Robust Mechanism Design

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Motivation:

A seller who

- wishes to sell an (indivisible) goods

- faces 1 consumer with (unobserved) willingness to pay, $v \in [0,1]$, drawn from $F(.)$

has a very simple procedure to maximize expected revenues. Indeed...

Motivation: The Simple Economics of Optimal Pricing:

- For price $p \in [0,1]$, consumer is willing to buy $1 - F(p)$ "units"
  - Total Revenues collected from are $p[1 - F(p)]$

- Compute "marginal revenues " (Bulow and Roberts, 1989)

$$\frac{\partial p[1 - F(p)]}{\partial [1 - F(p)]} = p - \frac{(1 - F(p))}{f(p)}$$

- Sell to i if, and only if, marginal revenues are positive

Motivation: The Simple Economics of Optimal Pricing, $N = 1$:

When F is regular (Myerson, 1981), optimal selling policy:

- Sell with probability 1 if $v \geq p^*$, where

$$p^* - \frac{(1 - F(p^*))}{f(p^*)} = 0$$

  and not sell to the other consumers

  – Posted price p^* is an optimal mechanism

Motivation:

... What if $F(.)$ – "demand " – is not fully known? How to compute marginal revenues?

- Designer could behave as econometrician (as in in Segal (2002))
  - Non-parametric estimation of F may be needed
    - $*$ Lots of data
    - $*$ Tedious computations (Fernandes, (2012))

Motivation:

Even if able to ontain an estimate or specify F:

- Optimal posted price p^* is too dependent on fine details of the problem (point-elasticity)
  - Insure agaisnt mispecification (or bias)?

Motivation:

Behaving as an econometrician is of no help

- with few observations of previous sales
- if the good will be just sold once (or infrequently)
- mispecification is an issue

What to do?

Motivation:

Experimentation à la Rotschild's (1974) multiarmed bandit problem?

- Can get stuck with the wrong distribution
  - Leads to poor design

Motivation:

Design for "multiple purposes" (Milgrom, 2005)

- doing well in a wide range of circumstances is of first order importance for designers
  - Executives
    - Concern with shareholders
  - Government agencies
    - Political economy implications of failures

The paper:

We assume that:

- Designer is uncertain about the distribution of a agent's private informationn (e.g., consumer's willingness to pay)

- In face of this uncertainty, designer has a maxmin objective

Related Literature:

- Robust Decion-Making/Delegation:
  - Frankel (2013), Carrasco and Moreira (2013)
- Robust Incentice Contracts:
  - Hurwicz and Shapiro (1978), Carroll (2013), Garret (2013)
- Full implementation (rule-out bad equilibria $\sim$ concern with worst case):
  - Maskin and Sjöström (2002)

First Model:

- Two agents: Seller, who can sell $K \geq 1$ indivisible goods, and one buyer

- Valuations: Seller has zero cost to produce the goods, buyer's valuation for the vector of goods is $v \in [0,1]^K$

- Seller only knows the expected value of v – set to $\mathbf{k} = (k_1, ..., k_K) > 0$

  - Maxmin objective

    * $\mathbf{k} > 0$ justified with a simple IA model (due to Carroll)

The Seller's Problem:

$$\max_{\{q(v),t(v)\}_v} \min_{\left\{F \mid \int v_j dF(v)=k_j, j=1,\ldots,k\right\}} \int t(v)\, dF(v)$$

subject to

$$\begin{aligned} v \cdot q(v) - t(v) &\geq 0 \text{ for all } v \\ v \cdot q(v) - t(v) &\geq v \cdot q\left(v'\right) - t\left(v'\right) \text{ for all } v, v' \end{aligned}$$

where

$$v \cdot q(v) = \sum_{j=1}^{K} v_j q_j(v)$$

Simplifying ICs

As usual, $\{q(v), t(v)\}_v$ is IC if, and only if,

$$\nabla U(v) = q(v) \text{ for a.e.} v \in [0,1]^K \quad \text{(Envelope)}$$

and

$$U(v) \text{ is convex (i.e., } q(v) \text{ is non-decreasing)}$$

Hence

$$t(v) = \underbrace{v \cdot q(v)}_{Total\ Surplus} - \underbrace{U(v)}_{\text{Buyer's Indirect Utility}} = v \cdot \nabla U(v) - U(v)$$

The Seller's Problem:

$$\max_{\{U(v)\text{ convex }\}} \; \min_{\left\{F \mid \int v_j dF(v) = k_j, j=1,\ldots,k\right\}} \int \underbrace{\left[v \cdot \nabla U(v) - U(v)\right]}_{l(v)} dF(v)$$

A Modified Min Problem:

Consider the problem:

$$\min_{\mu \in \mathcal{P}(k)} \int l(v)\, d\mu(v)$$

$$\mathcal{P}(k) = \left\{ \mu : \mu([0,1]) = 1 \text{ and } \int v_j d\mu(v) \geq k_j, j = 1, ..., k \right\}$$

The Modified Min Problem: First Fact

A solution ν exists:

- Proof uses standard arguments
  - The objective is a bounded linear functional (hence, continuous)
  - The choice set is weak-* compact

The Modified Min Problem: Second Fact

There exists $\lambda \geq 0, \boldsymbol{\theta} = (\theta_1, ..., \theta_K) \geq 0$ so that, at the solution η, $\int l(v)\, d\eta(v)$ equals

$$\min_{\mu} \int l(v)\, d\mu(v) - \lambda \left(1 - \int d\mu(v)\right) - \sum_{j=1}^{k} \theta_j \left(\int v_j d\mu(v) - k_j\right)$$

Conversely, if μ is in $\mathcal{P}(k)$ *and* minimizes

$$\begin{aligned} & \int l(v)\, d\mu(v) - \lambda \left(1 - \int d\mu(v)\right) - \sum_{j=1}^{k} \theta_j \left(\int v_j d\mu(v) - k_j\right) \\ = \; & -\lambda + \boldsymbol{\theta} \cdot \mathbf{k} + \int [l(v) + \lambda - \boldsymbol{\theta} \cdot v]\, d\mu(v) \end{aligned}$$

μ is a solution of the (relaxed) min problem

The Modified Min Problem: Solution

- From Fact 2, one has to minimize

$$\Phi(\mu) = \int \left[l(v) + \lambda - \boldsymbol{\theta} \cdot v \right] d\mu(v)$$

- A solution exists only if $l(v) + \lambda - \boldsymbol{\theta} \cdot v \geq 0$

  – if $l(\widehat{v}) + \lambda - \boldsymbol{\theta} \cdot \widehat{v} < 0$,

$$\Phi(N\delta_{\widehat{v}}) \to -\infty \text{ as } N \to \infty$$

  where $\delta_{\widehat{v}}$ is the Dirac Measure concentrated at $\widehat{v}$.

The Modified Min Problem: Solution

Letting $I = \{v | l(v) + \lambda - \boldsymbol{\theta} \cdot v = 0\}$ and $J = \{v | l(v) + \lambda - \boldsymbol{\theta} \cdot v > 0\}$, if

1. $$\int v_j d\mu(v) \geq k_j$$

2. $$\mu(I) = 1, \mu(J) = 0$$

   μ solves the relaxed Min Problem

Implications for Allocation: Part I: Exclusion Region

$E = \{v | \lambda - \boldsymbol{\theta} \cdot v > 0\}$. Then, $l(v) > 0 \Leftrightarrow v \in E^c$.

Sketch:

- If v is in E^c and $l(v) = 0$, $l(v) + \lambda - \boldsymbol{\theta} \cdot v < 0$ (which cannot hold)
- If $v \in E$,

$$l(v) + \lambda - \boldsymbol{\theta} \cdot v > 0$$

  - $E \subseteq J \Rightarrow l(v) > 0$ is suboptimal (no direct effect and negative indirect effect)

$\boldsymbol{\theta} \cdot v \geq \lambda \Rightarrow$ "minimum revenue" requirement for sales

Implications for Revenues:

- $I = \{v | \boldsymbol{\theta} \cdot v \geq \lambda\}$. Over $I, l(v) = -\lambda + \boldsymbol{\theta} \cdot v$

Lemma: Collected Revenues are *linear* in valuations in any Robust Mechanism:

$$l(v) = \begin{cases} 0 \text{ if } v \in I^c \\ -\lambda + \boldsymbol{\theta} \cdot v \text{ o.w.,} \end{cases}$$

- Robust design imposes restrictions on payoff *levels*!
  - Rational story for decisions based on payoff *levels* rather than *marginal* analysis

Implications for Allocation:

$$\underbrace{v \cdot \nabla U(v) - U(v)}_{l(v)} = -\lambda + \boldsymbol{\theta} \cdot v, v \in I \Rightarrow$$

$$\begin{aligned} v \cdot \nabla^2 U(v) &= \boldsymbol{\theta}, v \in I \Rightarrow \\ v \cdot \nabla q(v) &= \boldsymbol{\theta}, v \in I \end{aligned}$$

- System of Partial Differential Equations with boundary condition $q(v) = 0$ in ∂I

General Solution of the System of PDEs:

- The system of PDEs + boundary condition $\Rightarrow$

$$q_j(v) = \theta_j \ln\left(\frac{\boldsymbol{\theta}\cdot v}{\boldsymbol{\theta}\cdot\widetilde{v}}\right), j = 1, ..., K$$

  – where $\boldsymbol{\theta}\cdot\widetilde{v} = \lambda$ ("pasting condition" assuring $q(v) = 0$ in ∂I)

The Robust Mechanism, or: finding $\boldsymbol{\theta}$ and $\widetilde{v}$

- The seller's problem becomes:

$$\max_{\boldsymbol{\theta},\widetilde{v}} \int [-\lambda + \boldsymbol{\theta}\cdot v]\, d\mu = \boldsymbol{\theta}\cdot [\mathbf{k} - \widetilde{v}]$$

subject to

$$q_j(1) = \theta_j \ln\left(\frac{\boldsymbol{\theta}\cdot\mathbf{1}}{\boldsymbol{\theta}\cdot\widetilde{v}}\right) \leq \mathbf{1}, j = 1, ..., K$$

The Robust Mechanism when $k = 1$

- For a given $\widetilde{v} \leq k$ (never the case that $\widetilde{v} > k$) , pick the largest θ compatible with constraint:

$$\theta = \frac{1}{\ln\left(\frac{1}{\widetilde{v}}\right)}$$

- Seller's problem becomes

$$\max_{\widetilde{v}} \frac{1}{\ln\left(\frac{1}{\widetilde{v}}\right)} [k - \widetilde{v}]$$

  - denote solution by $\widetilde{v}^* \in (0, 1)$

Result:

Theorem 1: Let be $\widetilde{v}^*$ be the solution of FOC. The optimal robust selling mechanism has

$$q(v) = \begin{cases} 0 \text{ if } v < \widetilde{v}^* \\ \frac{\ln\left(\frac{v}{\widetilde{v}^*}\right)}{\ln\left(\frac{1}{\widetilde{v}^*}\right)} = -\frac{(\ln(v) - \ln(\widetilde{v}^*))}{\ln(\widetilde{v}^*)} \end{cases}$$

Properties:

- Sales with probability smaller than one for all $v < 1$
  - Distortions also in the "intensive" margin despite lack of curvature in the agents (ex-post) payoff
- Price discrimination:
  - Insures against uncertainty without reducing (much) what can be charged from high value consumers
    - standard type of argument
- No distortion at the top: $q(1) = 1$

Implementation:

Many ways to implement: tariifs (i.e., using Taxation Principle) or posting prices $p \in [z, 1]$ drawn from distribution

$$G(p) = q(p) \text{ for all } p \in [\widetilde{v}^*, 1]$$

Theorem 2: A non-degenerate distribution of posted prices is an optimal robust selling mechanism. Tariff

$$T(q) = vq(v) - \int_0^v q(\tau)\, d\tau$$

also implements robust mechanism

Take Home Message:

- Uncertainty leds to price discrimination even with linear payoffs
  - Insure against uncertainty by selling to low valuation consumers (as "nature" will certainly pick those guys)
  - distort their allocation to be able to keep on selling for high valuation consumers at higher prices
- Discrimination limits "Nature's" ability to hurt the seller
- "Pricewise", insurance takes the form of a *distribution* of prices

The Robust Mechanism when $k > 1$

Much as before:

- Seller's problem:

$$\max_{\boldsymbol{\theta},\widetilde{v}} \int [-\lambda + \boldsymbol{\theta}\cdot v]\, d\mu = \boldsymbol{\theta}\cdot [\mathbf{k} - \widetilde{v}]$$

subject to

$$q_j(1) = \theta_j \ln\left(\frac{\boldsymbol{\theta}\cdot\mathbf{1}}{\boldsymbol{\theta}\cdot\widetilde{v}}\right) \leq 1, j = 1, ..., K$$

- Solution $\theta_i = \theta_j = \theta > 0$ for all i,j, $\widetilde{v}_j \in (0, 1)$

Properties:

- Full bundling (despite separable environment)
  - sales of each good depend on $\sum v_j$, a measure of aggregate willingnes to pay
  - kinder-egg effect: all goods sold in a one-to-one same proportion *regardless* of single valuations

Take Home Message:

- Uncertainty leads to bundling even if ex-post payoffs (broadly defined) are separable
  - Insure against uncertainty by looking at "aggregate willingness to pay" and selling basket of goods!

- Bundling limits "Nature's" ability to hurt the seller

Some Conclusions:

What have we done?

- Re-wrote (in an anti-Pierre Menard way) a standard model imposing robustness

- Fully derived a non-strandard (multidimensional private info) model also imposing robustness

  - hard in an "expected utility" environment, surprisingly simple with maxmin

Some Conclusions:

Why should one care?

- Leads to realistic contractual features without any reliance on (unobservable) payoff or environmentalcharacteristics

Conceptually:

- leads to fully rational decision-making based on payoff levels (rather than margins)
  - No behavioral BS

Some Methodological Conclusions:

- Worst-case design as a tractable alternative to fully Bayesian objectives in Mechanism Design

- Developed three ways of solving robust design problems

  - as presented here (set-up Lagrangean + find exclusion region+solve ODEs + solve simple maximization problem)

  - as a Nash Equilibrium of a zero-sum game (extends Carrasco and Moreira (2013))