the initial weights are $(\kappa_{1,k}, \delta_{1,k}) \equiv (1-\delta, \delta)$.

We start by computing $\bar{E}^1[\theta_t]$ as

$$
\begin{aligned}
\bar{E}^1[\theta_t] &= \textstyle\sum_{j=0}^{\infty} \int_{\Lambda_j} E\left[\bar{E}^0[\theta_t] \mid I_{t-j}(z)\right] dz \\
&= \textstyle\sum_{j=0}^{\infty} \int_{\Lambda_j} E[\theta_t \mid I_{t-j}(z)] dz \\
&= \textstyle\sum_{j=0}^{\infty} \int_{\Lambda_j} [(1-\delta) x_{t-j}(z) + \delta\theta_{t-j-1}] dz \\
&= \lambda \textstyle\sum_{j=0}^{\infty} (1-\lambda)^j [(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}].
\end{aligned}
$$

We can use this result to obtain $\bar{E}^2[\theta_t]$ as

$$
\begin{aligned}
\bar{E}^2[\theta_t] &= \textstyle\sum_{m=0}^{\infty} \int_{\Lambda_m} E\left[\bar{E}^1[\theta_t] \mid I_{t-m}(z)\right] dz \\
&= \lambda \textstyle\sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{\infty} (1-\lambda)^j E[(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1} \mid I_{t-m}(z)] dz.
\end{aligned}
$$

We know that

$$E\left[\theta_{t-j} \mid I_{t-m}(z)\right] = \begin{cases} (1-\delta)\, x_{t-m}(z) + \delta\theta_{t-m-1} & : m \geq j \\ \theta_{t-j} & : m < j \end{cases}.$$

Thereafter

$$\begin{aligned}
\bar{E}^2[\theta_t] &= \lambda \textstyle\sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j \left\{(1-\delta)\, E\left[\theta_{t-j} \mid I_{t-m}(z)\right] + \delta E\left[\theta_{t-j-1} \mid I_{t-m}(z)\right]\right\} dz \\
&\quad + \lambda \textstyle\sum_{m=0}^{\infty} \int_{\Lambda_m} (1-\lambda)^m \left\{(1-\delta)\, E\left[\theta_{t-m} \mid I_{t-m}(z)\right] + \delta\theta_{t-m-1}\right\} dz \\
&\quad + \lambda \textstyle\sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=m+1}^{\infty} (1-\lambda)^j \left[(1-\delta)\, \theta_{t-j} + \delta\theta_{t-j-1}\right] dz \\
&= \lambda \textstyle\sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j \left[(1-\delta)\, x_{t-m}(z) + \delta\theta_{t-m-1}\right] dz \\
&\quad + \lambda \textstyle\sum_{m=0}^{\infty} \int_{\Lambda_m} (1-\lambda)^m \left[(1-\delta)\left[(1-\delta)\, x_{t-m}(z) + \delta\theta_{t-m-1}\right] + \delta\theta_{t-m-1}\right] dz \\
&\quad + \lambda \textstyle\sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=m+1}^{\infty} (1-\lambda)^j \left[(1-\delta)\, \theta_{t-j} + \delta\theta_{t-j-1}\right] dz \\
&= \lambda^2 \textstyle\sum_{m=0}^{\infty} (1-\lambda)^m \left[(1-\delta)\, \theta_{t-m} + \delta\theta_{t-m-1}\right] \sum_{j=0}^{m-1} (1-\lambda)^j \\
&\quad + \lambda^2 \textstyle\sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[(1-\delta)^2 \theta_{t-m} + \left[1-(1-\delta)^2\right] \theta_{t-m-1}\right] \\
&\quad + \lambda^2 \textstyle\sum_{j=1}^{\infty} (1-\lambda)^j \left[(1-\delta)\, \theta_{t-j} + \delta\theta_{t-j-1}\right] \sum_{m=0}^{j-1} (1-\lambda)^m \\
&= \lambda \textstyle\sum_{m=0}^{\infty} (1-\lambda)^m \left[(1-\delta)\, \theta_{t-m} + \delta\theta_{t-m-1}\right] \left[1-(1-\lambda)^m\right] \\
&\quad + \lambda^2 \textstyle\sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[(1-\delta)^2 \theta_{t-m} + \left[1-(1-\delta)^2\right] \theta_{t-m-1}\right] \\
&\quad + \lambda \textstyle\sum_{j=1}^{\infty} (1-\lambda)^j \left[(1-\delta)\, \theta_{t-j} + \delta\theta_{t-j-1}\right] \left[1-(1-\lambda)^j\right] \\
&= \lambda \textstyle\sum_{m=0}^{\infty} (1-\lambda)^m\, 2\left[1-(1-\lambda)^m\right] \left[(1-\delta)\, \theta_{t-m} + \delta\theta_{t-m-1}\right] \\
&\quad + \lambda^2 \textstyle\sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[(1-\delta)^2 \theta_{t-m} + \left[1-(1-\delta)^2\right] \theta_{t-m-1}\right].
\end{aligned}$$

We can write this expression as

$$\bar{E}^2[\theta_t] = \lambda \textstyle\sum_{j=0}^{\infty} (1-\lambda)^j \left[\kappa_{j,2}\theta_{t-j} + \delta_{j,2}\theta_{t-j-1}\right]$$

where

$$\begin{aligned}
\kappa_{j,2} &= \left(1-\delta^2\right)\left[1-(1-\lambda)^j\right] + (1-\delta)^2\left[1-(1-\lambda)^{j+1}\right] \\
&= \left[1-(1-\lambda)^{j+1}\right]\kappa_{j,1}^2 + \left[1-(1-\lambda)^j\right]\left(1-\delta_{j,1}^2\right), \\
\delta_{j,2} &= \delta^2\left[1-(1-\lambda)^j\right] + \left[1-(1-\delta)^2\right]\left[1-(1-\lambda)^{j+1}\right] \\
&= \left[1-(1-\lambda)^{j+1}\right]\left(1-\kappa_{j,1}^2\right) + \left[1-(1-\lambda)^j\right]\delta_{j,1}^2.
\end{aligned}$$

Note that

$$\kappa_{j,2} + \delta_{j,2} = \textstyle\sum_{n=0}^{1} \left[1-(1-\lambda)^j\right]^n \left[1-(1-\lambda)^{j+1}\right]^{1-n}.$$

We use induction to obtain the general case. Suppose that (15) holds for $k-1$. Then

$$\bar{E}^{k-1}\left[\theta_t\right] = \lambda \sum_{m=0}^{\infty} (1-\lambda)^m \left[\kappa_{m,k-1}\theta_{t-m} + \delta_{m,k-1}\theta_{t-m-1}\right],$$

where

$$\sum_{j=0}^{m-1} (1-\lambda)^j \left(\kappa_{j,k-1} + \delta_{j,k-1}\right) = \frac{1}{\lambda}\left[1-(1-\lambda)^m\right]^{k-1}.$$

As a result

$$\begin{aligned}
\bar{E}^k\left[\theta_t\right] &= \sum_{m=0}^{\infty} \int_{\Lambda_m} E\left[\bar{E}^{k-1}\left[\theta_t\right] \mid I_{t-m}(z)\right] dz \\
&= \sum_{m=0}^{\infty} \int_{\Lambda_m} E\left[\lambda \sum_{j=0}^{\infty} (1-\lambda)^j \left[\kappa_{j,k-1}\theta_{t-j} + \delta_{j,k-1}\theta_{t-j-1}\right] \mid I_{t-m}(z)\right] dz \\
&= \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j \left\{\kappa_{j,k-1} E\left[\theta_{t-j} \mid I_{t-m}(z)\right] + \delta_{j,k-1} E\left[\theta_{t-j-1} \mid I_{t-m}(z)\right]\right\} dz \\
&\quad + \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} (1-\lambda)^m \left\{\kappa_{m,k-1} E\left[\theta_{t-m} \mid I_{t-m}(z)\right] + \delta_{m,k-1}\theta_{t-m-1}\right\} dz \\
&\quad + \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=m+1}^{\infty} (1-\lambda)^j \left[\kappa_{j,k-1}\theta_{t-j} + \delta_{j,k-1}\theta_{t-j-1}\right] dz \\
&= \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j \left(\kappa_{j,k-1} + \delta_{j,k-1}\right) \left[(1-\delta) x_{t-m}(z) + \delta\theta_{t-m-1}\right] dz \\
&\quad + \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} (1-\lambda)^m \left[\kappa_{m,k-1}\left[(1-\delta) x_{t-m}(z) + \delta\theta_{t-m-1}\right] + \delta_{m,k-1}\theta_{t-m-1}\right] dz \\
&\quad + \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=m+1}^{\infty} (1-\lambda)^j \left[\kappa_{j,k-1}\theta_{t-j} + \delta_{j,k-1}\theta_{t-j-1}\right] dz \\
&= \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^m \left[(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}\right] \sum_{j=0}^{m-1} (1-\lambda)^j \left(\kappa_{j,k-1} + \delta_{j,k-1}\right) \\
&\quad + \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[\kappa_{m,k-1}(1-\delta)\theta_{t-m} + \left[\kappa_{m,k-1}\delta + \delta_{m,k-1}\right]\theta_{t-m-1}\right] \\
&\quad + \lambda^2 \sum_{j=1}^{\infty} (1-\lambda)^j \left[\kappa_{j,k-1}\theta_{t-j} + \delta_{j,k-1}\theta_{t-j-1}\right] \sum_{m=0}^{j-1} (1-\lambda)^m \\
&= \lambda \sum_{m=0}^{\infty} (1-\lambda)^m \left[1-(1-\lambda)^m\right]^{k-1} \left[(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}\right] \\
&\quad + \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[\kappa_{m,k-1}(1-\delta)\theta_{t-m} + \left[\kappa_{m,k-1}\delta + \delta_{m,k-1}\right]\theta_{t-m-1}\right] \\
&\quad + \lambda \sum_{m=0}^{\infty} (1-\lambda)^m \left[1-(1-\lambda)^m\right] \left[\kappa_{m,k-1}\theta_{t-m} + \delta_{m,k-1}\theta_{t-m-1}\right].
\end{aligned}$$

We can rewrite the last three as

$$\bar{E}^k\left[\theta_t\right] = \lambda \sum_{m=0}^{\infty} (1-\lambda)^m \left[\kappa_{m,k}\theta_{t-m} + \delta_{m,k}\theta_{t-m-1}\right],$$

where

$$\begin{aligned}
\kappa_{m,k} &\equiv (1-\delta)\left[1-(1-\lambda)^m\right]^{k-1} + \left[(1-\delta)\lambda(1-\lambda)^m + \left[1-(1-\lambda)^m\right]\right]\kappa_{m,k-1} \\
&= (1-\delta)\left[1-(1-\lambda)^m\right]^{k-1} \\
&\quad + \left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta\left[1-(1-\lambda)^m\right]\right]\kappa_{m,k-1} \\
\delta_{m,k} &\equiv \delta\left[1-(1-\lambda)^m\right]^{k-1} + \delta\lambda(1-\lambda)^m\kappa_{m,k-1} + \left[\lambda(1-\lambda)^m + \left[1-(1-\lambda)^m\right]\right]\delta_{m,k-1} \\
&= \delta\left[1-(1-\lambda)^m\right]^{k-1} \\
&\quad + \delta\left[\left[1-(1-\lambda)^{m+1}\right] - \left[1-(1-\lambda)^m\right]\right]\kappa_{m,k-1} + \left[1-(1-\lambda)^{m+1}\right]\delta_{m,k-1}
\end{aligned}$$

since

$$\lambda(1-\lambda)^m = \left[1-(1-\lambda)^{m+1}\right] - \left[1-(1-\lambda)^m\right].$$

Rewriting these weights in matrix format, we obtain

$$\begin{bmatrix}\kappa_{m,k+1}\\ \delta_{m,k+1}\end{bmatrix} = \begin{bmatrix}(1-\delta)\\ \delta\end{bmatrix}\left[1-(1-\lambda)^m\right]^k + A_m\begin{bmatrix}\kappa_{m,k}\\ \delta_{m,k}\end{bmatrix},$$

where the matrix A_m is given by

$$A_m \equiv \begin{bmatrix}\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta\left[1-(1-\lambda)^m\right]\right] & 0 \\ \delta\left[\left[1-(1-\lambda)^{m+1}\right] - \left[1-(1-\lambda)^m\right]\right] & \left[1-(1-\lambda)^{m+1}\right]\end{bmatrix},$$

which is exactly our result.